ON PARTITIONS AND CYCLOTOMIC POLYNOMIALS

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Abstract

Let m denote a squarefree number. Let $f_m(n)$ denote the number of partitions of n into parts that are relatively prime to m. Let $\Phi_m(z)$ denote the m^{th} cyclotomic polynomial. We obtain a generating function for $f_m(n)$ that involves factors $\Phi_m(z^n)$.

1. Introduction

If z is a complex variable, let $\Phi_m(z)$ denote the mth cyclotomic polynomial, that is

$$\Phi_m(z) = \prod_{d|m} (z^d - 1)^{\mu(m/d)}$$

where $\mu(n)$ denotes the Möbius function. If p is prime, let $b_p(n)$ denote the number of *p-regular* partitions of n, that is, the number of partitions of n such that no part occurs p or more times. It is well-known that $b_p(n)$ also counts the number of partitions of n into parts, k, such that $(k,p) = 1$. (See [1],[2], and [3].) Furthermore, $b_p(n)$ has a generating function given by

$$\sum_{n=0}^{\infty} b_p(n) z^n = \prod_{n=1}^{\infty} \frac{1 - z^{pn}}{1 - z^n} = \prod_{n=1}^{\infty} \Phi_p(z^n) \tag{1}$$

where $|z| < 1$. In particular, if $q(n)$ denotes the number of partitions of n into distinct parts (or odd parts), so that $q(n) = b_2(n)$, then

$$\sum_{n=0}^{\infty} q(n) z^n = \sum_{n=0}^{\infty} b_2(n) z^n = \prod_{n=1}^{\infty} (1 + z^n) = \prod_{n=1}^{\infty} \Phi_2(z^n). \tag{2}$$

In this note, we generalize (1) as follows. Let m be the product of r distinct primes. Let $f_m(n)$ denote the number of partitions of n into parts, k, such that $(k,m) = 1$. That is, $f_m(n)$

denotes the number of partitions of n into parts that are not divisible by any of the r distinct primes. We obtain a generating function for $f_m(n)$ as an infinite product of factors $\Phi_m(z^n)$ or $1/\Phi_m(z^n)$, accordingly as r is odd or even, respectively.

2. Preliminaries

Theorem 0 *If $H \subset N$, let $p_H(n)$ denote the number of partitions of n into parts belonging to H; let $q_H(n)$ denote the number of partitions of n into distinct parts belonging to H; let $q_H^E(n)$ denote the number of partitions of n into evenly many distinct parts from H; let $q_H^O(n)$ denote the number of partitions of n into oddly many distinct parts from H. Further, let $q_H^*(n) = q_H^E(n) - q_H^O(n)$ and define $p_H(0) = q_H(0) = q_H^E(0) = q_H^*(0) = 1$. Let z be a complex variable such that $|z| < 1$. Then*

$$\sum_{n=0}^{\infty} p_H(n) z^n = \prod_{n \in H} (1 - z^n)^{-1} \tag{3}$$

$$\sum_{n=0}^{\infty} q_H(n) z^n = \prod_{n \in H} (1 + z^n) \tag{4}$$

$$\sum_{n=0}^{\infty} q_H^*(n) z^n = \prod_{n \in H} (1 - z^n). \tag{5}$$

Remarks: Equation (3) is Theorem 1.1, (1.2.4) in [1]; (4) follows from the same theorem; (5) is proven for the case $H = N$ in [1]. The proof extends easily to the case: $H \subset N$.

3. The Main Results

Theorem 1 *Let $m = \prod_{i=1}^{r} p_i$, where $r \geq 1$ and the p_i are distinct primes. Let $f_m(n)$ be the number of partitions of n into parts, k, such that $(k, m) = 1$. Let $\Phi_m(z)$ denote the mth cyclotomic polynomial, where z is a complex variable, with $|z| < 1$. Then*

$$\sum_{n=0}^{\infty} f_m(n) z^n = \prod_{n=1}^{\infty} (\Phi_m(z^n))^{(-1)^{r-1}}. \tag{6}$$

Proof. If $r = 1$, then $f_m(n) = f_p(n) = b_p(n) =$ the number of p-regular partitions of n , so (by (1))

$$\sum_{n=0}^{\infty} f_m(n) z^n = \sum_{n=0}^{\infty} b_p(n) z^n = \prod_{n=1}^{\infty} \frac{1 - z^{pn}}{1 - z^n} = \prod_{n=1}^{\infty} \Phi_p(z^n).$$

Now suppose that m has r distinct prime factors, and p is a prime such that $p \not| m$. Then pm has $r+1$ distinct prime factors. By induction hypothesis,

$$\sum_{n=0}^{\infty} f_m(n) z^n = \prod_{n=1}^{\infty} (\Phi_m(z^n))^{(-1)^{r-1}}.$$

Now

$$\begin{aligned}\sum_{n=0}^{\infty} f_{pm}(n) z^n &= \prod_{(p,n)=1} (\Phi_m(z^n))^{(-1)^{r-1}} &= \prod_{n=1}^{\infty} \left(\frac{\Phi_m(z^n)}{\Phi_m(z^{pn})}\right)^{(-1)^{r-1}} \\ &= \prod_{n=1}^{\infty} (1/\Phi_{pm}(z^n))^{(-1)^{r-1}} &= \prod_{n=1}^{\infty} (\Phi_{pm}(z^n))^{(-1)^r},\end{aligned}$$

so we are done.

Remarks: Let $\omega(d)$ denote the number of distinct prime factors of d. Then (6) could be restated as:

$$\sum_{n=0}^{\infty} f_m(n) z^n = \prod_{n=1}^{\infty} \prod_{d|m} (1 - z^{dn})^{(-1)^{1+\omega(d)}}. \tag{7}$$

Since d is squarefree by hypothesis, we have $(-1)^{\omega(d)} = \mu(d)$. Thus (7) becomes:

$$\sum_{n=0}^{\infty} f_m(n) z^n = \prod_{n=1}^{\infty} \prod_{d|m} (1 - z^{dn})^{-\mu(d)}. \tag{8}$$

A shorter, alternate proof is based on the inclusion-exclusion principle, namely

$$\sum_{n=0}^{\infty} f_m(n) z^n = \prod_{n=1}^{\infty} (1-z^n)^{-1} \prod_{p|m} (1-z^{pn}) \prod_{p_1 p_2 | m} (1 - z^{p_1 p_2 n})^{-1} \prod_{p_1 p_2 p_3 | m} (1 - z^{p_1 p_2 p_3 n}) \cdots \tag{9}$$

$$= \prod_{n=1}^{\infty} \prod_{d|m} (1 - z^{dn})^{-\mu(d)}.$$

(In the products above, the p_i are distinct prime divisors of m.)

Furthermore, $f_m(n)$ may be computed recursively by the repeated use of Theorem 2 below, whose elementary proof is omitted.

Theorem 2 *Let m, r, $f_m(n)$, z be as in the hypothesis of Theorem 1. Let p be a prime such that $p \nmid m$. Then*

$$f_{pm}(n) + \sum_{j=1}^{[n/p]} f_{pm}(n-pj) f_m(j) = f_m(n).$$

For example, suppose we wish to compute the number of partitions of n into parts that are not divisible by 2, 3, or 5. That is, we wish to compute $f_{30}(n)$. According to Theorem 1, we have:

$$\sum_{n=0}^{\infty} f_{30}(n) z^n = \prod_{n=1}^{\infty} \Phi_{30}(z^n) = \prod_{n=1}^{\infty} (z^{8n} + z^{7n} - z^{5n} - z^{4n} - z^{3n} + z^n + 1).$$

We conclude with the following theorem, which follows easily from Theorems 1 and 0.

Theorem 3 *Let m, r, n, z be as in the hypothesis of Theorem 1. Let $q_m^E(n)$, $q_m^O(n)$ denote respectively the number of partitions of n into evenly, oddly many distinct parts, k, such that $(k, m) = 1$. Then*

$$\prod_{n=1}^{\infty} (\Phi_m(z^n))^{(-1)^r} = \sum_{n=0}^{\infty} (q_m^E(n) - q_m^O(n)) z^n.$$

Proof. This follows from the hypothesis, Theorem 1, and Theorem 0, part (iii).

References

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