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SOME COMBINATORIAL PROPERTIES OF THE LEAPING CONVERGENTS

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Abstract

Let $p_n/q_n = [a_0; a_1, \ldots, a_n]$ be the n-th convergent of the continued fraction expansion of $e^{1/s}$, where $s \geq 1$ is some integer. Elsner studied the case $s = 1$ in order to prove arithmetical properties of every third convergent p_{3n+1}/q_{3n+1} for Euler's number $e = [2; \overline{1, 2k, 1}]_{k=1}^{\infty}$. Komatsu studied the case for all $s \geq 2$ to prove those of every third convergent p_{3n}/q_{3n} for $e^{1/s} = [1; \overline{s(2k-1)-1, 1, 1}]_{k=1}^{\infty}$. He has also extended such results for some more general continued fractions. In spite of many properties of such leaping convergents, their explicit forms have not been known. In this paper we show some combinatorial properties of such leaping convergents.

1. Introduction

Let $p_n/q_n = [a_0; a_1, \ldots, a_n]$ be the n-th convergent of the continued fraction expansion of $[a_0; a_1, a_2, \ldots]$. The p_n's and q_n's satisfy the recurrence relation:

$$p_n = a_n p_{n-1} + p_{n-2} \quad (n \geq 0), \quad p_{-1} = 1, \quad p_{-2} = 0,$$
$$q_n = a_n q_{n-1} + q_{n-2} \quad (n \geq 0), \quad q_{-1} = 0, \quad q_{-2} = 1.$$

They also satisfy

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1},$$
$$p_n q_{n-2} - p_{n-2} q_n = (-1)^n a_n$$

and so on.

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The number $e^{1/s}$ $(s=1,2,\dots)$ has many significant arithmetical properties. Elsner [1] studied the case $s=1$ of Euler's number $e=[2;\overline{1,2k,1}]_{k=1}^{\infty}$, finding that every third convergent also has the similar characteristic relations to the original convergents. Putting $P_n=p_{3n+1}$, $Q_n=q_{3n+1}$ $(n\ge 0)$, $P_{-1}=P_{-2}=Q_{-1}=1$, $Q_{-2}=-1$, $P_{-n}=P_{n-3}$ and $Q_{-n}=-Q_{n-3}$ $(n\ge 0)$, then for any integer n,

$$P_n=2(2n+1)P_{n-1}+P_{n-2},\quad Q_n=2(2n+1)Q_{n-1}+Q_{n-2},$$
$$P_{n-1}Q_n-P_nQ_{n-1}=2(-1)^{n-1},$$
$$P_{n-2}Q_n-P_nQ_{n-2}=4(2n+1)(-1)^n$$

and some similar properties. Komatsu [3] studied the cases $s\ge 2$ of $e^{1/s}=[1;\overline{s(2k-1)-1,1,1}]_{k=1}^{\infty}$ to find similar properties. Putting $P_n=p_{3n}$, $Q_n=q_{3n}$ $(n\ge 0)$, $P_{-n}=P_{n-1}$ and $Q_{-n}=-Q_{n-1}$ $(n\ge 0)$, then for any integer n

$$P_n=2s(2n-1)P_{n-1}+P_{n-2},\quad Q_n=2s(2n-1)Q_{n-1}+Q_{n-2},$$
$$P_{n-1}Q_n-P_nQ_{n-1}=2(-1)^n,$$
$$P_{n-2}Q_n-P_nQ_{n-2}=4s(2n-1)(-1)^{n-1}$$

and some more similar properties. Though there are many properties of such leaping convergents, P_n or Q_n themselves have not been explicitly known. In this paper we exhibit combinatorially explicit forms of such leaping convergents.

Consider the continued fraction expansion of $\alpha=[1;\overline{ak+b,c,d}]_{k=1}^{\infty}$, where a and b are integers so that every $ak+b$ is a positive integer for $k=1,2,\dots$, and c and d are positive integers. It is known [2, Theorem 1] that if we put $P_n=p_{3n}$ and $Q_n=q_{3n}$ $(n\ge 0)$, then for $n\ge 2$ we have the recurrence relations

$$P_n=S_nP_{n-1}+P_{n-2},$$
$$Q_n=S_nQ_{n-1}+Q_{n-2}\,,$$

where $S_n=(cd+1)(an+b)+c+d$. However, none of $\hat{P}_n=p_{3n+1}$, $\hat{Q}_n=q_{3n+1}$, $\tilde{P}_n=p_{3n+2}$ and $\tilde{Q}_n=q_{3n+2}$ satisfies such recurrence relations. But they satisfy some different types of relation. It is impossible to express p_n or q_n in a closed form in a general way because $2(an+b+1)$ is not constant. Nevertheless, there exist general closed forms about P_n and Q_n, and even $\hat{P}_n$, $\hat{Q}_n$, $\tilde{P}_n$ and $\tilde{Q}_n$.

2. Combinatorial Expression of Leaping Convergents

As usual, binomial coefficients are defined by

$$\binom{n}{k}=\begin{cases}\frac{n(n-1)\dots(n-k+1)}{k!}, & k>0;\\ 1, & k=0;\\ 0, & k<0.\end{cases}$$

Our main result is the following theorem. For simplicity, put $S_n=(cd+1)(an+b)+c+d$. As usual, the empty product equals 1.

Theorem 1 Let $\alpha=[1;\overline{ak+b,c,d}]_{k=1}^\infty$. Then we have for $n\ge 1$

$$p_{3n}=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k}{k}\prod_{i=1}^{n-2k}S_{k+i}+(cd-c+1)\sum_{k=0}^{\lceil\frac{n}{2}\rceil-1}\binom{n-k-1}{k}\prod_{i=1}^{n-2k-1}S_{k+i+1}\,,$$

$$\begin{aligned}p_{3n-1}=&\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\left(\binom{n-k-1}{k-1}+\frac{c(a(k+1)+b+1)}{S_{k+1}}\binom{n-k-1}{k}\right)\prod_{i=1}^{n-2k}S_{k+i}\\&+\sum_{k=0}^{\lceil\frac{n}{2}\rceil-1}(S_{n-k}-a(n-2k-1))\binom{n-k-1}{k}S_{k+1}^{-1}\prod_{i=1}^{n-2k-1}S_{k+i}\,,\end{aligned}$$

$$\begin{aligned}p_{3n-2}=&\sum_{k=0}^{\lceil\frac{n}{2}\rceil-1}(a(k+1)+b+1)\binom{n-k-1}{k}\prod_{i=1}^{n-2k-1}S_{k+i+1}\\&+\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor-1}((n-k-1)ad-d+1)\binom{n-k-2}{k}\prod_{i=1}^{n-2k-2}S_{k+i+1}\end{aligned}$$

and

$$q_{3n}=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k}{k}\prod_{i=1}^{n-2k}S_{k+i}-c\sum_{k=0}^{\lceil\frac{n}{2}\rceil-1}\binom{n-k-1}{k}\prod_{i=1}^{n-2k-1}S_{k+i+1}\,,$$

$$\begin{aligned}q_{3n-1}=&\sum_{k=0}^{\lceil\frac{n}{2}\rceil-1}(c(a(k+1)+b)+1)\binom{n-k-1}{k}\prod_{i=1}^{n-2k-1}S_{k+i+1}\\&-a\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor-1}(n-k-1)\binom{n-k-2}{k}\prod_{i=1}^{n-2k-2}S_{k+i+1}\,,\end{aligned}$$

$$\begin{aligned}q_{3n-2}=&\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\left(\binom{n-k-1}{k-1}+\frac{a(k+1)+b+c}{S_{k+1}}\binom{n-k-1}{k}\right)\prod_{i=1}^{n-2k}S_{k+i}\\&-\sum_{k=0}^{\lceil\frac{n}{2}\rceil-1}(cS_{n-k}-ad(n-2k-1))\binom{n-k-1}{k}S_{k+1}^{-1}\prod_{i=1}^{n-2k-1}S_{k+i}\,.\end{aligned}$$

The 0-th convergent 1 may not be so important and some expressions may be awkward for this result. However, putting $c=d=1$ in Theorem 1, we have the following corresponding results between the p's and q's.

Corollary 1 Let $\alpha = [1; \overline{ak+b, 1, 1}]_{k=1}^{\infty}$. Then we have, for $n \geq 1$,

$$\begin{aligned}
p_{3n} &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k} \binom{n-k}{k} \prod_{i=1}^{n-2k} (a(k+i)+b+1) \\
&\quad + \sum_{k=0}^{\lceil \frac{n}{2} \rceil - 1} 2^{n-2k-1} \binom{n-k-1}{k} \prod_{i=1}^{n-2k-1} (a(k+i+1)+b+1), \\
p_{3n-1} &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k-1} \left(\binom{n-k}{k} + \binom{n-k-1}{k-1} \right) \prod_{i=1}^{n-2k} (a(k+i)+b+1) \\
&\quad + (a(n+1)+2(b+1)) \sum_{k=0}^{\lceil \frac{n}{2} \rceil - 1} 2^{n-2k-2} \binom{n-k-1}{k} \prod_{i=1}^{n-2k-2} (a(k+i+1)+b+1), \\
p_{3n-2} &= \sum_{k=0}^{\lceil \frac{n}{2} \rceil - 1} 2^{n-2k-1} \binom{n-k-1}{k} \prod_{i=1}^{n-2k} (a(k+i)+b+1) \\
&\quad + a \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} 2^{n-2k-2} (n-k-1) \binom{n-k-2}{k} \prod_{i=1}^{n-2k-2} (a(k+i+1)+b+1)
\end{aligned}$$

and

$$\begin{aligned}
q_{3n} &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k} \binom{n-k}{k} \prod_{i=1}^{n-2k} (a(k+i)+b+1) \\
&\quad - \sum_{k=0}^{\lceil \frac{n}{2} \rceil - 1} 2^{n-2k-1} \binom{n-k-1}{k} \prod_{i=1}^{n-2k-1} (a(k+i+1)+b+1), \\
q_{3n-1} &= \sum_{k=0}^{\lceil \frac{n}{2} \rceil - 1} 2^{n-2k-1} \binom{n-k-1}{k} \prod_{i=1}^{n-2k} (a(k+i)+b+1) \\
&\quad - a \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} 2^{n-2k-2} (n-k-1) \binom{n-k-2}{k} \prod_{i=1}^{n-2k-2} (a(k+i+1)+b+1), \\
q_{3n-2} &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k-1} \left(\binom{n-k}{k} + \binom{n-k-1}{k-1} \right) \prod_{i=1}^{n-2k} (a(k+i)+b+1) \\
&\quad - (a(n+1)+2(b+1)) \sum_{k=0}^{\lceil \frac{n}{2} \rceil - 1} 2^{n-2k-2} \binom{n-k-1}{k} \prod_{i=1}^{n-2k-2} (a(k+i+1)+b+1).
\end{aligned}$$

3. Examples

Put $a = 2s$ and $b = -s - 1$ in Corollary 1. Then we have the continued fraction expansion $e^{1/s} = [1; \overline{s(2k-1)-1, 1, 1}]_{k=1}^{\infty}$ $(s \geq 2)$. Let p_n/q_n be its n-th convergent. In addition, let p_n^*/q_n^* be the n-th convergent of the continued fraction expansion of $e = [2; \overline{1, 2k, 1}]_{k=1}^{\infty}$. Then, for $n \geq 1$, we have

$$p_{3n} = p_{3n-2}^* = \sum_{k=0}^{n} \frac{(2n-k)!}{k!(n-k)!} s^{n-k}, \quad q_{3n} = q_{3n-2}^* = \sum_{k=0}^{n} (-1)^k \frac{(2n-k)!}{k!(n-k)!} s^{n-k},$$

$$p_{3n-1} = p_{3n-3}^* = n \sum_{k=0}^{n} \frac{(2n-k-1)!}{k!(n-k)!} s^{n-k}, \quad q_{3n-1} = q_{3n-3}^* = \sum_{k=0}^{n-1} (-1)^k \frac{(2n-k-1)!}{k!(n-k-1)!} s^{n-k},$$

$$p_{3n-2} = p_{3n-4}^* = \sum_{k=0}^{n-1} \frac{(2n-k-1)!}{k!(n-k-1)!} s^{n-k}, \quad q_{3n-2} = q_{3n-4}^* = n \sum_{k=0}^{n} (-1)^k \frac{(2n-k-1)!}{k!(n-k)!} s^{n-k}.$$

Note that all the six formulas for p_{3n-2}^*, p_{3n-3}^*, p_{3n-4}^*, q_{3n-2}^*, q_{3n-3}^* and q_{3n-4}^* correspond to $s = 1$ in the continued fraction expansion of e.

Put $a = c = d = 0$ and $b = 1$ in Theorem 1. Then by $S_n = 1$ we have

$$Q_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}.$$

In fact, $Q_n = F_{n+1}$ $(n \geq 0)$, where F_n is the n-th Fibonacci number (see, e.g., [4, Theorem 12.4]). This is equivalent to the fact that the denominator of the n-th convergent of $[1; 1, 1, 1, \dots] = [1; \overline{1, 0, 0}\,]$ corresponds to F_n.

4. Proof of Theorem 1

We shall prove the results involving the q's by showing that the three recurrence relations

$$\begin{aligned} q_{3n} &= dq_{3n-1} + q_{3n-2} \\ q_{3n+1} &= (a(n+1)+b)q_{3n} + q_{3n-1} \\ q_{3n+2} &= cq_{3n+1} + q_{3n} \end{aligned}$$

are valid, provided that the identities from the theorem for the q's on the right sides are valid. Proof involving the p's can be done similarly. Note the addition formula

$$\binom{m}{l} + \binom{m}{l-1} = \binom{m+1}{l} \quad (l : \text{integer})$$

and the symmetry identity

$$\binom{m}{l} = \binom{m}{m-l} \quad (m \geq 0,\ l : \text{integers}).$$

For simplicity, put $K = a(k+1)+b$. It is easy to see that $q_0 = 1$, $q_1 = a+b$ and $q_2 = c(a+b)+1$. Suppose that the identities for q_{3n-2} and q_{3n-1} are valid for a positive integer n. Then we shall prove that $q_{3n} = dq_{3n-1} + q_{3n-2}$. This follows from

$$\begin{aligned}
& d\cdot(cK+1)\binom{n-k-1}{k}S_{k+2}\cdots S_{n-k} - d\cdot a(n-k-1)\binom{n-k-2}{k}S_{k+2}\cdots S_{n-k-1}\\
&\quad + \binom{n-k-1}{k-1}S_{k+1}\cdots S_{n-k} + (K+c)\binom{n-k-1}{k}S_{k+2}\cdots S_{n-k}\\
&\quad - c\binom{n-k-1}{k}S_{k+2}\cdots S_{n-k} + ad(n-2k-1)\binom{n-k-1}{k}S_{k+2}\cdots S_{n-k-1}\\
&= \binom{n-k-1}{k}S_{k+1}\cdots S_{n-k} + \binom{n-k-1}{k-1}S_{k+1}\cdots S_{n-k}\\
&\quad - c\binom{n-k-1}{k}S_{k+2}\cdots S_{n-k}\\
&= \binom{n-k}{k}S_{k+1}\cdots S_{n-k} - c\binom{n-k-1}{k}S_{k+2}\cdots S_{n-k}\,.
\end{aligned}$$

To see this, notice that if n is odd, then for $k = \lceil \frac{n}{2} \rceil - 1$ we have $ad(n-2k-1)\binom{n-k-1}{k} = 0$. If n is even, then for $k = \lfloor \frac{n}{2} \rfloor$ we have

$$\left(\binom{n-k-1}{k-1} + \frac{a(k+1)+b+c}{S_{k+1}}\binom{n-k-1}{k}\right)\prod_{i=1}^{n-2k} S_{k+i} = \binom{n-k}{k}\prod_{i=1}^{n-2k} S_{k+i} = 1\,.$$

Hence, we have the relation $q_{3n} = dq_{3n-1} + q_{3n-2}$.

Suppose that the identities for q_{3n-1} and q_{3n} are valid for a positive integer n. We shall prove that $q_{3n+1} = (a(n+1)+b)q_{3n} + q_{3n-1}$. We can see that

$$\begin{aligned}
(a(n+1)+b)\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n-k}{k}\prod_{i=1}^{n-2k} S_{k+i} &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\frac{K}{S_{k+1}}\binom{n-k}{k}\prod_{i=1}^{n-2k+1} S_{k+i} + a\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} k\binom{n-k}{k}\prod_{i=1}^{n-2k} S_{k+i}\\
&\quad + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} a(c+d)(n-2k)\binom{n-k}{k}S_{k+1}^{-1}\prod_{i=1}^{n-2k} S_{k+i}\\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\frac{K}{S_{k+1}}\binom{n-k}{k}\prod_{i=1}^{n-2k+1} S_{k+i}\\
&\quad + a\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1}(n-k-1)\binom{n-k-2}{k}\prod_{i=1}^{n-2k-2} S_{k+i+1}\\
&\quad + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} ca(n-k)\binom{n-k-1}{k}S_{k+1}^{-1}\prod_{i=1}^{n-2k} S_{k+i}\\
&\quad + \sum_{k=0}^{\lceil \frac{n-1}{2} \rceil} ad(n-2k)\binom{n-k}{k}S_{k+1}^{-1}\prod_{i=1}^{n-2k} S_{k+i}
\end{aligned}$$

and

$$
\begin{aligned}
&-(a(n+1)+b)c\sum_{k=0}^{\lceil\frac{n}{2}\rceil-1}\binom{n-k-1}{k}\prod_{i=1}^{n-2k-1}S_{k+i+1}+\sum_{k=0}^{\lceil\frac{n}{2}\rceil-1}(cK+1)\binom{n-k-1}{k}\prod_{i=1}^{n-2k-1}S_{k+i+1}\\
&=-\sum_{k=0}^{\lceil\frac{n}{2}\rceil-1}ca(n-k)\binom{n-k-1}{k}\prod_{i=1}^{n-2k-1}S_{k+i+1}+\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n-k}{k-1}\prod_{i=1}^{n-2k+1}S_{k+i}\,.
\end{aligned}
$$

Hence, if n is odd, then

$$
\begin{aligned}
(a(n+1)+b)q_{3n}+q_{3n-1}&=\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{n-k}{k-1}\prod_{i=1}^{n-2k+1}S_{k+i}+\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{K}{S_{k+1}}\binom{n-k}{k}\prod_{i=1}^{n-2k+1}S_{k+i}\\
&\quad+\sum_{k=0}^{\lceil\frac{n-1}{2}\rceil}ad(n-2k)\binom{n-k}{k}S_{k+1}^{-1}\prod_{i=1}^{n-2k}S_{k+i}\\
&=q_{3n+1},
\end{aligned}
$$

since $K\binom{n-k}{k}=0$ for $k=\lfloor\frac{n+1}{2}\rfloor$. If n is even, then by $ca(n-k)\binom{n-k-1}{k}=0$ for $k=\lfloor\frac{n}{2}\rfloor$, in a similar way we obtain

$$(a(n+1)+b)q_{3n}+q_{3n-1}=q_{3n+1}\,.$$

Suppose that the identities for q_{3n} and q_{3n+1} are valid for a positive integer n. Then we shall prove that $q_{3n+2}=cq_{3n+1}+q_{3n}$. Since

$$
\begin{aligned}
&c\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}\left(\binom{n-k}{k-1}+\frac{K}{S_{k+1}}\binom{n-k}{k}\right)\prod_{i=1}^{n-2k+1}S_{k+i}+\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k}{k}\prod_{i=1}^{n-2k}S_{k+i}\\
&=\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor-1}c\binom{n-k-1}{k}\prod_{i=1}^{n-2k-1}S_{k+i+1}+\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}cK\binom{n-k}{k}\prod_{i=1}^{n-2k}S_{k+i+1}\\
&\quad+\sum_{k=0}^{\lceil\frac{n+1}{2}\rceil-1}\binom{n-k}{k}\prod_{i=1}^{n-2k}S_{k+i+1}-\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(cd+1)a(n-2k)\binom{n-k}{k}\prod_{i=1}^{n-2k-1}S_{k+i+1}
\end{aligned}
$$

and

$$
\begin{aligned}
&c\sum_{k=0}^{\lceil\frac{n+1}{2}\rceil-1}ad(n-2k)\binom{n-k}{k}\prod_{i=1}^{n-2k-1}S_{k+i+1}-c\sum_{k=0}^{\lceil\frac{n}{2}\rceil-1}\binom{n-k-1}{k}\prod_{i=1}^{n-2k-1}S_{k+i+1}\\
&=cd\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}a(n-2k)\binom{n-k}{k}\prod_{i=1}^{n-2k-1}S_{k+i+1}-c\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor-1}\binom{n-k-1}{k}\prod_{i=1}^{n-2k-1}S_{k+i+1}\,,
\end{aligned}
$$

we have $cq_{3n+1}+q_{3n}=q_{3n+2}$. Notice that if n is odd, then $cK\binom{n-k}{k}=0$ for $k=\lfloor\frac{n+1}{2}\rfloor$; if n is even, then $(cd+1)a(n-2k)\binom{n-k}{k}=0$ for $k=\lfloor\frac{n}{2}\rfloor$.

5. Some More Recurrence Relations

We start by remarking that $\hat{P}_n = p_{3n+1}$, $\hat{Q}_n = q_{3n+1}$, $\tilde{P}_n = p_{3n+2}$ and $\tilde{Q}_n = q_{3n+2}$ do not satisfy any recurrence relations of the type $P_n = S_n P_{n-1} + P_{n-2}$. They do, however, satisfy some different types of relations. In fact, such relations also hold for a more general continued fraction of $[1; \overline{T_1(k), T_2(k), T_3(k)}]_{k=1}^{\infty}$.

Theorem 2 Let $\alpha = [1; \overline{T_1(k), T_2(k), T_3(k)}\,]_{k=1}^{\infty}$, where each $T_i(k)$ $(i = 1, 2, 3)$ takes a positive integer for $k = 1, 2, \dots$. Then for every integer $n \geq 2$ we have

$$(T_3(n-1)T_1(n) + 1)p_{3n+1} = U(n)p_{3n-2} + (T_3(n)T_1(n+1) + 1)p_{3n-5}\,,$$
$$(T_3(n-1)T_1(n) + 1)q_{3n+1} = U(n)q_{3n-2} + (T_3(n)T_1(n+1) + 1)q_{3n-5}\,,$$

where $U(n) = (T_3(n-1)T_1(n)+1)((T_2(n)T_3(n)+1)T_1(n+1)+T_2(n))+T_3(n-1)(T_3(n)T_1(n+1)+1)$, and

$$(T_1(n)T_2(n) + 1)p_{3n+2} = V(n)p_{3n-1} + (T_1(n+1)T_2(n+1) + 1)p_{3n-4}\,,$$
$$(T_1(n)T_2(n) + 1)q_{3n+2} = V(n)q_{3n-1} + (T_1(n+1)T_2(n+1) + 1)q_{3n-4}\,,$$

where $V(n) = (T_1(n)T_2(n) + 1)(T_3(n)(T_1(n+1)T_2(n+1) + 1) + T_2(n+1)) + T_1(n)(T_1(n+1)T_2(n+1) + 1)$.

Remark. If $T_1(n) = an + b$, $T_2(n) = c$, and $T_3(n) = d$, then

$$\begin{aligned}
U(n) &= a^2 d(cd+1)n^2 + a((a+2b)d(cd+1) + d^2 + 2cd + 1)n \\
&\quad + b(a+b)d(cd+1) + a(d^2 + cd + 1) + b(d^2 + 2cd + 1) + c + d\,, \\
U(n+1) &= a^2 d(cd+1)n^2 + a((3a+2b)d(cd+1) + d^2 + 2cd + 1)n \\
&\quad + (a+b)(2a+b)d(cd+1) + a(2d^2 + 3cd + 2) + b(d^2 + 2cd + 1) + c + d\,, \\
V(n) &= a^2 c(cd+1)n^2 + a((a+2b)c(cd+1) + c^2 + 2cd + 1)n \\
&\quad + b(a+b)c(cd+1) + acd + b(c^2 + 2cd + 1) + c + d\,, \\
V(n+1) &= a^2 c(cd+1)n^2 + a((3a+2b)c(cd+1) + c^2 + 2cd + 1)n \\
&\quad + (a+b)(2a+b)c(cd+1) + a(c^2 + 3cd + 1) + b(c^2 + 2cd + 1) + c + d\,.
\end{aligned}$$

Proof of Theorem 2. Since

$$\begin{pmatrix} p_{3n+1} & p_{3n} \\ q_{3n+1} & q_{3n} \end{pmatrix} = \begin{pmatrix} p_{3n-2} & p_{3n-3} \\ q_{3n-2} & q_{3n-3} \end{pmatrix} \begin{pmatrix} T_2(n) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} T_3(n) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} T_1(n+1) & 1 \\ 1 & 0 \end{pmatrix},$$

by comparing the $(1, 1)$ element, we have

$$p_{3n+1} = ((T_2(n)T_3(n) + 1)T_1(n+1) + T_2(n))p_{3n-2} + (T_3(n)T_(n+1) + 1)p_{3n-3}\,. \qquad (1)$$

The identity involving the q's is similar and omitted.

Since

$$\begin{pmatrix} p_{3n-2} & p_{3n-3} \\ q_{3n-2} & q_{3n-3} \end{pmatrix} = \begin{pmatrix} p_{3n-5} & p_{3n-6} \\ q_{3n-5} & q_{3n-6} \end{pmatrix} \begin{pmatrix} T_2(n-1) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} T_3(n-1) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} T_1(n) & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} p_{3n-5} & p_{3n-6} \\ q_{3n-5} & q_{3n-6} \end{pmatrix} \begin{pmatrix} (T_2(n-1)T_3(n-1)+1)T_1(n)+T_2(n-1) & T_2(n-1)T_3(n-1)+1 \\ T_3(n-1)T_1(n)+1 & T_3(n-1) \end{pmatrix},$$

by

$$\begin{pmatrix} p_{3n-5} & p_{3n-6} \\ q_{3n-5} & q_{3n-6} \end{pmatrix}$$
$$= \begin{pmatrix} p_{3n-2} & p_{3n-3} \\ q_{3n-2} & q_{3n-3} \end{pmatrix} \begin{pmatrix} -T_3(n-1) & T_2(n-1)T_3(n-1)+1 \\ T_3(n-1)T_1(n)+1 & -(T_2(n-1)T_3(n-1)+1)T_1(n)-T_2(n-1) \end{pmatrix},$$

we have

$$p_{3n-5} = -T_3(n-1)p_{3n-2} + (T_3(n-1)T_1(n)+1)p_{3n-3}\,. \tag{2}$$

Substituting (1) and (2) into

$$(T_3(n-1)T_1(n)+1)p_{3n+1} = U(n)p_{3n-2} + (T_3(n)T_1(n+1)+1)p_{3n-5}$$

and comparing the coefficient of p_{3n-2}, we get $U(n) = (T_3(n-1)T_1(n)+1)((T_2(n)T_3(n)+1)T_1(n+1)+T_2(n)) + T_3(n-1)(T_3(n)T_1(n+1)+1)$. Notice that the coefficient of p_{3n-3} is cancelled.

In a similar manner, substituting $p_{3n+2} = (T_3(n)(T_1(n+1)T_2(n+1)+1)+T_2(n+1))p_{3n-1} + (T_1(n+1)T(n+2)+1)p_{3n-2}$ and $p_{3n-4} = -T_1(n)p_{3n-1} + (T_1(n)T_2(n)+1)p_{3n-2}$ into $(T_1(n)T_2(n)+1)p_{3n+2} = V(n)p_{3n-1} + (T_1(n+1)T_2(n+1)+1)p_{3n-4}$ and comparing the coefficient of p_{3n-1}, we get $V(n) = (T_1(n)T_2(n)+1)(T_3(n)(T_1(n+1)T_2(n+1)+1)+T_2(n+1)) + T_1(n)(T_1(n+1)T_2(n+1)+1)$.

Comments. A further generalization on the leaping convergent was mentioned in [2]. It seems possible to obtain the explicit forms for $[1;\overline{T(k),c,d}\,]_{k=1}^{\infty}$ in Theorem 1, $[1;\overline{T_1(k),T_2(k),T_3(k),T_4(k),T_5(k)}\,]_{k=1}^{\infty}$ in Theorems 1 and 2, and so on. Unfortunately, for example $q_{3n} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \prod_{i=1}^{n-2k} S_{k+i} - c\sum_{k=0}^{\lceil \frac{n}{2} \rceil -1} \binom{n-k-1}{k} \prod_{i=1}^{n-2k-1} S_{k+i+1}$ does not hold if the general $T(k)$ replaces $ak+b$ with $S_n = (cd+1)T(n)+c+d$ in Theorem 1. We are currently preparing such results.

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