INTEGERS: ELECTRONIC JOURNAL OF COMBINATORIAL NUMBER THEORY 7(2) (2007), #A28

SEMIGROUPS WITH THE ERDŐS-TURÁN PROPERTY

J. Nešetřil
Department of Applied Mathematics Institute of Theoretical Computer Science (ITI), Charles University, Prague
nesetril@kam.ms.mff.cuni.cz

O. Serra[1]
Department of Applied Mathematics IV, Polytechnical University of Catalonia, Barcelona
oserra@ma4.upc.es

Received: 3/13/06, Accepted: 8/16/06

Abstract

A set X in a semigroup G has the Erdős-Turán property ET if, for any basis A of X, the representation function r_A is ubounded, where $r_A(x)$ counts the number of representations of x as a product two elements in A. We show that, under some conditions, operations on binary vectors whose value at each coordinate depends only on neighbouring coordinates of the factors give rise to semigroups with the ET–property. In particular countable powers of semigroups with no mutually inverse elements have the ET–property. As a consequence, for each k there is $N(k)$ such that, for every finite subset X of a group G with $X \cap X^{-1} = \{1\}$, the representation function of every basis of $X^N \subset G^N$, $N \geq N(k)$, is not bounded by k. This is in contrast with the known fact that each p–elementary group admits a basis of the whole group whose representation function is bounded by an absolute constant.

1. Introduction

Let $(G, *)$ be a set with a binary operation and $X \subset G$. A subset $A \subset X$ is a *basis* of X if $X \subset A * A$. When X is an infinite set, A is an *asymptotic basis* if $X \setminus (A * A)$ is finite.

For $g \in G$ we denote by $r_A(g)$ the number of pairs $(a, a') \in A \times A$ such that $g = a * a'$. The function r_A is the *representation function* of A.

[1]Supported by the Spanish Research Council under project BFM2002-00412

Given a positive integer k, we say that X has the *k-Erdős-Turán property* $ET(k)$ if, for any basis A of X, there is an element $x \in X$ with $r_A(x) \geq k$. We say that X has the *Erdős-Turán property* ET if it has $ET(k)$ for every $k \in \mathbb{N}$.

A famous conjecture of Erdős and Turán [5] formulated in 1942 states that the set of positive integers with addition has the ET property. The conjecture was proved to be true for the class of so-called d-bounded additive bases of $\mathbb{N}$, see [10]. However it is still wide open in its general formulation. Erdős showed that the function $r_A(n)$ can have logarithmic growth. Ruzsa [12] gives a construction of a basis A for which the number of representations is bounded in the square mean. These results indicate the difficulties involved in the conjecture and leads to the consideration of the problem in other semigroups.

The situation for the integers with multiplication is different. Erdős [1] proved in 1964 that $(\mathbb{N}, \cdot)$ does have the ET property. Nešetřil and Rödl [9] gave a simple proof of this result by using Ramsey Theorem. Puš [11] extended this result by showing that an abelian semigroup with an infinite set of primes and a finite number of units has the ET property.

On the negative side, it is not difficult to show that the group of integers with addition has basis with unique representation (up to commutativity), a result which can be extended to any abelian free group. Nathanson [8] even showed that the direct product of a countable semigroup with an infinite abelian group G such that $\{12g;\ g \in G\}$ is infinite, admits an asymptotic basis whose representation function can be arbitrarily prescribed. Ruzsa [12] shows that, for any prime p such that 2 is a square, there are bases of $\mathbb{Z}_p \times \mathbb{Z}_p$ whose representation function is bounded by 18. This result has been recently extended by Haddad and Helou [6] to the additive group of $F \times F$ for any finite field F of odd order. These authors also show [7] that there is an absolute constant C such that no cyclic group has the property $ET(k)$ for $k > C$. In particular, the p–elementary groups with $p > 2$ do not have the $ET(k)$–property for $k > 18C$.

In this paper we place the Ramsey argument of [9] in a natural broader setting by considering operations in the set of binary vectors. We show that locally bounded operations, a notion explained in Section 2, on binary vectors of length N do have the $ET(k)$ property for each positive integer k and large enough N. This result allows one to show that several classes of semigroups have the ET property. Among these there are the class of direct products of semigroups with no mutually inverse elements, the class of finite or cofinite sets of a countable set with union, or the class $\{(\mathbb{N}^k, +), k \in \mathbb{N}\}$ of powers of $\mathbb{N}$ with componentwise addition.

As a consequence of these results, combined with the negative results for abelian groups mentioned above, we can formulate the following statement:

Theorem 1 *Let G be a group of prime order $p > 2$ and k a positive integer. Let $R \subset G$ such that $R \cap (-R) = \{0\}$. There is $N(k)$ such that $R^N \subset G^N$ has the $ET(k)$ property for each $N \geq N(k)$, while G^N has a basis A with representation function bounded by $18C$.*

Theorem 1 shows that there are abelian groups which admit bases with bounded representation function, while containing asymmetric subsets with the *ET* property. The Erdős–Turán conjecture says that this is the case for the group of integers.

2. Locally Bounded Operations on Binary Vectors

Let $B_N = \{0,1\}^N$ denote the set of all binary vectors of length N. We denote by $s(x)$ the support of vector $x \in B_N$, that is, the set of nonzero coordinates of x. We denote by $B_{\mathbb{N}}$ the set of infinite binary sequences with finite support.

Given a nonnegative integer r we denote by $I_r = [-r, r]$ the integer interval of length $2r+1$ centered at 0. The *r–neighborhood* of a subset $U \subset [1, N]$ is defined as the set $(\overline{U})_r = (\bigcup_{i\in U}(i + I_r)) \cap [1, N]$. For a vector $x \in B_N$ we denote by $(x)_U = (x_i;\ i \in U)$ the vector of length $|U|$ of the entries of x in U.

Let $\alpha : \mathbb{N}_0 \to \mathbb{N}_0$ be a function. A binary operation '$*$' defined in B_N is said to be *α–locally bounded* if, for every three vectors x, x', y, and every i, $1 \le i \le N$, the following conditions hold:

$$\text{if } (x)_{(\bar{i})_r} = (x')_{(\bar{i})_r} \text{ then also } (x * y)_i = (x' * y)_i, \tag{1}$$

$$\text{if } (x)_{(\bar{i})_r} = (0, \cdots, 0) \text{ then } (x * y)_i = (y * x)_i, \tag{2}$$

$$s(x * y) \subset (\overline{s(x) \cup s(y)})_r \quad \text{and} \quad s(x) \cup s(y) \subset (\overline{s(x * y)})_r. \tag{3}$$

where $r = \alpha(|s(x*y)|)$. Condition (1) says that the i–th coordinate of $x * y$ depends only on the i-th coordinate of y and the coordinates of x in the r–neighborhood of i. Condition (3) specifies that a nonzero entry can occur in $x * y$ only in the r–neighborhood of the supports of x or y, and also that such nonzero entries must occur in a way that their r–neighborhoods cover the support of x and y; in particular the support of $x * y$ is bounded from above and from below, through the function α, by the supports of x and y. These two conditions are the essential properties of a locally bounded operation. Finally (2) states that the zero vector commutes 'locally' with every vector.

Theorem 2 *Let k be a positive integer and $\alpha : \mathbb{N}_0 \to \mathbb{N}_0$ a function. There is $N(k, \alpha)$ such that $(B_N, *)$ has the $ET(k)$–property for every α–locally bounded operation '$*$' on B_N and all $N \ge N(k, \alpha)$.*

Proof. Given k, let $t = 2\lceil \log_2 k + 1 \rceil$ and set $r = \alpha(t)$. Define $N = N(k, \alpha) = (5r + 1)(R(2t(t + 1), t, 2^{t(2r+1)}) + 2)$, where $R(m, t, l)$ denotes the Ramsey number which ensures the existence of an homogeneous subset of $[1, R(m, t, l)]$ of cardinality m all of whose t-subsets are monochromatic for any coloring of the t-subsets with l colors.

Let A be a basis of $(B_N, *)$.

Take the subset $Y \subset [1, N]$ of multiples of $5r$ if $r \geq 1$ and $Y = [1, N]$ if $r = 0$. Define a coloring c of the t-subsets of Y as follows. Given a t–subset $X = \{i_1 < i_2 < \cdots < i_t\}$, let x be the vector with the support X. Choose a pair $y, z \in A$ such that $x = y * z$. By (3) we have $s(x) \subset (\overline{s(y) \cup s(z)})_r$. Since two consecutive elements in $s(x)$ are at least at distance $4r + 1$, we can choose one of y and z, say y, such that $s(y)$ intersects at least $t/2$ of the disjoint intervals $(\overline{i_1})_r, (\overline{i_2})_r, \ldots, (\overline{i_t})_r$. Color X by the binary vector of length rt given by

$$c(X) = ((y)_{(\overline{i_1})_r}, (y)_{(\overline{i_2})_r}, \ldots, (y)_{(\overline{i_t})_r}).$$

By (3) again, we have $s(y) \subset (\overline{s(x)})_r$, so that this coloring gives an encoding of y except that we do not keep track of the position of its support in $[1, N]$.

Since A is a basis, c is a coloring of all t-subsets of Y and it uses at most $2^{t(2r+1)}$ colors. By the definition of N, there is a subset $Z \subset Y$ of cardinality $|Z| = 2t(t + 1)$ all of whose t-subsets have the same color $u = (u_1, u_2, \ldots, u_{t(2r+1)})$.

Let $u^{(1)} = (u_1, \ldots, u_r), u^{(2)} = (u_{r+1}, \ldots u_{2r}), \ldots, u^{(t)} = (u_{(t-1)r+1}, \ldots u_{tr})$. By the construction of the coloring, at least $t/2$ of the vectors $u^{(1)}, u^{(2)}, \ldots, u^{(t)}$ have nonempty support. Let $u^{(i_1)}, u^{(i_2)}, \ldots, u^{(i_s)}$ be these vectors.

Choose a subset $J = \{j_1 < j_2 < \cdots < j_{2s}\} \subset Z$ of cardinality $2s$ such that there are at least t elements of Z between two consecutive elements of J, and at least t elements of Z before j_1 and t after j_{2s}. Denote by $J_i = \{j_{2i-1}, j_{2i}\}$, $1 \leq i \leq s$. Recall that, by the choice of Y, every two consecutive elements in J are at distance at least $t(4r + 1)$.

Let $K = \{k_1 < \cdots < k_s\}$ be a s-subset of J obtained by picking one element in each J_i. Since there are t elements of Z between any two consecutive elements in J, this set can be completed to a t-subset by inserting an element from $Z \setminus J$ in each position i such that $u^{(i)}$ is the zero vector. In other words, we construct a t-set $K' = \{k'_1, \cdots < k'_t\} \subset Z$ such that $k'_{i_l} = k_l$ for $l = 1, \ldots s$.

Since Z is an homogeneous set, the vector $a = a(K) \in B_N$ whose support is contained in $(\overline{K})_r$ and

$$(a)_{(\overline{K})_r} = (u^{(i_1)}, u^{(i_2)}, \ldots, u^{(i_s)})$$

belongs to the basis A. By the same argument, the similarly defined vector $a(J \setminus K)$ also belongs to the basis. Denote by $b(K)$ the vector which coincides with $a(K)$ on the r–neighborhood of K and with $a(J \setminus K)$ in the r–neighborhood of $J \setminus K$ and has zero coordinates elsewhere (this is a correct definition as in our situation these r–neighborhoods are pairwise disjoint.) We have

$$(b(K))_{(\overline{J})_r} = (u^{(i_1)}, u^{(i_1)}, u^{(i_2)}, u^{(i_2)}, \ldots, u^{(i_s)}, u^{(i_s)}).$$

By (3), the support of $a(K) * a(J \setminus K)$ is contained in the r–neighborhood of $s(b(K)) = s(a(K)) \cup s(a(J \setminus K))$. This r–neighborhood is a subset of $(\overline{(\overline{J})_r})_r$. Since two consecutive

elements in J are at least at distance $4r+1$, the set $(\overline{s(b(K))})_r$ is again the disjoint union of the r–neighborhoods of each element in $s(b(K))$.

Let $K' \subset J$ be another choice of a s–subset with exactly one element in each J_i. The vectors $a(K)$ and $a(K')$ differ in the r–neighborhood of the symmetric difference $K\Delta K'$. On the other hand, $b(K) = b(K')$. By (1) and (2), for each coordinate i in the r–neighborhood of $(\overline{K'\Delta K})_r$ we have $((a(K') * a(J \setminus K'))_i = ((a(J \setminus K) * a(K))_i = (a(K) * (a(J \setminus K))_i$, while for the remaining coordinates in $(\overline{(\overline{J})_r})_r$, we have $((a(K') * a(J \setminus K'))_i = ((a(K) * a(J \setminus K))_i$. Therefore

$$a(K') * a(J \setminus K') = a(K) * a(J \setminus K).$$

There are 2^{s-1} choices leading to different pairs of vectors $\{a(K), a(J \setminus K)\}$. Hence, for $u = a(K) * a(J \setminus K)$, we have $r_A(u) \geq 2^{t/2-1} \geq k$. This completes the proof. □

3. Semigroups with the Erdős–Turán Property

Let $(G, *)$ be a semigroup with a distinguished idempotent element $e \in G$ which commutes with every element in G. We say that a subset $R \subset G$ is *antisymmetric* if $e \in R$ and the equation $x * y = e$ holds for $x, y \in R$ if and only if $x = y = e$. We will consider the direct product G^N. By the support of an vector $g \in G^n$ we mean the set of coordinates of g which are different from e. We denote by $G^{\mathbb{N}}$ the set of infinite sequences of elements of G with finite support, where the product is defined componentwise. As a direct application of Theorem 2 we have the following result.

Theorem 3 *Let $(G, *)$ be a semigroup and $e \in G$ an idempotent element which commutes with every element in G. Let R be a finite antisymmetric set of a semigroup with $|R| > 1$. For each positive integer k there is $N(k)$ such that R^N has the $ET(k)$ property for all $N \geq N(k)$. In particular, $R^{\mathbb{N}}$ has the ET property.*

Proof. Define an encoding ϕ of the elements of R by binary vectors of length $m = \lceil \log_2 |R| \rceil$ such that $\phi(e) = (0, \cdots, 0)$ and $\phi(x) = (0, \ldots, 0, 1)$ for some element $x \neq e$ in R. Let $B'_{Nm} = (\phi(R))^N \subset B_{Nm}$ and define an operation in B'_{Nm} according to the operation in G, that is, for $x, y \in B'_{Nm}$,

$$(x * y)_{[(i-1)m+1,im]} = \phi(\phi^{-1}((x)_{[(i-1)m+1,im]}) * \phi^{-1}((y)_{[(i-1)m+1,im]})),\ 1 \leq i \leq N.$$

In this way we have a locally bounded operation in B'_{Nm} with $\alpha(t) = mt$. Indeed, condition (1) is verified by definition, condition (3) follows from the asymmetry of R and (2) holds since e commutes with every element in G. We can now apply the proof of Theorem 2 even if the operation is not defined for all vectors in B_{Nm}: all vectors whose support lies in the set $Y \subset [1, Nm]$ of coordinates multiple of $5r$ in that proof do belong to $\phi(\{e, x\})^N \subset B'_{Nm}$,

and we only use the fact that each of these vectors belong to $A * A$ for any basis A, which is the case in our present situation. This shows that R^N has the $ET(k)$ property for $N \geq (5mt+1)(R(2t^2, t, 2^{t(2mt+1)})+2)$ where $t = \lceil \log_2 k + 1 \rceil$.

The result follows for $R^{\mathbb{N}}$ since each of its bases contains a basis of R^N for every N. □

The positive part of Theorem 1 follows from Theorem 3 by taking an antisymmetric set in $\mathbb{Z}/p\mathbb{Z}$.

Let $\mathcal{G}$ be an infinite class of semigroups. We say that $\mathcal{G}$ has the ET–property if, for every k, all but a finite number of members in $\mathcal{G}$ have the $ET(k)$–property. As a specialization of Theorem 3 we have the following examples of semigroups with the Erdős-Turán property.

Corollary 1 *The following classes have the ET property.*

1. $\{(P^N, \vee),\ N \in \mathbb{N}\}$ *and* $(P^{\mathbb{N}}, \vee)$*, where* P *is a finite semilattice with supremum and minimum elements.*
2. *The family of finite (or cofinite) subsets of a countable set* X *with respect to union,* $(2^X, \cup)$*, and with respect to intersection,* $(2^X, \cap)$*.*
3. $\{([0,m]^N, +),\ N \in \mathbb{N}\}$ *and* $([0,m]^{\mathbb{N}}, +)$*, where* $[0,m]$ *is the interval of integers* $0 \leq i \leq m$ *and the sum is componentwise.*
4. $\{(\mathbb{N}^N, +),\ N \in \mathbb{N}\}$ *and* $(\mathbb{N}^{\mathbb{N}}, +)$*, where the sum is componentwise.*

Proof. For $\{(P^N, \vee),\ N \in \mathbb{N}\}$ and for $(P^{\mathbb{N}}, \vee)$, the conditions of Theorem 3 are satisfied with $R = P$ and e the minimum element of P.

In particular, for $P = \{0, 1\}$ with the usual supremum function, $(P^k, \vee)$ corresponds to the family of subsets of $[1, N]$ with union and $(P^{\mathbb{N}}, \vee)$ to the family of finite subsets of an infinite countable set. By taking complements we get the result for cofinite sets with intersection. To prove (2) it remains to show that the class of finite subsets of $\mathbb{N}$ with intersection does have the ET–property. Let A be a basis of $(2^{\mathbb{N}}, \cap)$ and X a nonempty set in A. Since for each integer n the set $X \cup \{n\}$ must be obtained as the intersection of two sets in A, there are infinitely many sets in A containing X and, for each such set Y, X itself can be written as $X \cap Y$.

Part (3) follows directly from Theorem 3 with $R = [0, m]$. Finally, (4) follows from (3) since every basis of $(\mathbb{N}^k, +)$ contains a basis of $[0, m]^k$ for each $m \geq 1$. □

Note that the multiplicative semigroup of the positive integers can be viewed as $(\mathbb{N}^{\mathbb{N}}, +)$ by considering a vector $(x_i, i \in \mathbb{N})$ with finite support as the integer $\prod_{i \in N} p_i^{x_i}$ where $p_1, p_2, \ldots$ is the sequence of prime numbers. Thus Corollary 1 (4) includes the result that $(\mathbb{N}, \cdot)$ has the ET property.

4. Final Remarks

The condition in Theorem 3 that the set R is antisymmetric is essential for the proof. Although we were always more interested in the positive results (i.e., structures with the *ET* property) we add a few examples in the opposite direction. We say that a basis A of an abelian semigroup $(G,+)$ is a *unique representation* basis if every element $g \in G$ can be uniquely represented in $A+A$ (up to commutativity.) A simple greedy algorithm produces a unique representation basis of $G^{\mathbb{N}}$ for certain groups. Here we need the group assumption since Proposition 1 below does not hold for semigroups in general.

Proposition 1 *Let $(G,+)$ be a countable abelian group and $a \in G \setminus \{0\}$ such that $4a \neq 0$. Let $R \subset G$ containing $a, -a$ and 0. Then $R^{\mathbb{N}}$ has a unique representation basis.*

Proof. Take a linear ordering in R and consider the following ordering in $R^{\mathbb{N}}$: $x \leq y$ if and only if either $\max s(x) < \max s(y)$ or $\max s(x) = \max s(y) = m$ and x is smaller than y in lexicographic order.

Construct recursively a basis A as follows. Let $A_0 = \{0\}$. For each $i > 0$, let g be the minimum element which is not in $A_i + A_i$. Let m_i the largest element in the support of elements in $(A_i+A_i)\cup\{g\}$. Let g_a coincide with g except that the (m_i+1)–th coordinate of g_a is a. Let all the coordinates of g_b be 0 except $(g_b)_{m_i+1} = -a$. Now define $A_{i+1} = A_i \cup \{g_a, g_b\}$.

We have $g = g_a + g_b \in A_{i+1} * A_{i+1}$. Moreover, if every element in $A_i * A_i$ can be uniquely expressed, then the same is true in $A_{i+1} * A_{i+1}$, since each sum involving one of the two new elements has either a, $-a$, $2a$ or $-2a$ in the (m_i+1)-th coordinate. Therefore, $A = \cup_{i\in\mathbb{N}} A_i$ is a unique representation basis. □

A similar argument as in the above proof shows that finitely generated abelian free groups do not have the *ET*–property. On the other hand, the problem for free semigroups is as hard as for additive bases of positive integers.

Proposition 2 *Let X be a finite set. The free semigroup $FS(X)$ generated by X has the ET–property if and only if $(\mathbb{N},+)$ has the ET–property.*

Proof. For the *if* part, note that any basis A of $FS(X)$ contains a basis of the semigroup generated by a single element, which is isomorphic to $(\mathbb{N},+)$.

Suppose now that $FS(X)$ has the *ET*–property. For every basis $A \subset \mathbb{N}$ consider the set A' of words in $\mathbb{N}$ whose lengths belong to A, $A' = \{w \in F(X) : |w| \in A\}$. This is clearly a basis of $FS(X)$. If $w = x_1 * y_1 = \cdots = x_k * y_k$ are k different representations of a word $w \in FS(X)$ in elements of the basis A', then $|x_1| + |y_1| = \cdots = |x_k| + |y_k|$ are k different representations of $|w|$ in elements of A. This shows the *only if* part. □

We have dealt with bases of order two. More generally, for an integer $h \geq 2$, a subset $A \subset X$ is a *basis* of X of order h if $X \subset \underbrace{A * A * \cdots * A}_{h} = A^h$. Accordingly we say that X has the $ET_h(k)$ property if, for every basis of order h, there is an element in X with at least k representations in A^h. If X has $ET_h(k)$ for every $k \in \mathbb{N}$ then we say that it has the ET_h property. The proof of Theorem 2 can be easily extended to prove the following statement.

Theorem 4 *Let $h \geq 2$ and k be positive integers and $\alpha : N_0 \to \mathbb{N}_0$ an arbitrary function. There is $N(h, k, \alpha)$ such that $(B_N, *)$ has the $ET_h(k)$–property for every α–locally bounded operation '$*$' on B_N and all $N \geq N(h, k, \alpha)$. In particular, R^N has the $ET_h(k)$–property for every antisymmetric set R with respect to e in a semigroup G, where e is an idempotent element commuting with all elements in R, and all N large enough.*

In connection with the Erdős-Newman problem (cf. [3]) we may ask the following question. Let A be a basis with unbounded representation function and let $A = A_1 \cup \cdots \cup A_r$ be a finite partition of A. Is it true that one of the parts still has unbounded representation function? The answer is negative in general. It is shown in [10] that there are bases of $\mathbb{Z}$ with unbounded representation function which can be split into two B_2^1 sequences (every element can be uniquely written in each of the parts). It is perhaps true that the question has a positive answer for semigroups with the ET property. Again, the proof of Theorem 2 can be adapted to prove the following.

Theorem 5 *Let '$*$' be a locally bounded operation on $B_{\mathbb{N}}$. Let A be a basis of $B_{\mathbb{N}}$. For every finite partition of A one of the parts has unbounded representation function.*

Acknowledgements. We are very grateful to the comments and remarks of an anonymous referee who pointed out some inaccuracies in the original manuscript and were helpful in improving the presentation of the paper.

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