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A NOTE ON THE FIGURE OF MERIT OF 2-DIMENSIONAL RANK 2 LATTICE RULES

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Abstract

We give an explicit formula for the figure of merit ρ_N of 2-dimensional rank 2 lattice rules in terms of continued fractions for rational numbers. Further we generalize Fibonacci lattice rules to rank 2 Fibonacci lattice rules which have the same ratio of the figure of merit to the number of points as the classical Fibonacci lattice rule.

1. Introduction

For approximating integrals $\int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ one often uses a *quasi-Monte Carlo rule*, that is, one approximates the integral by the average of the function value at certain quadrature points. In this paper we restrict ourselves to dimension two and consider a special class of those quasi-Monte Carlo rules, so-called rank 2 lattice rules.

A *2-dimensional rank 2 lattice rule* is a quasi-Monte Carlo quadrature rule for functions f over the 2-dimensional unit cube $[0,1]^2$ of the form

$$Q(f) = \frac{1}{N}\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} f(\{k_1\mathbf{z}_1/n_1 + k_2\mathbf{z}_2/n_2\}), \tag{1}$$

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which cannot be re-expressed in an analogous form with a single sum. Here $n_1, n_2 \geq 2$ are integers such that $n_2 | n_1$, $N = n_1 n_2$ and $\mathbf{z}_1, \mathbf{z}_2$ are vectors in $\mathbb{Z}^2$ (note that for $n_2 = 1$ we obtain what is called a rank 1 lattice rule). The integers n_1, n_2 are called the invariants of the lattice rule. (For a vector $\mathbf{x} \in \mathbb{R}^2$ the fractional part $\{\mathbf{x}\}$ is defined component wise.) For the definition and the general theory of lattice rules for multivariate integration we refer to the books of Niederreiter [2] and of Sloan and Joe [4].

For a given rank 2 lattice rule with invariants n_1 and n_2, $N = n_1 n_2$ and with $\mathbf{z}_1 = (z_1, z_2)$ and $\mathbf{z}_2 = (\zeta_1, \zeta_2)$ for $z_i, \zeta_i \in \mathbb{Z}$ we define the quantity

$$\rho_N(\mathbf{z}_1, \mathbf{z}_2) := \min_{\substack{(h_1,h_2) \in \mathbb{Z}^2 \setminus \{(0,0)\} \\ h_1 z_1 + h_2 z_2 \equiv 0 \pmod{n_1} \\ h_1 \zeta_1 + h_2 \zeta_2 \equiv 0 \pmod{n_2}}} r(h_1) r(h_2),$$

where $r(h) = \max(1, |h|)$ for $h \in \mathbb{Z}$. This quantity is often called the *figure of merit* of the lattice rule. (If $n_2 = 1$ we obtain a rank 1 lattice rule and in this case we write $\rho_{n_1}(\mathbf{z}_1)$ for the figure of merit.) The figure of merit is an effective quality measure for lattice rules as will be explained in more detail in the following.

For reals $\alpha > 0$ and $C > 0$ let $\mathcal{E}_\alpha^2(C)$ be the class of all continuous periodic functions $f : [0,1]^2 \longrightarrow \mathbb{R}$ with period 1 in each variable and with Fourier-coefficients $\hat{f}(\mathbf{h})$, $\mathbf{h} = (h_1, h_2) \in \mathbb{Z}^2$, satisfying $|\hat{f}(\mathbf{h})| \leq C r(\mathbf{h})^{-\alpha}$ for any $\mathbf{h} \in \mathbb{Z}^2$, where $r(\mathbf{h}) = \prod_{i=1}^2 r(h_i)$. Then for the worst-case error for integration in the class $\mathcal{E}_\alpha^2(C)$ using a rank 2 lattice rule (1) we have the relation

$$\frac{2C}{\rho_N(\mathbf{z}_1, \mathbf{z}_2)^\alpha} \leq \max_{f \in \mathcal{E}_\alpha^2(C)} \left| \int_{[0,1]^2} f(\mathbf{x}) \, d\mathbf{x} - Q(f) \right| \leq C \cdot c_\alpha \frac{1 + \log \rho_N(\mathbf{z}_1, \mathbf{z}_2)}{\rho_N(\mathbf{z}_1, \mathbf{z}_2)^\alpha}, \tag{2}$$

where the constant c_α depends only on α. So the quantity ρ_N determines – up to a log factor – the exact order of the worst-case error for integration in the class $\mathcal{E}_\alpha^2(C)$ using a rank 2 lattice rule. For a proof of the above result see [2, Theorem 5.34].

Furthermore one can use the figure of merit of a lattice rule to estimate the discrepancy D_N of the corresponding node set

$$\{k_1 \mathbf{z}_1 / n_1 + k_2 \mathbf{z}_2 / n_2\}, \text{ for } 1 \leq k_i \leq n_i \text{ and } i = 1, 2. \tag{3}$$

(For the definition of the discrepancy D_N see for example [1] or [2].) The discrepancy of the point set (3) can be estimated by

$$\frac{1}{4\rho_N(\mathbf{z}_1, \mathbf{z}_2)} \leq D_N < \frac{2}{N} + \frac{1}{\rho_N(\mathbf{z}_1, \mathbf{z}_2)} \frac{2}{\log 2} \left((\log N)^2 + \frac{3}{2} \log N \right). \tag{4}$$

For a proof see [3] or [2].

From (2) and (4) it follows that for a given number of points N one would like to obtain a figure of merit as large as possible. In dimensions larger than two there is no theoretical foundation how this can be achieved, whereas in dimension two this is known if the number of points is a Fibonacci number, see [5]. These lattice rules are called Fibonacci lattice rules and are obtained by setting $n_2 = 1$, $n_1 = F_k$, the k-th Fibonacci number, $\mathbf{z}_1 = (1, F_{k-1})$, where F_{k-1} is the $k-1$-th Fibonacci number, and $\mathbf{z}_2 = (0,0)$ in (1).

Note that a trivial upper and lower bound on the figure of merit is given by

$$1 \leq \rho_N(\mathbf{z}_1, \mathbf{z}_2) \leq n_1. \tag{5}$$

In this paper we prove an explicit formula for the figure of merit $\rho_N(\mathbf{z}_1, \mathbf{z}_2)$ in terms of continued fractions for rational numbers. Our result is the rank 2 analogue to Niederreiters formula [2, Theorem 5.15] for the figure of merit of rank 1 lattice rules. Using this result we can give a rank 2 version of the well known Fibonacci lattice rules, which achieve the same ratio of the figure of merit to the number of points as do rank 1 Fibonacci lattice rules. Further, using our formula we can give good bounds for the figure of merit of rank 2 lattice rules.

2. The figure of merit of rank 2 lattice rules

We consider 2-dimensional rank 2 lattice rules with invariants n_1 and n_2, $N = n_1 n_2$ and with $\mathbf{z}_1 = (z_1, z_2)$ and $\mathbf{z}_2 = (\zeta_1, \zeta_2)$ with $z_i, \zeta_i \in \mathbb{Z}$ and $\gcd(z_i, n_1) = 1$ and $\gcd(\zeta_i, n_2) = 1$ for $i = 1, 2$. In this case we may assume w.l.o.g. that $z_1 = \zeta_1 = 1$. For simplicity we write in the following z for z_2 and ζ for ζ_2.

Before we state our main result we introduce some notation which is used throughout the paper. For $z, \widetilde{n} \in \mathbb{Z}$ with $\gcd(z, \widetilde{n}) = 1$ let

$$\frac{z}{\widetilde{n}} = [a_0; a_1, \ldots, a_l]$$

be the continued fraction expansion of $z/\widetilde{n}$, where $a_j \in \mathbb{N}$ for $1 \leq j \leq l$ and where $a_l = 1$. The convergents to $z/\widetilde{n}$ are defined by

$$\frac{p_j}{q_j} = [a_0; a_1, \ldots, a_j], \tag{6}$$

for $0 \leq j \leq l$. The integers p_j and q_j are uniquely determined if we impose the conditions $q_j \geq 1$ and $\gcd(p_j, q_j) = 1$. Further let $K(z/\widetilde{n})$ denote the largest partial quotient a_j, $1 \leq j \leq l$, in the continued fraction expansion of $z/\widetilde{n}$.

Remark 1 With this notation we can improve the lower bound in (5). We have

$$\rho_N(\mathbf{z}_1, \mathbf{z}_2) \geq \min_{\substack{(h_1,h_2) \in \mathbb{Z}^2 \setminus \{(0,0)\} \\ h_1 + h_2 z \equiv 0 \pmod{n_1}}} r(h_1) r(h_2) \geq \frac{n_1}{K(z/n_1) + 2}, \tag{7}$$

where $K(z/n_1)$ denotes the largest partial quotient in the continued fraction expansion of z/n_1. Here the last inequality follows from [2, Theorem 5.17].

The following theorem is a rank 2 version of [2, Theorem 5.15] and expresses the figure of merit using the convergents to $z/\widetilde{n}$, where the integer $\widetilde{n}$ depends on z, ζ, n_1, n_2 and is defined in the subsequent theorem.

Theorem 1 *Let $n_1, n_2 \in \mathbb{N}$, $n_2|n_1$ and let $z, \zeta \in \mathbb{Z}$ such that $\gcd(n_1, z) = 1$ and $\gcd(n_2, \zeta) = 1$. Let $d := \gcd(z - \zeta, n_2)$, $n_1^* := n_1/n_2$ and $\widetilde{n} := n_1^* d$. Then we have*

$$\rho_N(\mathbf{z}_1, \mathbf{z}_2) = \min\left(n_1, \frac{n_2^2}{d^2} \min_{0 \le j < l} q_j |q_j z - p_j \widetilde{n}|\right),$$

where p_j/q_j, $0 \le j \le l$ are the convergents to $z/\widetilde{n}$ as defined in (6).

Remark 2 1. From [2, Theorem 5.17] we obtain

$$\frac{N}{d(K(z/\widetilde{n}) + 2)} \le \frac{n_2^2}{d^2} \min_{0 \le j < l} q_j |q_j z - p_j \widetilde{n}| \le \frac{N}{dK(z/\widetilde{n})}.$$

2. If the parameters are chosen in such a way that $d = n_2$, then we have

$$\frac{n_1}{K(z/n_1) + 2} \le \rho_N(\mathbf{z}_1, \mathbf{z}_2) \le \frac{n_1}{K(z/n_1)}.$$

Compare this result with Remark 1.

Proof of Theorem 1. Since $\rho_N(\mathbf{z}_1, \mathbf{z}_2) \le n_1$ we have

$$\rho_N(\mathbf{z}_1, \mathbf{z}_2) = \min_{\substack{(h_1,h_2) \in \mathbb{Z}^2 \setminus \{(0,0)\} \\ h_1 + h_2 z \equiv 0 \pmod{n_1} \\ h_1 + h_2 \zeta \equiv 0 \pmod{n_2}}} r(h_1) r(h_2) = \min\left(n_1, \min_{\substack{1 \le h_2 < n_1 \\ h_1 \in \mathbb{Z} \\ h_1 + h_2 z \equiv 0 \pmod{n_1} \\ h_1 + h_2 \zeta \equiv 0 \pmod{n_2}}} r(h_1) r(h_2)\right).$$

Define

$$\widetilde{\rho}_N(\mathbf{z}_1, \mathbf{z}_2) := \min_{\substack{1 \le h_2 < n_1 \\ h_1 \in \mathbb{Z} \\ h_1 + h_2 z \equiv 0 \pmod{n_1} \\ h_1 + h_2 \zeta \equiv 0 \pmod{n_2}}} r(h_1) r(h_2).$$

For $h_2 \in \mathbb{Z}$ the system

$$h_1 + h_2 z \equiv 0 \pmod{n_1} \tag{8}$$
$$h_1 + h_2 \zeta \equiv 0 \pmod{n_2} \tag{9}$$

has a solution h_1 iff $\gcd(n_1, n_2) = n_2$ is a divisor of $-h_2 z + h_2 \zeta$, i.e., iff

$$h_2(z - \zeta) \equiv 0 \pmod{n_2}. \tag{10}$$

Let $d = \gcd(z-\zeta, n_2)$. Then (10) has d incongruent (modulo d) solutions $x_0, \ldots, x_{d-1}$ with $0 \le x_i < d$. In fact it is clear that $x_i = in_2/d$ for $0 \le i < d$. It follows that h_2 must be of the form

$$h_2 = x_i + \widetilde{h}_2 n_2.$$

Now the system (8 & 9) becomes

$$h_1 + (x_i + \widetilde{h}_2 n_2)z \equiv 0 \pmod{n_1} \tag{11}$$

$$h_1 + (x_i + \widetilde{h}_2 n_2)\zeta \equiv 0 \pmod{n_2}. \tag{12}$$

From (12) we obtain

$$h_1 + x_i\zeta \equiv 0 \pmod{n_2}$$

and hence h_1 is of the form

$$h_1 = -x_i\zeta + \widetilde{h}_1 n_2.$$

Inserting this back into (11) gives

$$-x_i\zeta + \widetilde{h}_1 n_2 + x_i z + \widetilde{h}_2 n_2 z \equiv 0 \pmod{n_1}.$$

This is equivalent to

$$\widetilde{h}_1 n_2 + \widetilde{h}_2 z n_2 \equiv -x_i(z-\zeta) \pmod{n_1}.$$

Since $x_i = in_2/d$ and since $d = \gcd(n_2, z-\zeta)$, the last congruence is equivalent to

$$\widetilde{h}_1 + \widetilde{h}_2 z \equiv -i\frac{z-\zeta}{d} \pmod{n_1^*},$$

where $n_1^* = n_1/n_2$. Define $a := \frac{z-\zeta}{d}$. Therefore we get

$$\begin{aligned}\widetilde{\rho}_N(\mathbf{z}_1, \mathbf{z}_2) &= \min_{0\le i<d} \min_{\substack{0\le \widetilde{h}_2 < n_1^* - i/d \\ \max(i,\widetilde{h}_2)\ne 0, \widetilde{h}_1\in\mathbb{Z} \\ \widetilde{h}_1 + \widetilde{h}_2 z \equiv -ia \pmod{n_1^*}}} (x_i + \widetilde{h}_2 n_2)|-x_i\zeta + \widetilde{h}_1 n_2| \\ &= \min_{0\le i<d} \min_{\substack{0\le \widetilde{h}_2 < n_1^* - i/d \\ \max(i,\widetilde{h}_2)\ne 0 \\ t\in\mathbb{Z}}} (x_i + \widetilde{h}_2 n_2)|-x_i\zeta + (tn_1^* - ia - z\widetilde{h}_2)n_2|.\end{aligned}$$

We have

$$-x_i\zeta + (tn_1^* - ia - z\widetilde{h}_2)n_2 = tn_1^* n_2 - n_2 z\left(\frac{i}{d} + \widetilde{h}_2\right).$$

Hence

$$\begin{aligned}\widetilde{\rho}_N(\mathbf{z}_1, \mathbf{z}_2) &= \min_{0\le i<d} \min_{\substack{0\le \widetilde{h}_2 < n_1^* - i/d \\ \max(i,\widetilde{h}_2)\ne 0 \\ t\in\mathbb{Z}}} (x_i + \widetilde{h}_2 n_2)\left|tn_1^* n_2 - n_2 z\left(\frac{i}{d} + \widetilde{h}_2\right)\right| \\ &= \frac{n_2^2}{d^2} \min_{0\le i<d} \min_{\substack{0\le \widetilde{h}_2 < n_1^* - i/d \\ \max(i,\widetilde{h}_2)\ne 0 \\ t\in\mathbb{Z}}} (\widetilde{h}_2 d + i)|z(\widetilde{h}_2 d + i) - tn_1^* d| \\ &= \frac{n_2^2}{d^2} \min_{\substack{1\le h < n_1^* d \\ t\in\mathbb{Z}}} h|hz - tn_1^* d|. \end{aligned} \tag{13}$$

We define $\widetilde{n} = n_1^* d$ and show that $\gcd(z, \widetilde{n}) = 1$. Trivially we have $\gcd(z, n_1^*) = 1$ since $\gcd(z, n_1) = 1$. Now assume that $\gcd(z, d) = d_1 > 1$. Then d_1 is a divisor of z, of $z - \zeta$ and hence of ζ. Further d_1 is a divisor of n_2 and hence it follows that d_1 is a divisor of $\gcd(n_2, \zeta) = 1$. Thus $\gcd(z, d) = 1$ and also $\gcd(z, \widetilde{n}) = 1$.

Let p_j/q_j, $0 \leq j \leq l$, be the convergents to $z/\widetilde{n}$ as defined in (6). Then it is easy to see that

$$\widetilde{\rho}_N(\mathbf{z}_1, \mathbf{z}_2) = \frac{n_2^2}{d^2} \min_{0 \leq j < l} q_j |q_j z - p_j \widetilde{n}|$$

and we are done. □

In the following lemma we state a few special cases not included in the Theorem 1 which will be usefull in the next section. The proofs of those results can be obtained using similar arguments as in Theorem 1.

Lemma 1 Let $n_1, n_2 \in \mathbb{N}$, $n_2 | n_1$, $z, \zeta \in \mathbb{Z}$ and $n_1^* = n_1/n_2$.

a. If z is odd and $\zeta = 0$, then

$$\rho_N(\mathbf{z}_1, \mathbf{z}_2) = \min\left(n_1, n_2^2 \min_{0 \leq j < l} q_j |q_j z - p_j n_1^*|\right),$$

b. If $n_2 = 2$, $\gcd(z, n_1) = 2$ and $\zeta = 0$, then

$$\rho_N(\mathbf{z}_1, \mathbf{z}_2) = 2\rho_{n_1^*}((1, z/2)),$$

where $\rho_{n_1^*}((1, z/2))$ denotes the figure of merit of a rank 1 lattice rule with generating vector $(1, z/2)$ and n_1^* points.

c. If $n_2 = 2$, $\gcd(z, n_1) = 2$ and $\zeta = 1$, then

$$\rho_N(\mathbf{z}_1, \mathbf{z}_2) = 4\rho_{n_1^*}((1, z)),$$

where $\rho_{n_1^*}((1, z))$ denotes the figure of merit of a rank 1 lattice rule with generating vector $(1, z)$ and n_1^* points.

3. Rank 2 lattice rules with a large figure of merit

In order to be able to compare the quality of lattice rules with different number of points we consider the relative figure of merit which is given by the ratio of ρ to the number of points, i.e., ρ_N/N. In the following we investigate how good rank 2 lattice rules can be in terms of the relative figure of merit compared to rank 1 lattice rules.

Note that the best rank 1 lattice rules are obtained by choosing the number of points a Fibonacci number (see Section 1). Recall that Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. For this case we have $\rho_N(\mathbf{z}_1, \mathbf{z}_2) = F_{k-2}$ (see Zaremba [5]) and

$$\frac{\rho_N}{N} = \frac{F_{k-2}}{F_k}.$$

We remark that $\lim_{k\to\infty} F_{k-2}/F_k = (3 - \sqrt{5})/2 = 0,3819\ldots$ and

$$\frac{\rho_N}{N} = \frac{F_{k-2}}{F_k} = \frac{F_{k-2}}{2F_{k-2} + F_{k-3}} > \frac{F_{k-2}}{2F_{k-2} + F_{k-2}} = \frac{1}{3}.$$

We search through all rank 2 lattice rules by considering all choices for n_2. First observe that for $n_2 = 1$ the lattice rule we obtain is actually a rank 1 lattice rule. Hence we consider only the cases where $n_2 > 1$.

Now consider the case where $n_2 > 2$. Then we have $\rho_N(\mathbf{z}_1, \mathbf{z}_2) \leq n_1$ and hence

$$\frac{\rho_N}{N} \leq \frac{1}{n_2} \leq \frac{1}{3}.$$

Thus in this case the relative figure of merit is worse than the relative figure of merit of rank 1 Fibonacci lattice rules.

We are left with the case $n_2 = 2$. As $n_2 = 2$ it follows that n_1 has to be even and the value of d in Theorem 1 can either be 1 or 2.

We first consider the case where $\gcd(z, n_1) = 1$. Now if $\zeta = 1$ we can use Theorem 1. It follows that $d = 2$ and by Remark 2 we have

$$\rho_N(\mathbf{z}_1, \mathbf{z}_2) = \frac{n_2^2}{d^2} \min_{0\leq j<l} q_j|q_j z - p_j\widetilde{n}| = \min_{0\leq j<l} q_j|q_j z - p_j\widetilde{n}| = \rho_{\widetilde{n}}((1, z)),$$

that is, the figure of merit for our rank 2 lattice rule coincides with the figure of merit of a rank 1 lattice rule with $\widetilde{n}$ points and generating vector $(1, z)$. Note that $\widetilde{n} = dn_1/n_2 = n_1$ and hence in this case we have

$$\frac{\rho_N((1, z), (1, \zeta))}{N} = \frac{\rho_N((1, z), (1, \zeta))}{2n_1} = \frac{1}{2}\frac{\rho_{\widetilde{n}}((1, z))}{\widetilde{n}}.$$

This means that under the above assumptions the best rank 2 lattice rule can only be half as good as the best rank 1 lattice rule with an even number of points in terms of the relative figure of merit (note that $\widetilde{n}$ has to be even as $n_2|n_1$ and $\widetilde{n} = n_1$).

On the other hand if we choose $\zeta = 0$ in the above case we can use Lemma 1. Then we have $d = 1$ and $n_1^* = n_1/n_2 = n_1/2$ and hence by [2, Lemma 5.8] it follows that

$$\min_{0\leq j<l} q_j|q_j z - p_j n_1^*| = \rho_{n_1^*}((1, z)) \leq \frac{n_1^*}{2} = \frac{n_1}{4}.$$

Hence we have

$$n_2^2 \min_{0 \le j < l} q_j |q_j z - p_j n_1^*| = 4 \min_{0 \le j < l} q_j |q_j z - p_j n_1^*| \le n_1$$

and thus in this case

$$\rho_N((1,z),(1,\zeta)) = 4 \min_{0 \le j < l} q_j |q_j z - p_j n_1^*| = 4\rho_{n_1^*}((1,z)),$$

with $\rho_{n_1^*}((1,z))$ being the figure of merit of a rank 1 lattice rule with n_1^* points and generating vector $(1,z)$. As $N = n_1 n_2 = 2n_1 = 4n_1^*$ we have

$$\frac{\rho_N((1,z),(1,\zeta))}{N} = \frac{4\rho_{n_1^*}((1,z))}{4n_1^*} = \frac{\rho_{n_1^*}((1,z))}{n_1^*},$$

that is, for any rank 1 lattice rule with an even number of points we can always find a rank 2 lattice rule with four times the number of points and the same relative figure of merit.

Now let z be such that $\gcd(z, n_1) = 2$ (note that as $n_2 | n_1$ and $n_2 = 2$ it follows that $2 | n_1$). If we choose $\zeta = 0$ we obtain from Lemma 1 that

$$\frac{\rho_N(\mathbf{z}_1, \mathbf{z}_2)}{N} = \frac{2\rho_{n_1^*}((1,z/2))}{2n_1} = \frac{1}{2}\frac{\rho_{n_1^*}((1,z/2))}{n_1^*},$$

again showing that in this case rank 2 lattice rules are worse than rank 1 lattice rules. If we choose $\zeta = 1$ on the other hand, then Lemma 1 yields

$$\frac{\rho_N(\mathbf{z}_1, \mathbf{z}_2)}{N} = \frac{4\rho_{n_1^*}((1,z))}{2n_1} = \frac{\rho_{n_1^*}((1,z))}{n_1^*}$$

and thus in this case we can obtain a rank 2 lattice rule with the same relative figure of merit as a rank 1 lattice rule.

Observe that for $n_2 = 2$ we can obtain the figure of merit for the rank 2 lattice rule via a rank 1 lattice rule with $n_1^* = n_1/2$ points. The above results can hence also be used the other way round. For a given rank 1 lattice rule with n_1^* points and generating vector $(1,z)$ with $\gcd(z, n_1^*) = 1$ we can always construct a rank 2 lattice rule with four times the number of points and the same relative figure of merit.

We have shown the following theorem.

Theorem 2 *Let a rank 2 lattice rule with generating vectors $(1,z)$ and $(1,\zeta)$, $z,\zeta \in \mathbb{Z}$, and $N = n_1 n_2$ with $n_2 | n_1$ be given. For $n_2 > 2$ we have*

$$\frac{\rho_N((1,z),(1,\zeta))}{N} \le \frac{1}{3}.$$

If $\gcd(z, n_1)$ *is either* 1 *or* 2, $n_2 = 2$ *and* $z - \zeta$ *odd we have*

$$\frac{\rho_N((1,z),(1,\zeta))}{N} = \frac{\rho_{n_1/2}((1,z))}{n_1/2},$$

where $\rho_{n_1/2}((1,z))$ *denotes the figure of merit of a rank 1 lattice rule with* $n_1/2$ *points and generating vector* $(1,z)$*. If* $z-\zeta$ *is even we obtain*

$$\frac{\rho_N((1,z),(1,\zeta))}{N} = \frac{1}{2}\frac{\rho_{n_1/2}((1,y))}{n_1/2},$$

where $y = z$ *if* z *is odd and* $y = z/2$ *if* z *is even.*

Remark 3 Theorem 2 shows that the best rank 2 lattice rules are obtained by choosing $n_2 = 2$ and ζ such that $z-\zeta$ odd. In all other cases Fibonacci rank 1 lattice rules achieve a larger relative figure of merit.

Theorem 2 further shows that for every rank 1 lattice rule with m points and generating vector $(1,z)$ with $\gcd(z,m) = 1$ there exists a rank 2 lattice rule with four times the number of points which has the same relative figure of merit. This rank 2 lattice rule is obtained by choosing $n_1 = 2m$, $n_2 = 2$ and $\zeta \in \{0,1\}$ such that $z-\zeta$ is odd.

In particular, if we choose $n_1 = 2F_k$, $z = F_{k-1}$ and ζ such that $z-\zeta$ is odd, then we obtain

$$\frac{\rho_{4F_k}((1,F_{k-1}),(1,\zeta))}{4F_k} = \frac{F_{k-2}}{F_k},$$

which is the same ratio as for the relative figure of merit of rank 1 Fibonacci lattice rules. Hence we shall name those point sets rank 2 Fibonacci lattice rules. (See Figures 1, 2 and 3 for examples.)

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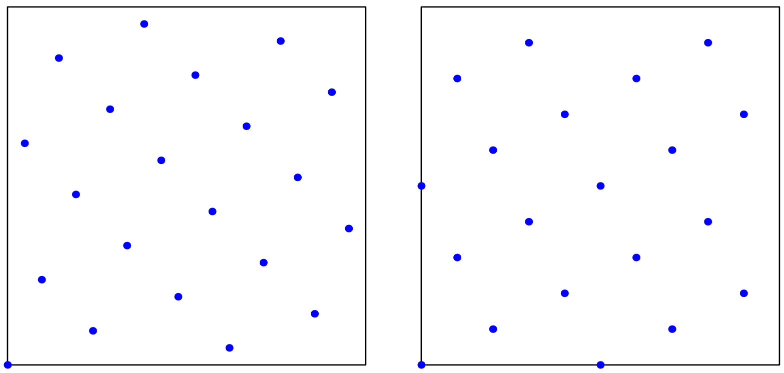

Figure 1: Nodes of a rank 1 Fibonacci rule with 21 points (left) and of a rank 2 Fibonacci rule with $20 = 4 \cdot 5$ points (right).

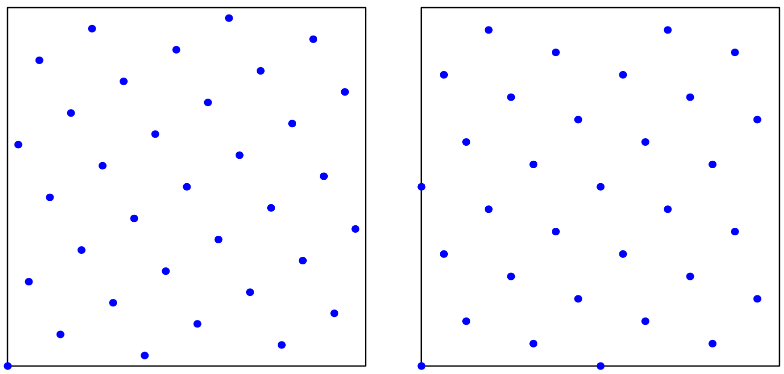

Figure 2: Nodes of a rank 1 Fibonacci rule with 34 points (left) and of a rank 2 Fibonacci rule with $32 = 4 \cdot 8$ points (right).

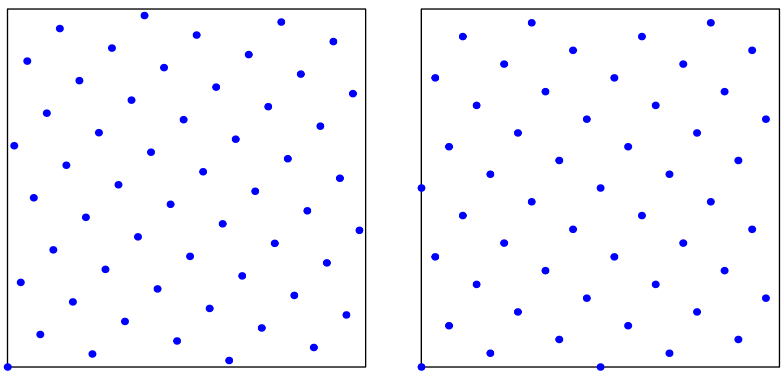

Figure 3: Nodes of a rank 1 Fibonacci rule with 55 points (left) and of a rank 2 Fibonacci rule with $52 = 4 \cdot 13$ points (right).