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OUTCOMES OF PARTIZAN EUCLID

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Abstract

PARTIZAN EUCLID is a game based on the Euclidean Algorithm. The outcome of any position (p, q) is determined by a single path of the game tree; this path has connections to the furthest integer continued fraction of p/q. We convert the question of 'Who wins?' to a word problem, then give a list of reductions that reduces the word/position to one of 9 positions.

1. Introduction

Suggested by 'Euclid'[3] and Richard K. Guy, the game of PARTIZAN EUCLID is played by two players, Left and Right, and starts with a pair of positive integers (p, q) with $p > q$. Let $p = kq + t$ where $0 \leq t < q$. If $q \mid p$ (i.e. $t = 0$) then the game is over; otherwise, Left moves to (q, t) and Right moves to $(q, q - t)$. The game may seem trivial as there is only one move available for each player. However, as we shall show, answering the question 'Who wins?' reveals some of the interesting structure of the game. We would like to answer the question of who wins in the *disjunctive sum* of this game, but this appears to be difficult. See the last section for a discussion of that problem.

In the (impartial) game EUCLID, which is also played with (p, q), a pair of positive integers, a player is allowed to remove any multiple of the smaller from the larger provided the remainder is positive. Lengyel [7] reports that Schwartz first found that

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EUCLID is the sequential sum [10] of nim-heaps: given (p, q), suppose the normal continued fraction of $\frac{p}{q}$ is $[a_1, a_2, \ldots, a_n]$ ($a_n > 1$ except if Fibonacci numbers are involved) then the EUCLID position (p, q) corresponds to playing the sequential sum of NIM with nim-heaps $a_1, a_2, \ldots, a_n$. EUCLID has attracted much attention and has been generalized, see [4, 5] for example. PARTIZAN EUCLID is related to *nearest* and *farthest* integer continued fractions (NICF and FICF) (see [8]).

In the case of both NICFs and FICFs we write rational numbers as a sum or difference of an integer and a rational less than 1. For example, the FICF for $\frac{11}{8}$ is obtained by rewriting, noting that

- $\frac{11}{8} = 2 - \frac{1}{8/5}$ since 2 is further away from $\frac{11}{8}$ than 1;
- $\frac{8}{5} = 1 + \frac{1}{5/3}$ since 1 is further away than 2;
- $\frac{5}{3} = 1 + \frac{1}{3/2}$ since 1 is further away than 2;
- and $\frac{3}{2} = 2 - \frac{1}{2/1} = 1 + \frac{1}{2/1}$ since 1 and 2 are equally distant.

We are not interested in the continued fraction itself but in noting that during the calculation (i) 'integer subtract fraction' corresponds to a move by Right and (ii) 'fraction subtract integer' corresponds to a Left move. We'll use the word *rlle* to represent this where e is the common move to $(2, 1)$. Section 2 reports on the structure of the game tree and shows there is one path, the path obtained from the FICF algorithm, that determines the whole game tree.

For example, in Figure 1 the path formed by the moves (edges)

$$(11, 8) \overset{Right}{\to} (8, 5) \overset{Left}{\to} (5, 3) \overset{Left}{\to} (3, 2)$$

is the important path. Why is it important? Non-trivial parts of the tree that are not on that path are isomorphic to parts rooted on the path; '(8, 3)' is isomorphic to '(5, 3)', '(5, 2)' is isomorphic to '(3, 2)', and '(3, 1)' is isomorphic to '(2, 1)'. All the information needed to determine the outcome and value of (11, 8) is found on this path.

A game tree (position) is represented as a word from the alphabet r, l ending in e. In Lemma 13, we find reduction rules that preserve the outcome class of the word, moreover, any word reduces to one of just 9 words each with length at most 4. This can be accomplished in time linear in the length of the corresponding FICF. Unfortunately, these reductions most of the time do not preserve the value.

We will denote a game position as $E(p, q)$. Also, we will use $p\%q$ for $p \bmod q$.

We try to present a sufficient amount of game theory to make the paper self-contained (with the exception of the last two sections). For terms not defined in the paper we follow [1]. A *position*, say h, is defined in terms of its options as follows: $h = \{h^L | h^R\}$. For example, where $t = p\%q$, $E(p, q) = \{E(q, t) | E(q, q - t)\}$.

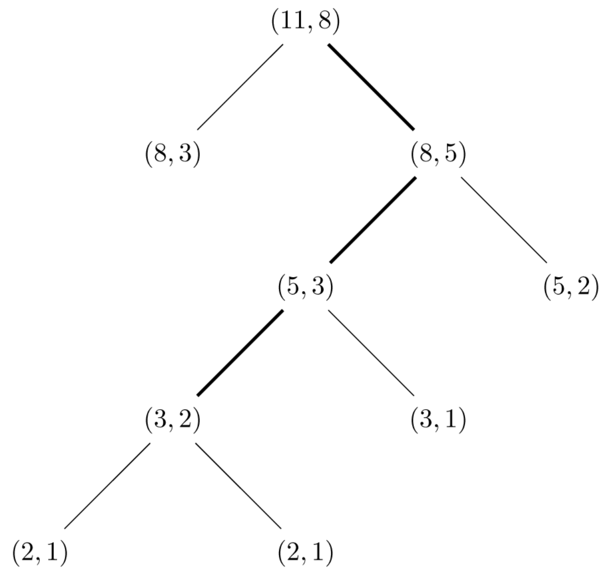

Figure 1: Some of the game tree of $(11, 8)$

The *outcome* of a position is **L**eft, **R**ight, **N**ext or **P**revious depending on (under perfect play), respectively, whether Left can win going first and second, Right can win going first and second, the next player to move wins regardless if this Left or Right, the next player cannot win regardless if this Left or Right. We phrase this more formally.

Lemma 1. *Let h be a position. The outcome of h is determined by the outcomes of its options. Specifically:*

- $o(h) = \mathcal{L}$ *iff* $\exists h^L$, $o(h^L) \in \{\mathcal{L}, \mathcal{P}\}$ *and* $\forall h^R$, $o(h^R) \in \{\mathcal{L}, \mathcal{N}\}$*;*
- $o(h) = \mathcal{P}$ *iff* $\forall h^L$, $o(h^L) \in \{\mathcal{N}, \mathcal{R}\}$ *and* $\forall h^R$, $o(h^R) \in \{\mathcal{L}, \mathcal{N}\}$*;*
- $o(h) = \mathcal{N}$ *iff* $\exists h^L$, $o(h^L) \in \{\mathcal{L}, \mathcal{P}\}$ *and* $\exists h^R$, $o(h^r) \in \{\mathcal{P}, \mathcal{R}\}$*;*
- $o(h) = \mathcal{R}$ *iff* $\forall h^L$, $o(h^L) \in \{\mathcal{N}, \mathcal{R}\}$ *and* $\exists h^R$, $o(h^r) \in \{\mathcal{P}, \mathcal{R}\}$*.*

Let $g = E(p, q)$. Since there is at most one option for each player we will abuse notation and write $o(g) = o(\{g^L|g^R\})$ as $\{o(g^L)|o(g^R)\}$. For example, $\{\mathcal{N}|\mathcal{P}\} = \mathcal{R}$.

2. Game Tree Structure

Lemmas 2 and 3 each show that for every position there are infinitely many positions with the same game tree. We call positions *equivalent* if they have the same game tree.

Lemma 2. *For all k, $E(kp, kq) = E(p, q)$.*

Proof. Recall that $q \mid p$ if and only if $kq \mid kp$. Thus, $E(p, q) = \{\cdot|\cdot\}$ if and only if $E(kp, kq) = \{\cdot|\cdot\}$. Let $t = p\%q$, then by induction $E(p, q) = \{E(q, t)|E(q, q - t)\} = \{E(kq, kt)|E(kq, kq - kt)\} = E(kp, kq)$. □

Note that if $h = E(n, m)$ is a follower of a position $g = E(p, q)$ with $\gcd(p, q) = 1$, then $\gcd(n, m) = 1$. In the rest of the paper we will assume that every position has $\gcd(p, q) = 1$ and thus $p\%q = 0$ if and only if $q = 1$.

Lemma 3. *If $p > 2q$ then $E(p, q) = E(p - q, q)$.*

Proof. Let $p = kq + t$, $0 \le t < q$ and $k \ge 2$. Then $p - q = (k - 1)q + t$, $0 \le t < q$ and $k - 1 \ge 1$. Consider the options of both positions:

$$E(p, q) = \{E(q, t)|E(q, q - t)\}$$
$$E(p - q, q) = \{E(q, t)|E(q, q - t)\}.$$

Since they have identical options the two positions are equivalent. □

A position $E(p, q)$ will be called *standard* if $q < p < 2q$. All positions $E(p, q)$ with $q > 2$ are equivalent to some standard position (which is reachable with repeated applications of Lemma 3). Notably $E(2, 1)$ is not standard and positions of the form $E(k, 1)$ are neither standard nor equivalent to some standard position. A follower of a standard position may not be standard. For example, $E(3, 2)$ has only one proper follower, $E(2, 1)$, which is not standard.

Lemma 4. *Let $g = E(p, q)$ and $t = p\%q$. If $t \neq 0$ then g has exactly one standard option except when $q = 2t$ (i.e. $E(3, 2)$). Moreover;*

- *if $2t > q$ then exactly g^L is standard,*
- *if $0 < 2t < q$ then exactly g^R is standard.*

Proof. As $t > 0$, $g^L = E(q, t)$ and $g^R = E(q, q - t)$. Recall by the definition of standard; that $t > \frac{q}{2}$ if and only if g^L is standard and $t < \frac{q}{2}$ if and only if g^R is standard. Otherwise, $2t = q$, implying that $p = 3t$ and subsequently that $g = E(3, 2)$; from $E(3, 2)$ both players have the option $E(2, 1)$, which is not standard. □

There is a unique standard position with a given left or right option.

Lemma 5. *Let $g = E(p, q)$ and $0 < t < q$. If g is standard then:*

- *If $g^L = E(q, t)$ then $g = E(q + t, q)$.*

- *If* $g^R = E(q, q-t)$ *then* $g = E(q+t, q)$.

Proof. If $g^L = E(q,t)$ then $p = kq + t$; as g is standard, $k = 1$. If $g^R = E(q, q-t)$ then $p = kq + (t-q) = (k-1)q + t$; as g is standard, $k = 2$. □

The essential game tree structure is given by Theorem 6.

Theorem 6. *Let* $g = E(p,q)$ *and* $t = p\%q$.

- *If* $t = 0$ *then* $g = \{\cdot|\cdot\}$.
- *If* $2t = q$ *then* $g^L = g^R$.
- *If* $2t > q$ *then* $g^{LL} = g^R$.
- *If* $0 < 2t < q$ *then* $g^L = g^{RL}$.

Proof. If $t = 0$ then the game is over and $g = \{\cdot|\cdot\}$. Thus we suppose $t > 0$ and hence $g^L = E(q,t)$ and $g^R = E(q, q-t)$.

Suppose $2t = q$; then $g^L = E(q,t) = E(q, q-2t+t) = E(q, q-t) = g^R$. Moreover, $q - t \mid q$, so $E(q,t) = E(q, q-t) = 0$. (For the purists, $g = \{0|0\} = *$.)

Suppose $2t > q$; then $q > 2(q-t)$ and $t > q-t$ so $g^R = E(q, q-t) = E(q-(q-t), q-t) = E(t, q-t)$ by Lemma 3, and $g^{LL} = E(q,t)^L = E(t, q-t)$, giving

$$g^R = E(t, q-t) = g^{LL}.$$

Suppose $2t < q$; then $g^L = E(q,t) = E(q-t, t)$ by Lemma 3. Since $g^R = E(q, q-t)$ we have that $g^{RL} = E(q-t, t)$, giving $g^{RL} = g^L$. □

Corollary 7. *Let* $p > q$ *with* $p = kq + t$, $0 < t < q$ *and let* $g = E(p,q)$.

- *If* $2t > q$ *then* $g = \{g^L|g^{LL}\}$.
- *If* $2t < q$ *then* $g = \{g^{RL}|g^R\}$.

Proof. This is a simplification of Theorem 6. □

We note the similarity of Lemma 4 and Corollary 7. In Corollary 7, the two options are the standard option and its left option. The case $2t > q$ is when the lower integer is the 'farthest' integer when calculating the FICF and $2t < q$ is when the higher integer is the 'farthest'.

This motivates our next definition, the signature of a position, in which we highlight the *important* option at each stage. Recall that when we refer to $E(p,q)$ we are assuming that $\gcd(p,q) = 1$.

Definition 8. Let $g = E(p, q)$. The *signature* of g, denoted S_g, is defined as follows. If $q = 1$ then $S_g = \lambda$, the empty word. If $q = 2$ then $S_g = e$. Otherwise, let h be the standard option from g. If $g^L = h$ then $S_g = lS_h$. If $g^R = h$ then $S_g = rS_h$.

The position g and the standard positions that are successively the standard option (as per Lemma 4) starting from g are the *spine* of g.

For example, if $g = E(12, 7)$ then the signature of g is $lrle$ and the spine of g is $\{E(12, 7), E(7, 5), E(5, 3), E(3, 2)\}$. Often, we will write the signature with superscripts; for example, $S = lllrrllre$ is the same as $S = l^3r^2l^3e$.

If two positions have the same signature then they have the same game tree. We use signatures liberally to represent positions. Furthermore, we use αf to denote the position g where $S_g = \alpha S_f$.

The position $E(3, 2)$ is the unique standard position with signature e. This position is at the bottom of every spine for every position other than $E(k, 1)$ and $E(2k + 1, 2)$ for $k \geq 2$.

Theorem 9. *Let g be a* PARTIZAN EUCLID *position. Every follower of g not of the form $E(k, 1)$ is equivalent to some position on the spine of g.*

Proof. A position is on its spine, so we only need consider *proper* followers. If the length of the signature is 0 then there are no proper followers. If the length of the signature is 1 then $S_g = e$ and $g = E(3, 2)$ where the only proper follower is $E(2, 1)$. We proceed by induction on the length of signature. If the length of the signature is at least 2 then the standard option is on the spine; the non-standard option is (by Theorem 6) either $g^L = g^{RL}$ or $g^R = g^{LL}$, which is the left option of the standard option and is by induction on the spine of the standard option or of the form $E(k, 1)$. As the spine of the standard option is part of the spine of g, this completes the proof. □

Corollary 10. *Consider the position g, and let k represent an unfixed non-negative integer.*

We can write S_g as either $r^k l\alpha e$ or $r^k e$; Left's move has signature αe or λ, respectively.

We can also write S_g as either $r\alpha e$, $lr^k l\alpha e$, or $lr^k e$; Right's move has signature αe, αe, or λ, respectively.

Proof. If S_g is $r^k l\alpha e$ or $r^k e$, then S_{g^L} is αe or λ, respectively, because $g^L = g^{RL} = g^{RRL} = g^{RRRL} = \cdots = g^{R^k L}$.

If $S_g = r\alpha e$ then $S_{g^R} = \alpha e$. Otherwise, $g^R = g^{LL}$ so S_{g^R} is the signature of the position resulting from two Left moves namely αe or λ, as seen by the first part. □

In the examples below, we repeatedly use Theorem 6, but using Corollary 10 one can easily jump from the leftmost term in a line of equalities to the rightmost. Let

$g = llrle$ then

$$\begin{aligned}
e &= \{e^L|e^R\} = \{\lambda|\lambda\},\\
le &= \{le^L|le^R\} = \{e|le^{LL}\} = \{e|e^L\} = \{e|\lambda\},\\
rle &= \{rle^L|rle^R\} = \{rle^{RL}|le\} = \{le^L|le\} = \{e|le\},\\
lrle &= \{lrle^L|lrle^R\} = \{rle|lrle^{LL}\} = \{rle|rle^L\} = \{rle|rle^{RL}\} = \{le^L|e\} = \{rle|e\},\\
llrle &= \{llrle^L|llrle^R\} = \{lrle|llrle^{LL}\} = \{lrle|lrle^L\} = \{lrle|rle\}.
\end{aligned}$$

3. Reducing the Signature

The *paired outcome* of a position g (or signature S_g) is the pair $(o(g^L), o(g))$, denoted by $po(g)$ or $po(S_g)$. For example, if $S_g = e$, then $po(g) = po(e) = (\mathcal{P}, \mathcal{N})$. Note that $po(\lambda)$ is not defined.

The paired outcome of a position depends upon which option is standard and the paired outcome of that option.

Lemma 11. *If $S_g = lS_h$ then $po(g) = (o(h), \{o(h)|o(h^L)\})$. If $S_g = rS_h$ then $po(g) = (o(h^L), \{o(h^L)|o(h)\})$.*

Proof. Follows immediately from Theorem 6. □

That is, the paired outcome of a position with a standard option is determined by the paired outcome of the standard option. We use l and r to denote functions on paired outcomes; we write $l \circ po(S_g)$ to mean $po(l \circ S_g)$ and $r \circ po(S_g)$ to mean $po(r \circ S_g)$.

There are $4 \times 4 = 16$ ordered pairs of outcome classes. However, as stated in Lemma 1, there are relationships between the outcome of a position and the outcome of its options; there are only 8 ordered pairs that are *paired outcomes* of positions.

Figure 2 has a vertex for each of the 8 paired outcomes. The directed edges labelled l and r from a paired outcome, say x, lead to $l \circ x$ and $r \circ x$, respectively.

The outcome of a position is given by the paired outcome of the signature. Provided we know the paired outcome of some suffix of the signature, we can find the paired outcome of the next larger suffix using Figure 2 and eventually the desired paired outcome. As e is the suffix of every non-empty signature, we only need to know that $po(e) = po(E(3, 2)) = (\mathcal{P}, \mathcal{N})$ is where we start and to read the signature from the right starting after e.

We have now described a relatively efficient method to determine the outcome of a position given its signature, but we give a better way to determine the outcome than a walk through the graph for each letter in the signature.

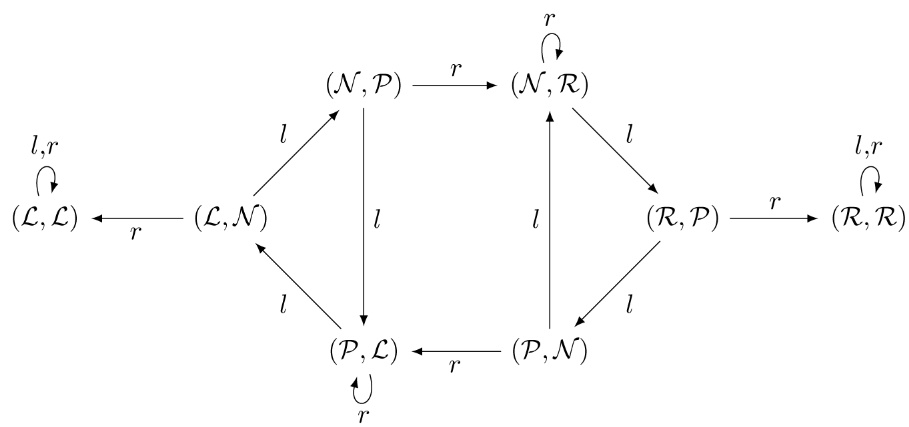

Figure 2: Paired outcome of signatures.

Just as l and r are functions on paired outcomes, we use words (denoted by Greek letters) from $\{l,r\}^*$ such as $\alpha = lrr$ as functions on paired outcomes in the natural way where $\alpha \circ x = l \circ r \circ r \circ x$. For any such β, $\beta \circ po(S_h) = po(\alpha S_h)$.

We give reduction rules by which we can simplify a word (signature) that preserves outcome, in the sense that if two positions have the same reduced signature then they have the same outcome. The main goal of this section is to prove Theorem 15, in which we give a short list of words to which any signature will reduce.

Lemma 12. *If $\alpha \in \{l,r\}^*$ then $\alpha \circ (\mathcal{L},\mathcal{L}) = (\mathcal{L},\mathcal{L})$ and $\alpha \circ (\mathcal{R},\mathcal{R}) = (\mathcal{R},\mathcal{R})$.*

Proof. Immediate from Figure 2. □

Lemma 13. *Let x be a paired outcome.*

1. $l^3 \circ x = x$;
2. $r^2 \circ x = r \circ x$;
3. $\alpha rlr \circ x = rlr \circ x$;
4. $(rl^2)^2 r \circ x = r \circ x$;
5. $rllr \circ (\mathcal{P},\mathcal{N}) = l \circ (\mathcal{P},\mathcal{N})$;
6. $\alpha rll \circ (\mathcal{P},\mathcal{N}) = rll \circ (\mathcal{P},\mathcal{N})$;
7. $rl \circ (\mathcal{P},\mathcal{N}) = l \circ (\mathcal{P},\mathcal{N})$;

Proof. In this proof we make extensive use of Figure 2. The most common use is to find a vertex with the desired paired outcome then look up the paired outcomes reached by the directed edges. For the first 4 rules, we need to show that these equation holds for all paired outcomes x.

Rule 1: we apply l to each x three times to observe that we return to the x with which we started.

Rule 2: following two edges marked r and, regardless, the second edge is a loop.

Rule 3: following three edges marked r, l and r results in $(\mathcal{L}, \mathcal{L})$ or $(\mathcal{R}, \mathcal{R})$ so the first part of the signature it is irrelevant.

Rule 4: following the walk with edges marked $rllrllr$ either (i) goes once around the 6-cycle in Figure 2 ending at the starting vertex; (ii) or if the starting vertex is $(\mathcal{L}, \mathcal{L})$ or $(\mathcal{R}, \mathcal{R})$ then remains at $(\mathcal{L}, \mathcal{L})$ or $(\mathcal{R}, \mathcal{R})$ respectively; (iii) starts at $(\mathcal{L}, \mathcal{N})$ or $(\mathcal{R}, \mathcal{P})$ then the first r-edge goes to $(\mathcal{L}, \mathcal{L})$, $(\mathcal{R}, \mathcal{R})$ respectively. In all cases the effect of following the last r edge in the walk is the same as following the first.

Rule 5: obvious from the figure.

Rule 6: from $(\mathcal{P}, \mathcal{N})$ the walk l, l, r ends at $(\mathcal{R}, \mathcal{R})$ and any further edges does not change the paired outcome.

Rule 7: obvious from the figure. □

Lemma 14. *If x is a paired outcome and $\gamma \circ x = \delta \circ x$, then $po(\alpha\gamma\beta e) = po(\alpha\delta\beta e)$.*

Proof. $po(\alpha\gamma\beta e) = \alpha\gamma \circ po(\beta e) = \alpha\delta \circ po(\beta e) = po(\alpha\delta\beta e)$. □

In applying Lemma 14 to reduce a word, we make reference to particular rules from 13.

We call a word *irreducible* if none of the reduction rules are applicable. Reduction rules may be applied in any order to the signature of a position, say g, to derive an irreducible word; the irreducible word corresponds to some other position, say h. The outcome of g is the same as the outcome of h, as they have the same paired outcome, which is a stronger condition. As there are a finite number of irreducible words, we will be able to compute and store the outcomes of those positions.

Theorem 15. *There are 9 irreducible words: λ, e, re, le, lle, lre, $rlre$, $llre$, and $rlle$.*

For the proof of Theorem 15 we need the following Lemma:

Lemma 16. *A word containing 4 rs is reducible.*

Proof. Suppose α is an irreducible word containing 4 rs. By Rule 2, each pair of consecutive rs is separated by at least one l. By Rule 3, each pair of rs except possibly the leftmost, is separated by more than one l. By Rule 1, each pair of rs is separated by at most 2 ls. That is, the rightmost 3 rs form the pattern $rllrllr$, which contradicts the assumption that α is irreducible by Rule 4. □

S_g	$o(g)$	g
λ	$\mathcal{P}$	$E(2,1)$
e	$\mathcal{N}$	$E(3,2)$
re	$\mathcal{L}$	$E(4,3)$
le	$\mathcal{R}$	$E(5,3)$
lle	$\mathcal{P}$	$E(8,5)$
lre	$\mathcal{N}$	$E(7,4)$
$rlre$	$\mathcal{L}$	$E(10,7)$
$rlle$	$\mathcal{R}$	$E(11,8)$
$llre$	$\mathcal{P}$	$E(11,7)$

Table 1: Irreducible signatures with corresponding positions and outcomes

Proof of Theorem 15. The irreducible words that do not end in e are easy to list and count with the help of Lemma 16; such words have at most 3 rs. In what follows, α and β are one of either λ, l, or ll.

Words with 3 rs are of the form $rlrllr\beta$, of which there are 3.

Words with 2 rs are of the form $\alpha rllr\beta$ or $rlr\beta$, of which there are 12.

Words with 1 r are of the form $\alpha r\beta$, of which there are 9.

Words with no r are of the form α, of which there are 3.

There are a total of 27 such words. The only irreducible signatures are among the set containing these strings but with a trailing e appended, and the empty word.

We show that 19 of the 27 strings reduce to the remaining 8: e, le, lle, re, lre, $llre$, $rlle$ and $rlre$.

- The 7 strings of the form $\gamma rlle$ where γ is non-empty reduce by Rule 6 to $rlle$.
- The 4 words of the form $\gamma rllrle$ reduce to $\gamma rllle$ and then to γre by Rules 7 and 1, respectively, leaving re, lre, $llre$, and $rlre$.
- The 4 words of the form $\gamma rllre$ reduce to γle by Rule 5, leaving the 3 strings with no r (note $llle = e$) and $rlle$.
- The 4 words of the form γrle reduce to γle by Rule 7, leaving the 3 strings with no r (note $llle = e$) and $rlle$.

We note that of the none of the reductions apply to the 9 claimed irreducible words (8 from above and λ). □

3.1. Algorithm

We present an algorithm that efficiently determines the outcome of a PARTIZAN EUCLID position.

Step 0: Let S be the signature of $E(p,q)$. Let S' be the empty string.

Step 1: If S is non-empty, remove the first letter of S and add it to the end of S'; go to Step 2. Otherwise, go to Step 3.

Step 2:
- If you added l to S', use Rule 1 on the suffix of S' if applicable. Go to Step 1.
- If you added r to S', use Rule 2, 3 or 4 on the suffix of S' if applicable; at most one will apply. Go to Step 1.
- If you added e to S', use Rule 5, 6 or 7 on the suffix of S' if applicable, at most one will apply. If you applied Rule 5 or 7, then use Rule 1 if applicable. Go to Step 3.

Step 3: The outcome of $E(p, q)$ is the outcome of S' given in Table 1.

Reductions occur at the end of the word and the application of a reduction does not cause another reduction, except possibly in Step 2: part 3, with an lll reduction. As such, Step 2 finishes in constant time (as do Steps 1 and 3). Step 1 takes about as long as the Euclidean algorithm. Steps 1 and 2 have to be performed at most p times; Steps 0 and 3 are each performed once.

By Lemma 16, S' reaches a length of at most 8, as demonstrated by $llrllrll$. That is, if at any point the length of S' is 9, then it will be irreducible in the next step. In the algorithm as given above, S is computed in full at the beginning, for ease of description. However, we can easily modify our algorithm to be an on-line algorithm by computing the next letter of S as we need it to add to S'. In that case, to run the algorithm we store at most 3 integers no larger than p and a string of length at most 9.

4. Outcome Observations

There are several interesting observations that can be made about the outcomes which may be useful in actual play.

Observation 1. If $S_g = r\beta e$ then $o(g) \in \{\mathcal{L}, \mathcal{R}\}$.

All signatures in Table 1 starting with r are in $\mathcal{L}$ or $\mathcal{R}$. The reductions (from Lemma 13) change signatures starting with r to shorter signatures starting with r or to le, which is in $\mathcal{R}$.

Observation 2. Let $g = E(p, q)$ be a standard position. If $o(g) \in \{\mathcal{N}, \mathcal{P}\}$, then $\frac{2p}{3} \geq q > \frac{p}{2}$.

If $o(g) \in \{\mathcal{N}, \mathcal{P}\}$, then S_g is λ (in which case g is not standard), e (in which case $\frac{2p}{3} = q$), or starts with l. As g is standard, if $t = p\%q$, then $t = p - q$ and $q > \frac{p}{2}$.

For the signature to start with l, we need $2t > q$, which is $2(p-q) > q$ or $\frac{2p}{3} > q$; combining this with $q > \frac{p}{2}$ gives the result.

Lemma 13 can be restated in terms of positions in the game.

Observation 3. If a and b are integers with $a > b > 0$ then

(1) $o(E(a+b,a)) = o(E(5a+3b,3a+2b))$; and

(2) if $o(E(2a+b,a+b)) = \mathcal{P}$ and $a > 2b$ then $E(2a+3b,a+2b) = \mathcal{P}$.

For **(1)** let $E(a+b,a) = \alpha e$ and consider $E(5a+3b,3a+2b)$. First suppose $a > b$; the consecutive left options $E(3a+2b,2a+b)$, $E(2a+b,a+b)$, and $E(a+b,a)$ are standard and are on the spine so $E(5a+3b,3a+2b) = lll\alpha e$. By Rule 1 $o(E(5a+3b,3a+2b)) = o(E(a+b,a))$. Now suppose $a < b$; from $E(5a+3b,3a+2b)$, both $E(3a+2b,2a+b)$, $E(2a+b,a+b)$ are still on the spine and now $E(a+b,b)$ is too. Since $E(a+b,a) = \mathcal{P}$ then by Theorem 6 $o(E(a+b,b)^L) = \mathcal{P}$ thus $o(E(2a+b,a+b)) = \mathcal{L}$. Now $o(E(3a+2b,2a+b)^L) = o(E(2a+b,a+b)) = \mathcal{L}$ and again by Theorem 6 $o(E(3a+2b,2a+b)^R) = o(E(a+b,a)) = \mathcal{P}$ thus $o(E(3a+2b,2a+b)) = \mathcal{N}$. Finally $o(E(5a+3b,3a+2b)^L) = o(E(3a+2b,2a+b)) = \mathcal{N}$ and again by Theorem 6 $o(E(5a+3b,3a+2b)^R) = o(E(2a+b,a+b)) = \mathcal{L}$ and thus $o(E(5a+3b,3a+2b)) = \mathcal{P}$.

For **(2)** Since $a > 2b$ then $E(2a+b,a+b) = lr\alpha e$ where the left option is $E(a+b,a)$ and its right option is $E(a,a-b)$ are both standard and on the spine. The left option of $E(2a+3b,a+2b)$ is $E(a+2b,a+b)$ and its right option is $E(a+b,a)$ again both on the spine. Thus $E(2a+3b,a+2b) = lrr\alpha e$. Therefore, from Lemma 13 $o(E(2a+3b,a+2b)) = o(E(2a+b,a+b))$.

5. Open Questions

Our main work is describing the structure of postitions of PARTIZAN EUCLID and giving an efficient algorithm for determining the outcome. Thus we arrive at one main open question.

Question 1. Is there an efficient method to play disjunctive sums of PARTIZAN EUCLID positions?

For some families of positions (signatures) we can give the value easily. Observations 4 and 5 happen to correspond to the extreme cases of the Euclidean algorithm.

Observation 4. Positions of the form $E(k+1,k)$ have value $*+(k-2){\uparrow}*$ for $k \geq 2$.

The signature $r^k e$ corresponds to $E(k+1,k)$. When $k \geq 2$ the Left option is to $E(k,1)$ which is equal to 0, and the Right option is to $E(k,k-1)$.

Observation 5. Let f_n be the nth Fibonacci number where $f_0 = 0$ and $f_1 = 1$. The position $E(f_k, f_{k-1})$ has the signature $l^k e$ and the value is periodic in k; $E(f_{3k}, f_{3k-1}) = \uparrow$, $E(f_{3k+1}, f_{3k}) = *$, and $E(f_{3k+2}, f_{3k+1}) = 0$. Starting with $E(2,1) = 0$, $E(3,2) = *$, and $E(5,3) = \uparrow$, an easy induction gives the result.

It seems unlikely that we would find a heuristic method to play a sum; we expect a solution would require first computing the values of the summands and a method to play on a sum of such values (see [1] for more on values). Values are harder to calculate and there are few easy reductions of the signature that allow short cuts. However, we present a general rule that we have found.

Note from Corollary 10 that a Left move from a position whose signature has at least one l in it, removes exactly one l; and a Right move from a position whose signature has at least two ls in it, removes exactly one r or exactly two ls.

Observation 6. Let two positions g and h have signatures $\alpha l r^a l \beta e$ and $\alpha l r^b l \gamma e$ respectively. If $r^a l \beta e = r^b l \gamma e$ and $\beta e = \gamma e$ then $g = h$.

To see this, if $\alpha = \lambda$, then $g^L = r^a l \beta e = r^b l \gamma e = h^L$ and $g^R = g^{LL} = \beta e = \gamma e = h^{LL} = h^R$. If $\alpha = l$, then $g^L = l r^a l \beta e = l r^b l \gamma e = h^L$ and $g^R = \beta e = \gamma e = h^R$. If $\alpha = r$, then $g^L = r^a l \beta e = r^b l \gamma e = h^L$ and $g^R = l r^a l \beta e = l r^b l \gamma e = h^R$.

As there are many non-trivial $\mathcal{P}$ positions in PARTIZAN EUCLID, and all $\mathcal{P}$ positions have value 0, we think it is reasonable to expect many other values to occur repeatedly, perhaps with similar patterns to that of the $\mathcal{P}$ positions.

Question 2. Which signature reductions preserve value in which instances?

Canonical forms become messy with even small values of p and q. Atomic weights (see [1]) are an approximation to the value of a position. The atomic weights of Table 5 were generated by CGSuite [9]. (In version 1.0 of CGSuite PARTIZAN EUCLID is used as an example and tables of both canonical forms and atomic weights can be generated easily.)

	$q = 11$	12	13	14	15	16	17	18	19	20
$p = 10$	$\{6\|2\}$	$3/2$	2	0	-1	0	2	0	0	0
11		$\{7\|2\}$	0	-3	0	4	3	0	-4	0
12			$\{8\|2\}$	$\{2\|2\}$	1	0	-3	-1	-1	0
13				$\{9\|2\}$	0	2	3	0	5	4
14					$\{10\|2\}$	$\{3\|2\}$	-3	0	0	2
15						$\{11\|2\}$	0	$3/2$	0	0

Table 2: $E(q,p)$, $q = 11, \ldots, 20$, $p = 10, \ldots, 15$.

The mean values of atomic weights show some regularity on a large scale, see Figure 3, where the figure is cutoff above 25, removing the points corresponding to $E(k+1,k)$ for $k > 27$.

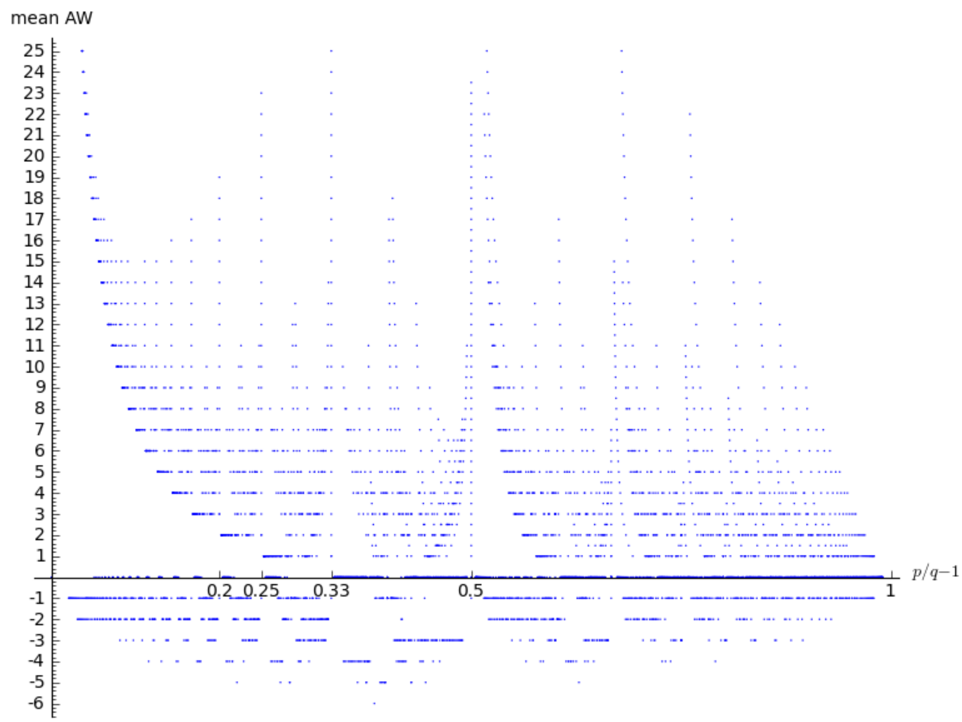

Figure 3: Graph showing mean atomic weights of positions against ratio of p to q.

In addition to the variation of the means of atomic weights of positions there is great complexity among the atomic weights, which include positions such as $\{6|9\frac{1}{2}\}$, $+_3$, $8\Uparrow *$, and $\{7||6|5\}$.

Question 3. Is there an efficient way to determine the atomic weight of a position from its signature?

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