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LINEAR QUANTUM ADDITION RULES

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Abstract

The quantum integer $[n]_q$ is the polynomial $1 + q + q^2 + \cdots + q^{n-1}$. Two sequences of polynomials $\mathcal{U} = \{u_n(q)\}_{n=1}^{\infty}$ and $\mathcal{V} = \{v_n(q)\}_{n=1}^{\infty}$ define a *linear addition rule* $\oplus$ on a sequence $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ by $f_m(q) \oplus f_n(q) = u_n(q)f_m(q) + v_m(q)f_n(q)$. This is called a *quantum addition rule* if $[m]_q \oplus [n]_q = [m+n]_q$ for all positive integers m and n. In this paper all linear quantum addition rules are determined, and all solutions of the corresponding functional equations $f_m(q) \oplus f_n(q) = f_{m+n}(q)$ are computed.

–To Ron Graham on his 70th birthday

1. Multiplication and Addition of Quantum Integers

We consider polynomials $f(q)$ with coefficients in a commutative ring with 1. A sequence $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ of polynomials is nonzero if $f_n(q) \neq 0$ for some integer n. For every positive integer n, the *quantum integer* $[n]_q$ is the polynomial $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$. These polynomials appear in many contexts. In quantum calculus (Cheung-Kac [2]), for example, the q derivative of $f(x) = x^n$ is

$$f'(x) = \frac{f(qx) - f(x)}{qx - x} = [n]_q x^{n-1}.$$

The quantum integers are ubiquitous in the study of quantum groups (Kassel [3]).

Let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a sequence of polynomials. Nathanson [5] observed that the multiplication rule

$$f_m(q) * f_n(q) = f_m(q) f_n(q^m)$$

induces a natural multiplication on the sequence of quantum integers, since $[m]_q * [n]_q = [mn]_q$ for all positive integers m and n. He asked what sequences $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ of polynomials,

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rational functions, and formal power series satisfy the multiplicative functional equation

$$f_m(q) * f_n(q) = f_{mn}(q) \tag{1}$$

for all positive integers m and n. Borisov, Nathanson, and Wang [1] proved that the only solutions of (1) in the field $\mathbf{Q}(q)$ of rational functions with rational coefficients are essentially quotients of products of quantum integers. More precisely, let $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ be a nonzero solution of (1) in $\mathbf{Q}(q)$, and let $\text{supp}(\mathcal{F})$ be the set of all integers n with $f_n(q) \neq 0$. They proved that there is a finite set R of positive integers and a set $\{t_r\}_{r\in R}$ of integers such that, for all $n \in \text{supp}(\mathcal{F})$,

$$f_n(q) = \lambda(n) q^{t_0(n-1)} \prod_{r\in R} [n]_{q^r}^{t_r},$$

where $\lambda(n)$ is a completely multiplicative arithmetic function and t_0 is a rational number such that $t_0(n-1) \in \mathbf{Z}$ for all $n \in \text{supp}(\mathcal{F})$. Nathanson [6] also proved that if $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ is any solution of the functional equation (1) in polynomials or formal power series with coefficients in a field, and if $f_n(0) = 1$ for all $n \in \text{supp}(\mathcal{F})$, then there exists a formal power series $F(q)$ such that

$$\lim_{\substack{n\to\infty \\ n\in\text{supp}(\mathcal{F})}} f_n(q) = F(q).$$

Nathanson [7] also defined the addition rule

$$f_m(q) \oplus f_n(q) = f_m(q) + q^m f_n(q) \tag{2}$$

on a sequence $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$of polynomials, and considered the additive functional equation

$$f_m(q) \oplus q^m f_n(q) = f_{m+n}(q). \tag{3}$$

He noted that

$$[m]_q \oplus [n]_q = [m+n]_q \tag{4}$$

for all positive integers m and n, and proved that every solution of the additive functional equation (3) is of the form $f_n(q) = h(q)[n]_q$, where $h(q) = f_1(q)$. This implies that if a nonzero sequence of polynomials $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ satisfies both the multiplicative functional equation (1) and additive function equation (3), then $f_n(q) = [n]_q$ for all positive integers n.

In this paper we consider other binary operations $f_m(q)\oplus f_n(q)$ on sequences of polynomials that induce the natural addition of quantum integers or, equivalently, that satisfy (4). The goal of this paper is to prove that the addition rule (2) is essentially the only linear quantum addition rule, and to find all solutions of the associated additive functional equation.

2. Linear Addition Rules

A general *linear quantum addition rule* is defined by two doubly infinite sequences of polynomials $\mathcal{U} = \{u_{m,n}(q)\}_{m,n=1}^{\infty}$ and $\mathcal{V} = \{v_{m,n}(q)\}_{m,n=1}^{\infty}$ such that

$$[m+n]_q = u_{m,n}(q)[m]_q + v_{m,n}(q)[n]_q \tag{5}$$

for all positive integers m and n. If the sequences $\mathcal{U}$ and $\mathcal{V}$ satisfy (5), then $\mathcal{U}$ determines $\mathcal{V}$, and conversely. It is not known for what sequences $\mathcal{U}$ there exists a complementary sequence $\mathcal{V}$ satisfying (5).

A *linear zero identity* is determined by two sequences of polynomials $\mathcal{S} = \{s_{m,n}(q)\}_{m,n=1}^{\infty}$ and $\mathcal{T} = \{t_{m,n}(q)\}_{m,n=1}^{\infty}$ such that $s_{m,n}(q)[m]_q + t_{m,n}(q)[n]_q = 0$ for all positive integers m and n.

We can construct new addition rules from old rules by adding zero identities and by taking affine combinations of addition rules. For example, the simplest quantum addition rule is

$$[m+n]_q = [m]_q + q^m[n]_q. \tag{6}$$

Then

$$[m]_q + q^m[n]_q = [m+n]_q = [n+m]_q = [n]_q + q^n[m]_q$$

for all positive integers m and n, and we obtain the zero identity

$$(1-q^n)[m]_q + (q^m-1)[n]_q = 0. \tag{7}$$

Adding (6) and (7), we obtain

$$[m+n]_q = (2-q^n)[m]_q + (2q^m-1)[n]_q. \tag{8}$$

An affine combination of (6) and (8) gives

$$[m+n]_q = (4-3q^n)[m]_q + (4q^m-3)[n]_q. \tag{9}$$

We can formally describe this process as follows.

Theorem 1. *For $i = 1,\ldots,k$, let $\mathcal{U}^{(i)} = \{u_{m,n}^{(i)}(q)\}_{m,n=1}^{\infty}$ and $\mathcal{V}^{(i)} = \{v_{m,n}^{(i)}(q)\}_{m,n=1}^{\infty}$ be sequences of polynomials that determine a quantum addition rule. If $\alpha_1,\ldots,\alpha_k$ are elements of the coefficient ring such that $\alpha_1+\cdots+\alpha_k = 1$, and if the sequences $\mathcal{U} = \{u_{m,n}(q)\}_{m,n=1}^{\infty}$ and $\mathcal{V} = \{v_{m,n}(q)\}_{m,n=1}^{\infty}$ are defined by*

$$u_{m,n}(q) = \sum_{i=1}^{k} \alpha_i u_{m,n}^{(i)}(q)$$

and

$$v_{m,n}(q) = \sum_{i=1}^{k} \alpha_i v_{m,n}^{(i)}(q)$$

for all positive integers m and n, then $\mathcal{U}$ and $\mathcal{V}$ determine a quantum addition rule. Similarly, if $\mathcal{U} = \{u_{m,n}(q)\}_{m,n=1}^{\infty}$ and $\mathcal{V} = \{v_{m,n}(q)\}_{m,n=1}^{\infty}$ are sequences of polynomials that determine a quantum addition rule, and if $\mathcal{S} = \{s_{m,n}(q)\}_{m,n=1}^{\infty}$ and $\mathcal{T} = \{t_{m,n}(q)\}_{m,n=1}^{\infty}$ are sequences of polynomials that determine a zero identity, then the sequences $\mathcal{U}+\mathcal{S} = \{u_{m,n}(q)+s_{m,n}(q)\}_{m,n=1}^{\infty}$ and $\mathcal{V}+\mathcal{T} = \{v_{m,n}(q)+t_{m,n}(q)\}_{m,n=1}^{\infty}$ determine a quantum addition rule.

3. The Fundamental Quantum Addition Rule

In this paper we consider sequences $\mathcal{U}$ and $\mathcal{V}$ that depend only on m or n. We shall classify all linear zero identities and all linear quantum addition rules.

Theorem 2. *Let* $\mathcal{S} = \{s_n(q)\}_{n=1}^{\infty}$ *and* $\mathcal{T} = \{t_m(q)\}_{m=1}^{\infty}$ *be sequences of polynomials. Then*

$$s_n(q)[m]_q + t_m(q)[n]_q = 0 \tag{10}$$

for all positive integers m *and* n *if and only if there exists a polynomial* $z(q)$ *such that*

$$s_n(q) = z(q)[n]_q \qquad \text{for all } n \geq 1 \tag{11}$$

and

$$t_m(q) = -z(q)[m]_q \qquad \text{for all } m \geq 1. \tag{12}$$

If

$$s_m(q)[m]_q + t_m(q)[n]_q = 0 \tag{13}$$

for all positive integers m *and* n*, or if*

$$s_m(q)[m]_q + t_n(q)[n]_q = 0 \tag{14}$$

for all positive integers m *and* n*, then* $s_n(q) = t_n(q) = 0$ *for all* n*.*

Proof. If there exists a polynomial $z(q)$ such that the sequences $\mathcal{S}$ and $\mathcal{T}$ satisfy identities (11) and (12), then $s_n(q)[m]_q + t_m(q)[n]_q = z(q)[n]_q[m]_q - z(q)[m]_q[n]_q = 0$ for all m and n.

Conversely, suppose that the sequences $\mathcal{S}$ and $\mathcal{T}$ define a linear zero identity of the form (10). Letting $m = n = 1$ in (10), we have

$$s_1(q) + t_1(q) = s_1(q)[1]_q + t_1(q)[1]_q = 0.$$

Let $z(q) = s_1(q) = -t_1(q)$. For all positive integers n we have

$$s_n(q)[1]_q + t_1(q)[n]_q = s_n(q) - z(q)[n]_q = 0,$$

and so $s_n(q) = z(q)[n]_q$. Similarly,

$$s_1(q)[m]_q + t_m(q)[1]_q = z(q)[m]_q + t_m(q) = 0,$$

and so $t_m(q) = -z(q)[m]_q$ for all positive integers m.

If the sequences $\mathcal{S}$ and $\mathcal{T}$ define a linear zero identity of the form (13), then

$$t_m(q)[n]_q = -s_m(q)[m]_q = t_m(q)[n+1]_q = t_m(q)([n]_q + q^n),$$

and so $t_m(q)q^n = 0$. It follows that $t_m(q) = 0$ for all m, and so $s_m(q) = 0$ for all m.

Suppose that the sequences $\mathcal{S}$ and $\mathcal{T}$ define a linear zero identity of the form (14). Then

$$s_m(q)[m]_q = -t_n(q)[n]_q$$

for all m and n. This implies that if $s_m(q) \neq 0$ for some m, then $t_n(q) \neq 0$ for all n and $s_m(q) \neq 0$ for all m. If $\mathcal{S}$ and $\mathcal{T}$ are not the zero sequences, then, denoting the degree of a polynomial f by $\deg(f)$, we obtain $\deg(s_m) + m - 1 = \deg(t_n) + n - 1 \geq n - 1$, and so $\deg(s_m) \geq n - m$ for all positive integers n, which is absurd. Therefore, $\mathcal{S}$ and $\mathcal{T}$ are the zero sequences. This completes the proof. □

Theorem 3. *Let $\mathcal{U} = \{u_n(q)\}_{n=1}^{\infty}$ and $\mathcal{V} = \{v_m(q)\}_{m=1}^{\infty}$ be sequences of polynomials. Then*

$$[m+n]_q = u_n(q)[m]_q + v_m(q)[n]_q \tag{15}$$

for all positive integers m and n if and only if there exists a polynomial $z(q)$ such that

$$u_n(q) = 1 + z(q)[n]_q \tag{16}$$

and

$$v_m(q) = q^m - z(q)[m]_q \tag{17}$$

for all positive integers m and n. Moreover, $z(q) = u_1(q) - 1 = q - v_1(q)$.

Proof. Let $z(q)$ be any polynomial, and define the sequences $\mathcal{U} = \{u_n(q)\}_{n=1}^{\infty}$ and $\mathcal{V} = \{v_m(q)\}_{m=1}^{\infty}$ by (16) and (17). Then

$$\begin{aligned} u_n(q)[m]_q + v_m(q)[m]_q &= (1 + z(q)[n]_q)[m]_q + (q^m - z(q)[m]_q)[n]_q \\ &= ([m]_q + q^m[n]_q) + (z(q)[n]_q[m]_q - z(q)[m]_q[n]_q) \\ &= [m]_q + q^m[n]_q \\ &= [m+n]_q. \end{aligned}$$

Conversely, let $\mathcal{U} = \{u_n(q)\}_{n=1}^{\infty}$ and $\mathcal{V} = \{v_m(q)\}_{m=1}^{\infty}$ be a solution of (15). We define $z(q) = u_1(q) - 1$. Since $1 + q = [2]_q = [1+1]_q = u_1(q) + v_1(q) = 1 + z(q) + v_1(q)$, it follows that $v_1(q) = q - z(q)$. For all positive integers m we have $[m+1]_q = u_1(q)[m]_q + v_m(q)$, so

$$\begin{aligned} v_m(q) &= [m+1]_q - u_1(q)[m]_q \\ &= q^m + [m]_q - u_1(q)[m]_q \\ &= q^m - z(q)[m]_q. \end{aligned}$$

Similarly, for all positive integers n we have $[n+1]_q = [1+n]_q = u_n(q) + v_1(q)[n]_q$, and so

$$\begin{aligned} u_n(q) &= [n+1]_q - v_1(q)[n]_q \\ &= 1 + q[n]_q - (q - z(q))[n]_q \\ &= 1 + z(q)[n]_q. \end{aligned}$$

This completes the proof. □

For example, we can rewrite the quantum addition rule (9) in the form

$$\begin{aligned} [m+n]_q &= (4 - 3q^n)[m]_q + (4q^m - 3)[n]_q \\ &= (1 + z(q)[n]_q)[m]_q + (q^m - z(q)[m]_q)[n]_q, \end{aligned}$$

where $z(q) = 3 - 3q$.

Theorem 4. *Let $\mathcal{U} = \{u_m(q)\}_{m=1}^{\infty}$ and $\mathcal{V} = \{v_n(q)\}_{n=1}^{\infty}$ be sequences of polynomials. Then*

$$[m+n]_q = u_m(q)[m]_q + v_m(q)[n]_q \tag{18}$$

for all positive integers m and n if and only if $u_m(q) = 1$ and $v_m(q) = q^m$ for all m. There do not exist sequences of polynomials $\mathcal{U} = \{u_m(q)\}_{m=1}^{\infty}$ and $\mathcal{V} = \{v_n(q)\}_{n=1}^{\infty}$ such that

$$[m+n]_q = u_m(q)[m]_q + v_n(q)[n]_q \tag{19}$$

for all positive integers m and n.

Proof. Suppose that for every positive integer m we have

$$[m+1]_q = u_m(q)[m]_q + v_m(q)[1]_q = u_m(q)[m]_q + v_m(q),$$

and

$$[m+2]_q = u_m(q)[m]_q + v_m(q)[2]_q = u_m(q)[m]_q + (1+q)v_m(q).$$

Subtracting, we obtain

$$q^{m+1} = [m+2]_q - [m+1]_q = qv_m(q),$$

and so $v_m(q) = q^m$. Then

$$u_m(q)[m]_q = [m+1]_q - v_m(q) = [m+1]_q - q^m = [m]_q,$$

and so $u_m(q) = 1$ for all m. This proves the first assertion of the theorem.

If (19) holds for $n = 1$ and all m, then

$$[m+1]_q = u_m(q)[m]_q + v_1(q)[1]_q = u_m(q)[m]_q + v_1(q),$$

and so

$$u_m(q)[m]_q = [m+1]_q - v_1(q).$$

We also have

$$[m+2]_q = u_m(q)[m]_q + v_2(q)[2]_q = [m+1]_q - v_1(q) + (1+q)v_2(q),$$

and so

$$q^{m+1} = [m+2]_q - [m+1]_q = (1+q)v_2(q) - v_1(q)$$

for all positive integers m, which is absurd. □

Theorems 3 and 4 show that all linear quantum addition rules are of the form $[m+n]_q = u_n(q)[m]_q + v_m(q)[n]_q$. The following result shows that the sequence of quantum integers is essentially the only solution of the corresponding functional equation.

Theorem 5. *Let $\mathcal{U} = \{u_n(q)\}_{n=1}^{\infty}$ and $\mathcal{V} = \{v_m(q)\}_{m=1}^{\infty}$ be sequences of polynomials such that $[m+n]_q = u_n(q)[m]_q + v_m(q)[n]_q$ for all positive integers m and n. Then $\mathcal{F} = \{f_n(q)\}_{n=1}^{\infty}$ is a solution of the functional equation $f_{m+n}(q) = u_n(q)f_m(q) + v_m(q)f_n(q)$ if and only if there is a polynomial $h(q)$ such that $f_n(q) = h(q)[n]_q$ for all $n \geq 1$.*

Proof. By Theorem 3, there exists a polynomial $z(q)$ such that $u_n(q) = 1 + z(q)[n]_q$ and $v_m(q) = q^m - z(q)[m]_q$ for all positive integers m and n. The proof is by induction on n. Let $h(q) = f_1(q)$. Suppose that $f_n(q) = h(q)[n]_q$ for some integer $n \geq 1$. Then

$$\begin{aligned} f_{n+1}(q) &= u_1(q)f_n(q) + v_n(q)f_1(q) \\ &= (1+z(q))h(q)[n]_q + (q^n - z(q)[n]_q)h(q) \\ &= h(q)([n]_q + q^n) \\ &= h(q)[n+1]_q. \end{aligned}$$

This completes the proof. □

Remark. The only property of polynomials used in this paper is the degree of a polynomial, which occurs in the proof that there is no nontrivial zero identity of the form (14). It follows that Theorems 3 and 5 hold in any algebra that contains the quantum integers, for example, the polynomials, the rational functions, the formal power series, or the formal Laurent series with coefficients in a ring or field.

4. Nonlinear Addition Rules

A. V. Kontorovich observed that the quantum integers satisfy the following two nonlinear addition rules:

$$[m+n]_q = [m]_q + [n]_q - (1-q)[m]_q[n]_q$$

and

$$[m+n]_q = q^n[m]_q + q^m[n]_q + (1-q)[m]_q[n]_q.$$

These give rise to the functional equations

$$f_m(q) \oplus f_n(q) = f_m(q) + f_n(q) - (1-q)f_m(q)f_n(q)$$

and

$$f_m(q) \oplus f_n(q) = q^n f_m(q) + q^m f_n(q) + (1-q)f_m(q)f_n(q),$$

whose solutions are, respectively,

$$f_n(q) = \frac{1}{q-1}\sum_{k=1}^{n}\binom{n}{k}((q-1)f_1(q))^k = \frac{1-(1+(q-1)f_1(q))^n}{1-q}.$$

and

$$f_n(q) = \frac{1}{q-1}\sum_{k=1}^{n}\binom{n}{k}q^{n-k}((1-q)f_1(q))^k = \frac{(q+(1-q)f_1(q))^n - q^n}{1-q}.$$

Kontorovich and Nathanson [4] have recently described all quadratic addition rules for the quantum integers. It would be interesting to classify higher order nonlinear quantum addition rules.

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