SOME THETA FUNCTIONS IDENTITIES ASSOCIATED WITH THE MODULAR EQUATIONS OF DEGREE 5

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Abstract

In this paper the author proves a general theta functions identity by using the theory of elliptic functions. This identity allows the author to derive four interesting theta functions identities. These identities lead to new proofs of some well-known identities of Ramanujan associated with the modular equations of degree 5. Some new identities are also discussed.

1. Introduction

We suppose throughout that $q := e^{2\pi i\tau}$, Im $(\tau) > 0$; this condition ensures that all the sums and products that appear here converge. We will use the standard q-notations:

$$\begin{aligned}(a;q)_\infty &= \prod_{n=1}^{\infty}(1-aq^{n-1}),\\ (a_1,a_2\dots a_m;q)_\infty &= \prod_{k=1}^{m}(a_k;q)_\infty.\end{aligned}$$

The well-known Jacobi's triple product identity [1, pp. 21-22], [2, p. 35], [9, 10] is

$$(q,z,q/z;q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(n-1)}{2}} z^n. \tag{1.1}$$

Using the above identity, M. D. Hirschhorn [9] derived the following septagonal numbers identity

$$\begin{aligned}&(a^3-a-a^{-1}+a^{-3})(a^2q,a^{-2}q,a^4q,a^{-4}q;q)_\infty(q;q)_\infty^2\\ &=(q^2,q^3,q^5;q^5)_\infty\sum_{n=-\infty}^{\infty}(-1)^n(a^{10n-3}+a^{-10n+3})q^{\frac{n(5n-3)}{2}}\\ &-(q,q^4,q^5;q^5)_\infty\sum_{n=-\infty}^{\infty}(-1)^n(a^{10n-1}+a^{-10n+1})q^{\frac{n(5n-1)}{2}}.\end{aligned} \tag{1.2}$$

Using this identity, Hirschhorn provided a simpler proof of the following identity of Ramanujan [13, p. 139], [4, 9, 16]:

$$1-5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n}=\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty}, \tag{1.3}$$

where $\left(\frac{n}{5}\right)$ denote the Legendre symbol.

In [7], H. M. Farkas and I. Kra rediscovered (1.2) using the theory of theta functions with rational characteristics. F. G. Garvan [6] provided an interesting generalization of (1.2) and his proof depends only on the triple product identity.

Using the Jacobi theta function $\theta_1(z|q)$ ((2.1) below), (1.2) can be reformulated as

$$\begin{aligned} iq^{-\frac{1}{2}}\theta_1(x|q)\theta_1(2x|q) = {} & \theta_1(2\pi\tau|q^5)\left\{e^{2ix}\theta_1(5x+\pi\tau|q^5)-e^{-2ix}\theta_1(5x-\pi\tau|q^5)\right\} \\ & -\theta_1(\pi\tau|q^5)\left\{e^{4ix}\theta_1(5x+2\pi\tau|q^5)-e^{-4ix}\theta_1(5x-2\pi\tau|q^5)\right\}. \end{aligned} \tag{1.4}$$

In Section 3 of this paper we will prove a very general identity (identity (3.2) below) involving theta functions.

In Section 4, we will derive (1.4) from identity (3.2), and then prove (1.3) using (1.4). We will also prove the following important result of Ramanujan:

$$\frac{1}{R(q^5)}-1-R(q^5)=\frac{(q;q)_\infty}{q(q^{25};q^{25})_\infty}, \tag{1.5}$$

where

$$q^{-\frac{1}{5}}R(q)=\frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty}. \tag{1.6}$$

The following identity will be established in Section 5 using (3.2):

$$\begin{aligned} 5\theta_1(x|q^5)\theta_1(2x|q^5) &= \theta_1(\frac{2\pi}{5}|q)\left\{\theta_1(x+\frac{\pi}{5}|q)-\theta_1(x-\frac{\pi}{5}|q)\right\} \\ &- \theta_1(\frac{\pi}{5}|q)\left\{\theta_1(x+\frac{2\pi}{5}|q)-\theta_1(x-\frac{2\pi}{5}|q)\right\}. \end{aligned} \tag{1.7}$$

From this identity, we can obtain the following identity of Ramanujan [13, p. 139], [4, 16]:

$$\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2}=q\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty}. \tag{1.8}$$

Using (1.7), we can also derive the identity

$$\frac{\theta_1(\frac{2\pi}{5}|q)}{\theta_1(\frac{\pi}{5}|q)}-\frac{\theta_1(\frac{\pi}{5}|q)}{\theta_1(\frac{2\pi}{5}|q)}=1+5q\frac{(q^{25};q^{25})_\infty}{(q;q)_\infty}. \tag{1.9}$$

In Section 6, we will prove

$$C(q)\theta_1(x|q)\theta_1(2x|q) = \theta_1^5(\frac{2\pi}{5}|q)\left\{\theta_1^5(x+\frac{\pi}{5}|q)-\theta_1^5(x-\frac{\pi}{5}|q)\right\} -\theta_1^5(\frac{\pi}{5}|q)\left\{\theta_1^5(x+\frac{2\pi}{5}|q)-\theta_1^5(x-\frac{2\pi}{5}|q)\right\}, \tag{1.10}$$

where

$$C(q) = 250q(q;q)_\infty^4(q^5;q^5)_\infty^4 + 3125q^2\frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^2}, \tag{1.11}$$

Using (1.10), we can obtain

$$\frac{\theta_1^5(\frac{2\pi}{5}|q)}{\theta_1^5(\frac{\pi}{5}|q)} - \frac{\theta_1^5(\frac{\pi}{5}|q)}{\theta_1^5(\frac{2\pi}{5}|q)} = 11 + 125q\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^6}. \tag{1.12}$$

Section 7 is devoted to the proof of the following identity:

$$C(q)\theta_1(x|q^5)\theta_1(2x|q^5) = \theta_1^5(2\pi\tau|q^5)\left\{e^{2ix}\theta_1^5(x+\pi\tau|q^5) - e^{-2ix}\theta_1^5(x-\pi\tau|q^5)\right\} \tag{1.13}$$

$$-\theta_1^5(\pi\tau|q^5)\left\{e^{4ix}\theta_1^5(x+2\pi\tau|q^5) - e^{-4ix}\theta_1^5(x-2\pi\tau|q^5)\right\}, \tag{1.14}$$

where

$$q^{\frac{5}{2}}C(q) = 10q(q;q)_\infty^4(q^5;q^5)_\infty^4 + \frac{(q;q)_\infty^{10}}{(q^5;q^5)_\infty^2}. \tag{1.15}$$

Using this identity, we can rederive the following identity of Ramanujan:

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{(q;q)_\infty^6}{q(q^5;q^5)_\infty^6}. \tag{1.16}$$

It should be pointed out that we also need the method of L. -C. Shen [17] in deriving (1.4), (1.7), (1.10) and (1.14).

Identities (1.5) and (1.16) can be found in Ramanujan's second notebook [14, pp. 265-267] and were first proved by G. N. Watson [18] for the purpose of establishing some of Ramanujan's claims about $R(q)$ made in his first two letters to Hardy [15, pp. xxvii, xxviii]. They were used by B. C. Berndt, H. H. Chan, and L. -C. Zhang [3] in deriving general formulas for the explicit evaluation of $R(q)$. They were also used by the author and R. P. Lewis [11] to provide simpler proofs of two Lambert series identities of Ramanujan.

In [4], Chan utilized the Hecke correspondence between Dirichlet series and Fourier expansions of modular forms to show that (1.3) and (1.4) are equivalent.

2. Some Basic Facts About $\theta_1(z|q)$

For $q = e^{2\pi i\tau}$, with $\mathrm{Im}(\tau) > 0$, the Jacobi theta function $\theta_1(z|q)$ is defined by [19, p. 463]

$$\begin{aligned}\theta_1(z|q) :&= -iq^{\frac{1}{8}} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n(n+1)} e^{(2n+1)iz} \\ &= 2q^{\frac{1}{8}} \sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(n+1)} \sin(2n+1)z. \end{aligned} \tag{2.1}$$

In terms of infinite products [19, p. 469],

$$\begin{aligned}\theta_1(z|q) &= 2q^{\frac{1}{8}} (\sin z)(q;q)_\infty (qe^{2iz};q)_\infty (qe^{-2iz};q)_\infty \\ &= iq^{\frac{1}{8}} e^{-iz} (q;q)_\infty (e^{2iz};q)_\infty (qe^{-2iz};q)_\infty. \end{aligned} \tag{2.2}$$

One deduces easily from (2.1) that (see, for example, [17])

$$\frac{\partial^2 \theta_1}{\partial z^2} = -8q \frac{\partial \theta_1}{\partial q}. \tag{2.3}$$

In the following, θ_1' and θ_1'' denote the first and second partial derivatives of θ_1 with respect to z. We can derive the following two important identities [17] from (2.2):

$$\frac{\theta_1'}{\theta_1}(z|q) = \cot z + 4 \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} \sin 2nz, \tag{2.4}$$

$$\begin{aligned}\frac{\theta_1''}{\theta_1}(z|q) &= -1 + 8 \sum_{n=1}^{\infty} \frac{nq^n e^{2iz}}{1-q^n e^{2iz}} + 8 \sum_{n=1}^{\infty} \frac{nq^n e^{-2iz}}{1-q^n e^{-2iz}} + 8 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \\ &= -1 + 16 \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} \cos 2nz + 8 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}. \end{aligned} \tag{2.5}$$

Differentiating (2.4) with respect to z, we obtain

$$\left(\frac{\theta_1'}{\theta_1}\right)'(z|q) = -\csc^2 z + 8 \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \cos 2nz. \tag{2.6}$$

Using these identities and the simple differential identity

$$\left(\theta_1'/\theta_1\right)^2 = \theta_1''/\theta_1 - \left(\theta_1'/\theta_1\right)', \tag{2.7}$$

an interesting proof of the following trigonometric series identity of Ramanujan [12], [8, pp. 134-135] has recently been given by Shen [17]:

$$\begin{aligned}&\left\{\frac{1}{4}\cot z + \sum_{n=1}^{\infty} \frac{q^n \sin 2nz}{1-q^n}\right\}^2 \\ &= \left\{\frac{1}{4}\cot z\right\}^2 + \sum_{n=1}^{\infty} \frac{q^n \cos 2nz}{(1-q^n)^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{nq^n(1-\cos 2nz)}{1-q^n}. \end{aligned} \tag{2.8}$$

By some elementary calculations we find the following trigonometric functions identities:

$$\csc^2\frac{\pi}{5}-\csc^2\frac{2\pi}{5} = \frac{4}{\sqrt{5}}, \tag{2.9}$$

$$\cos\frac{2n\pi}{5}-\cos\frac{4n\pi}{5} = \frac{\sqrt{5}}{2}\left(\frac{n}{5}\right). \tag{2.10}$$

Replacing z by $\frac{\pi}{5}$ and $\frac{2\pi}{5}$ in (2.4) respectively and then subtracting the two resulting equations and finally using (2.9) we find that

$$\frac{\theta_1''}{\theta_1}\left(\frac{\pi}{5}\Big|q\right)-\frac{\theta_1''}{\theta_1}\left(\frac{2\pi}{5}\Big|q\right)=8\sqrt{5}\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2}. \tag{2.11}$$

Replacing z by $\frac{\pi}{5}$ and $\frac{2\pi}{5}$ in (2.5) respectively and then subtracting the two resulting equations and finally using (2.9), (2.10) and (1.3) we find that

$$\begin{aligned}\left(\frac{\theta_1'}{\theta_1}\right)'\left(\frac{\pi}{5}\Big|q\right)-\left(\frac{\theta_1'}{\theta_1}\right)'\left(\frac{2\pi}{5}\Big|q\right) &= -\frac{4}{\sqrt{5}}\left\{1-5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n}\right\}\\ &= -\frac{4}{\sqrt{5}}\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty}.\end{aligned} \tag{2.12}$$

Applying logarithmic differentiation to (2.2), we find that

$$\frac{\theta_1'}{\theta_1}(z|q)=-i-2i\sum_{n=0}^{\infty}\frac{q^ne^{2iz}}{1-q^ne^{2iz}}+2i\sum_{n=1}^{\infty}\frac{q^ne^{-2iz}}{1-q^ne^{-2iz}}. \tag{2.13}$$

Differentiating (2.13) with respect to z, we find that

$$\left(\frac{\theta_1'}{\theta_1}\right)'(z|q)=4\sum_{n=0}^{\infty}\frac{q^ne^{2iz}}{(1-q^ne^{2iz})^2}+4\sum_{n=1}^{\infty}\frac{q^ne^{-2iz}}{(1-q^ne^{-2iz})^2}. \tag{2.14}$$

Replacing q by q^5, setting $z=\pi\tau$ and $z=2\pi\tau$ respectively, we obtain

$$\begin{aligned}\left(\frac{\theta_1'}{\theta_1}\right)'(\pi\tau|q^5) &= 4\sum_{n=0}^{\infty}\frac{q^{5n+1}}{(1-q^{5n+1})^2}+4\sum_{n=0}^{\infty}\frac{q^{5n+4}}{(1-q^{5n+4})^2},\\ \left(\frac{\theta_1'}{\theta_1}\right)'(2\pi\tau|q^5) &= 4\sum_{n=0}^{\infty}\frac{q^{5n+2}}{(1-q^{5n+2})^2}+4\sum_{n=0}^{\infty}\frac{q^{5n+3}}{(1-q^{5n+3})^2}.\end{aligned}$$

Therefore, we have

$$\left(\frac{\theta_1'}{\theta_1}\right)'(\pi\tau|q^5)-\left(\frac{\theta_1'}{\theta_1}\right)'(2\pi\tau|q^5)=4\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2}. \tag{2.15}$$

From (2.1), we find that

$$e^{2iz}\theta_1(5z+\pi\tau|q^5) = iq^{\frac{1}{8}}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(5n-3)}{2}}e^{(10n-3)iz}, \tag{2.16}$$

$$e^{4iz}\theta_1(5z+2\pi\tau|q^5) = iq^{-\frac{3}{8}}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(5n-1)}{2}}e^{(10n-1)iz}. \tag{2.17}$$

Differentiating both sides of (2.16) twice with respect to z, and then setting $z=0$ in the resulting equation, we obtain

$$\begin{aligned} & -4\theta_1(\pi\tau|q^5)+20i\theta_1'(\pi\tau|q^5)+25\theta_1''(\pi\tau|q^5) \\ = & -iq^{\frac{1}{8}}\sum_{n=-\infty}^{\infty}(-1)^n(10n-3)^2 q^{\frac{n(5n-3)}{2}} \\ = & -iq^{\frac{1}{8}}\left(9+40q\frac{d}{dq}\right)\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(5n-3)}{2}} \\ = & -iq^{\frac{1}{8}}\left(9+40q\frac{d}{dq}\right)(q,q^4,q^5;q^5)_\infty. \end{aligned} \tag{2.18}$$

By logarithmic differentiation, we have

$$\begin{aligned} & (q,q^4,q^5;q^5)_\infty^{-1}\left(9+40q\frac{d}{dq}\right)(q,q^4,q^5;q^5)_\infty \\ = & \ 9-40\sum_{n=1}^{\infty}\left(\frac{5nq^{5n}}{1-q^{5n}}+\frac{(5n-4)q^{5n-4}}{1-q^{5n-4}}+\frac{(5n-1)q^{5n-1}}{1-q^{5n-1}}\right). \end{aligned} \tag{2.19}$$

Substituting (2.19) into (2.18) and using (2.24) below we obtain

$$\begin{aligned} & 4-20i\frac{\theta_1'}{\theta_1}(\pi\tau|q^5)-25\frac{\theta_1''}{\theta_1}(\pi\tau|q^5) \\ = & \ 9-40\sum_{n=1}^{\infty}\left(\frac{5nq^{5n}}{1-q^{5n}}+\frac{(5n-4)q^{5n-4}}{1-q^{5n-4}}+\frac{(5n-1)q^{5n-1}}{1-q^{5n-1}}\right). \end{aligned} \tag{2.20}$$

Similarly, from (2.17) we obtain

$$\begin{aligned} & 16-40i\frac{\theta_1'}{\theta_1}(2\pi\tau|q^5)-25\frac{\theta_1''}{\theta_1}(2\pi\tau|q^5) \\ = & \ 1-40\sum_{n=1}^{\infty}\left(\frac{5nq^{5n}}{1-q^{5n}}+\frac{(5n-3)q^{5n-3}}{1-q^{5n-3}}+\frac{(5n-2)q^{5n-2}}{1-q^{5n-2}}\right). \end{aligned} \tag{2.21}$$

Substracting (2.20) from (2.21) we obtain

$$\begin{aligned} & 12+20i\left\{\frac{\theta_1'}{\theta_1}(\pi\tau|q^5)-2\frac{\theta_1'}{\theta_1}(2\pi\tau|q^5)\right\}+25\left\{\frac{\theta_1''}{\theta_1}(\pi\tau|q^5)-\frac{\theta_1''}{\theta_1}(2\pi\tau|q^5)\right\} \\ & = -8\left\{1-5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n}\right\}. \end{aligned} \tag{2.22}$$

Differentiating (2.2) with respect to z and then putting $z = 0$ we obtain the identity

$$\theta_1'(q) := \theta_1'(0|q) = 2q^{\frac{1}{8}}(q;q)_\infty^3. \tag{2.23}$$

Replacing q by q^5 in (2.2) and then taking $z = \pi\tau$, and $2\pi\tau$ in the resulting equation respectively we find that

$$\theta_1(\pi\tau|q^5) = iq^{\frac{1}{8}}(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty, \tag{2.24}$$

$$\theta_1(2\pi\tau|q^5) = iq^{-\frac{3}{8}}(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty. \tag{2.25}$$

Multiplying the above two equations, after some manipulation we find that

$$\theta_1(\pi\tau|q^5)\theta_1(2\pi\tau|q^5) = -q^{-\frac{1}{4}}(q;q)_\infty(q^5;q^5)_\infty. \tag{2.26}$$

From (2.2) we have the following special values of $\theta_1(z|q)$

$$\theta_1(\frac{2\pi}{5}|q) = 2q^{\frac{1}{8}}(\sin\frac{\pi}{5})(q;q)_\infty(qe^{\frac{2\pi i}{5}};q)_\infty(qe^{\frac{-2\pi i}{5}};q)_\infty, \tag{2.27}$$

$$\theta_1(\frac{2\pi}{5}|q) = 2q^{\frac{1}{8}}(\sin\frac{\pi}{5})(q;q)_\infty(qe^{\frac{4\pi i}{5}};q)_\infty(qe^{\frac{-4\pi i}{5}};q)_\infty. \tag{2.28}$$

Multiplying the above equations and then using the elementary facts

$$\sin\frac{\pi}{5}\sin\frac{2\pi}{5} = \frac{\sqrt{5}}{4}$$

and

$$(1-x)(1-xe^{\frac{2\pi i}{5}})(1-xe^{\frac{2\pi i}{5}})(1-xe^{\frac{2\pi i}{5}})(1-xe^{\frac{2\pi i}{5}}) = 1-x^5,$$

in the resulting equation, we find that

$$\theta_1(\frac{\pi}{5}|q)\theta_1(\frac{2\pi}{5}|q) = \sqrt{5}q^{\frac{1}{4}}(q;q)_\infty(q^5;q^5)_\infty. \tag{2.29}$$

From the definition of $\theta_1(z|q)$ we readily find the following functional equations

$$\theta_1(z+n\pi|q) = (-1)^n\theta_1(z|q), \tag{2.30}$$

$$\theta_1(z+n\pi\tau|q) = (-1)^n q^{-\frac{n^2}{2}}e^{-2n\pi iz}\theta_1(z|q), \tag{2.31}$$

where n is any integer. Differentiating the above equations with respect to z and then setting $z = 0$ in the resulting equations, we find that

$$\theta_1'(n\pi|q) = (-1)^n\theta_1'(q) \quad \text{and} \quad \theta_1'(n\pi\tau|q) = (-1)^n q^{-\frac{n^2}{2}}\theta_1'(q). \tag{2.32}$$

3. A General Identity For $\theta_1(z|q)$

In this section we will prove the following theta functions identity.

Theorem 1 *If $f_1(z)$ and $f_2(z)$ are two different entire functions satisfying the functional equations*

$$f(z+\pi) = -f(z) \qquad \text{and} \qquad f(z+\pi\tau) = -q^{-\frac{5}{2}}e^{-10iz}f(z), \tag{3.1}$$

with $f_1(0) \neq 0$, $f_2(0) \neq 0$, then there is a constant $C(q)$ such that

$$C(q)\theta_1(x|q)\theta_1(2x|q) = f_2(0)\left(f_1(x) + f_1(-x)\right) - f_1(0)\left(f_2(x) + f_2(-x)\right). \tag{3.2}$$

To prove the above identity we require the following fundamental theorem of elliptic functions [5, p. 22, Theorem 2]:

Theorem 2 *The sum of all the residues of an elliptic function vanishes in the period parallelogram.*

The idea is to construct an elliptic function whose poles are known and then compute the residues of the elliptic function at these poles. Set the sum of the residues to zero to obtain the desired identity for theta functions. We will use L'Hôpital's rule to compute the residues.

Proof. Let $0 < x, y < \pi$ be two distinct parameters different from the zeros of $f_1(z)$. By (2.31) and (2.32) we readily verify that

$$\theta_1(z|q)\theta_1(z-x|q)\theta_1(z+x|q)\theta_1(z-y|q)\theta_1(z+y|q)$$

satisfies (3.1). Therefore,

$$E(z) = \frac{f_1(z)}{\theta_1(z|q)\theta_1(z-x|q)\theta_1(z+x|q)\theta_1(z-y|q)\theta_1(z+y|q)},$$

is an elliptic function with periods π and $\pi\tau$. The poles of $E(z)$ are $0, x, \pi - x, y$, and $\pi - y$, all of which are simple poles. Let $\operatorname{res}(E;\alpha)$ denote the residue of $E(z)$ at α. From Theorem 2, we have

$$\operatorname{res}(E;0) + \operatorname{res}(E;x) + \operatorname{res}(E;\pi-x) + \operatorname{res}(E;y) + \operatorname{res}(E;\pi-y) = 0. \tag{3.3}$$

Now

$$\begin{aligned}
\operatorname{res}(E;0) &= \lim_{z\to 0} zE(z) \\
&= \lim_{z\to 0}\frac{f_1(z)}{\theta_1(z-x|q)\theta_1(z+x|q)\theta_1(z-y|q)\theta_1(z+y|q)} \times \lim_{z\to 0}\frac{z}{\theta_1(z|q)} \\
&= \frac{f_1(0)}{\theta_1'(q)\theta_1^2(x|q)\theta_1^2(y|q)}.
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
\operatorname{res}(E;x) &= \lim_{z\to x}(z-x)E(z)\\
&= \lim_{z\to x}\frac{f_1(z|q)}{\theta_1(z|q)\theta_1(z+x|q)\theta_1(z-y|q)\theta_1(z+y|q)}\times\lim_{z\to x}\frac{z-x}{\theta_1(z-x|q)}\\
&= \frac{f_1(x)}{\theta_1'(q)\theta_1(x|q)\theta_1(2x|q)\theta_1(x-y|q)\theta_1(x+y|q)}. \qquad (3.5)
\end{aligned}$$

$$\begin{aligned}
\operatorname{res}(E;\pi-x) &= \lim_{z\to\pi-x}(z-\pi+x)E(z)\\
&= \lim_{z\to\pi-x}\frac{f_1(z)}{\theta_1(z|q)\theta_1(z-x|q)\theta_1(z-y|q)\theta_1(z+y|q)}\times\lim_{z\to\pi-x}\frac{z-\pi+x}{\theta_1(z+x|q)}\\
&= -\frac{f_1(\pi-x)}{\theta_1'(\pi|q)\theta_1(\pi-x|q)\theta_1(\pi-2x|q)\theta_1(\pi-x-y|q)\theta_1(\pi-x+y|q)}\\
&= \frac{f_1(-x)}{\theta_1'(q)\theta_1(x|q)\theta_1(2x|q)\theta_1(x-y|q)\theta_1(x+y|q)}. \qquad (3.6)
\end{aligned}$$

Similarly, we have

$$\operatorname{res}(E;y) = -\frac{f_1(y)}{\theta_1'(q)\theta_1(y|q)\theta_1(2y|q)\theta_1(x-y|q)\theta_1(x+y|q)}, \qquad (3.7)$$

$$\operatorname{res}(E;\pi-y) = -\frac{f_1(-y)}{\theta_1'(q)\theta_1(y|q)\theta_1(2y|q)\theta_1(x-y|q)\theta_1(x+y|q)}. \qquad (3.8)$$

Substituting (3.5), (3.6), (3.7) and (3.8) into (3.3) we obtain

$$\frac{f_1(y)+f_1(-y)}{f_1(0)\theta_1(y|q)\theta_1(2y|q)} - \frac{f_1(x)+f_1(-x)}{f_1(0)\theta_1(x|q)\theta_1(2x|q)} = \frac{\theta_1(x+y|q)\theta_1(x-y|q)}{\theta_1^2(x|q)\theta_1^2(y|q)}. \qquad (3.9)$$

In the same way we can obtain the identity for $f_2(z)$:

$$\frac{f_2(y)+f_2(-y)}{f_2(0)\theta_1(y|q)\theta_1(2y|q)} - \frac{f_2(x)+f_2(-x)}{f_2(0)\theta_1(x|q)\theta_1(2x|q)} = \frac{\theta_1(x+y|q)\theta_1(x-y|q)}{\theta_1^2(x|q)\theta_1^2(y|q)}. \qquad (3.10)$$

By comparing (3.9) and (3.10), we obtain

$$\begin{aligned}
&\frac{f_1(x)+f_1(-x)}{f_1(0)\theta_1(x|q)\theta_1(2x|q)} - \frac{f_2(x)+f_2(-x)}{f_2(0)\theta_1(x|q)\theta_1(2x|q)}\\
&= \frac{f_1(y)+f_1(-y)}{f_1(0)\theta_1(y|q)\theta_1(2y|q)} - \frac{f_2(y)+f_2(-y)}{f_2(0)\theta_1(y|q)\theta_1(2y|q)}. \qquad (3.11)
\end{aligned}$$

From this identity we readily find that

$$\frac{f_1(x)+f_1(-x)}{f_1(0)\theta_1(x|q)\theta_1(2x|q)} - \frac{f_2(x)+f_2(-x)}{f_2(0)\theta_1(x|q)\theta_1(2x|q)}$$

is independent of x, so it must be a constant, say $\frac{C(q)}{f_1(q)f_2(q)}$. This completes the proof of Theorem 1. By analytic continuation, we know (3.2) holds for all x.

Identity (3.2) is a very general identity involving theta functions. In the next sections we will choose some special functions $f_1(z)$ and $f_2(z)$ to obtain some interesting identities for theta functions.

4. The Proofs of (1.3), (1.4), and (1.5)

Using (2.31), it is easy to verify that $f_1(z) = e^{2iz}\theta_1(5z + \pi\tau|q^5)$ and $f_2(z) = e^{4iz}\theta_1(5z + 2\pi\tau|q^5)$ satisfy all the conditions of Theorem 1. Taking $f_1(z) = e^{2iz}\theta_1(5z + \pi\tau|q^5)$ and $f_2(z) = e^{2iz}\theta_1(5z + 2\pi\tau|q^5)$ in Theorem 1, we find that

$$\begin{aligned} C(q)\theta_1(x|q)\theta_1(2x|q) = \theta_1(2\pi\tau|q^5)\left\{e^{2ix}\theta_1(5x+\pi\tau|q^5) - e^{-2ix}\theta_1(5x-\pi\tau|q^5)\right\} \\ -\theta_1(\pi\tau|q^5)\left\{e^{4ix}\theta_1(5x+2\pi\tau|q^5) - e^{-4ix}\theta_1(5x-2\pi\tau|q^5)\right\}. \end{aligned} \tag{4.1}$$

To determine $C(q)$, we take $x = \frac{\pi}{5}$ in (4.1). After some simple calculations we find that

$$C(q)\theta_1(\frac{\pi}{5}|q)\theta_1(\frac{2\pi}{5}|q) = 2i(\cos\frac{4\pi}{5} - \cos\frac{2\pi}{5})\theta_1(\pi\tau|q^5)\theta_1(2\pi\tau|q^5). \tag{4.2}$$

Substituting (2.26) and (2.29) into the above identity we find that

$$C(q) = \frac{2}{\sqrt{5}}i(\cos\frac{2\pi}{5} - \cos\frac{4\pi}{5})q^{-\frac{1}{2}} = iq^{-\frac{1}{2}}. \tag{4.3}$$

Using (4.3) in (4.1) we obtain (1.4).

Replacing q by q^5, setting $x = \pi\tau$, and using (2.25) we obtain (1.5).

Differentiating both sides of (1.4) twice with respect to x and setting $x = 0$, we find that

$$\begin{aligned} 12 + 20i\left\{\frac{\theta_1'}{\theta_1}(\pi\tau|q^5) - 2\frac{\theta_1'}{\theta_1}(2\pi\tau|q^5)\right\} + 25\left\{\frac{\theta_1''}{\theta_1}(\pi\tau|q^5) - \frac{\theta_1''}{\theta_1}(2\pi\tau|q^5)\right\} \\ = \frac{4q^{-\frac{1}{4}}\theta_1'(q)^2}{\theta_1(\pi\tau|q^5)\theta_1(2\pi\tau|q^5)}. \end{aligned} \tag{4.4}$$

Substituting (2.22), (2.23), and (2.26) into the above identity we obtain (1.3).

5. The Proofs of (1.7), (1.8), and (1.9)

Taking $f_1(z) = \theta_1(z + \frac{\pi}{5}|q^{\frac{1}{5}})$ and $f_2(z) = \theta_1(z + \frac{2\pi}{5}|q^{\frac{1}{5}})$ in Theorem 1, we find that

$$\begin{aligned} C(q)\theta_1(x|q)\theta_1(2x|q) = &\ \theta_1(\frac{2\pi}{5}|q^{\frac{1}{5}})\left\{\theta_1(x+\frac{\pi}{5}|q^{\frac{1}{5}}) - \theta_1(x-\frac{\pi}{5}|q^{\frac{1}{5}})\right\} \\ &-\theta_1(\frac{\pi}{5}|q^{\frac{1}{5}})\left\{\theta_1(x+\frac{2\pi}{5}|q^{\frac{1}{5}}) - \theta_1(x-\frac{2\pi}{5}|q^{\frac{1}{5}})\right\}. \end{aligned} \tag{5.1}$$

Replacing q by q^5 we obtain

$$\begin{aligned} C(q^5)\theta_1(x|q^5)\theta_1(2x|q^5) = &\ \theta_1(\frac{2\pi}{5}|q)\left\{\theta_1(x+\frac{\pi}{5}|q) - \theta_1(x-\frac{\pi}{5}|q)\right\} \\ &-\theta_1(\frac{\pi}{5}|q)\left\{\theta_1(x+\frac{2\pi}{5}|q) - \theta_1(x-\frac{2\pi}{5}|q)\right\}. \end{aligned} \tag{5.2}$$

To determine $C(q^5)$, we take $x = \pi\tau$ in (5.2). Using (2.31), after some simple calculations, we find that

$$C(q^5)\theta_1(\pi\tau|q^5)\theta_1(2\pi\tau|q^5) = 2q^{-\frac{1}{2}}(\cos\frac{4\pi}{5} - \cos\frac{2\pi}{5})\theta_1(\frac{\pi}{5}|q)\theta_1(\frac{2\pi}{5}|q). \tag{5.3}$$

Substituting (2.26) and (2.29) into the above identity we find that

$$C(q^5) = 2\sqrt{5}(\cos\frac{2\pi}{5} - \cos\frac{4\pi}{5})q^{-\frac{1}{2}} = 5. \tag{5.4}$$

Using (5.4) in (5.2) we obtain (1.7).

Setting $x = \frac{\pi}{5}$ in (1.7) and using (2.29) we obtain (1.8).

Differentiating both sides of (1.7) twice with respect to x and setting $x = 0$, we find that

$$\frac{\theta_1''}{\theta_1}\left(\frac{\pi}{5}\Big|q\right) - \frac{\theta_1''}{\theta_1}\left(\frac{2\pi}{5}\Big|q\right) = \frac{10\theta_1'(q^5)^2}{\theta_1(\frac{\pi}{5}|q)\theta_1(\frac{2\pi}{5}|q)}. \tag{5.5}$$

Substituting (2.11), (2.23), and (2.29) in the above identity we obtain (1.9)

6. The Proofs of (1.10) and (1.12)

Taking $f_1(z) = \theta_1^5(z+\frac{\pi}{5}|q)$ and $f_2(z) = \theta_1^5(z+\frac{2\pi}{5}|q)$ in Theorem 1, we find that

$$\begin{aligned} C(q)\theta_1(x|q)\theta_1(2x|q) = &\ \theta_1^5(\frac{2\pi}{5}|q)\left\{\theta_1^5(x+\frac{\pi}{5}|q) - \theta_1^5(x-\frac{\pi}{5}|q)\right\} \\ &-\theta_1^5(\frac{\pi}{5}|q)\left\{\theta_1^5(x+\frac{2\pi}{5}|q) - \theta_1^5(x-\frac{2\pi}{5}|q)\right\}. \end{aligned} \tag{6.1}$$

Differentiating both sides of (6.1) twice with respect to x and setting $x = 0$, we find that

$$\frac{\theta_1''}{\theta_1}\left(\frac{\pi}{5}\Big|q\right) - \frac{\theta_1''}{\theta_1}\left(\frac{2\pi}{5}\Big|q\right) + 4\left(\frac{\theta_1'}{\theta_1}\right)^2\left(\frac{\pi}{5}\Big|q\right) - 4\left(\frac{\theta_1'}{\theta_1}\right)^2\left(\frac{2\pi}{5}\Big|q\right) = \frac{2C(q)\theta_1'(q)^2}{5\theta_1^5(\frac{\pi}{5}|q)\theta_1^5(\frac{2\pi}{5}|q)}. \tag{6.2}$$

Using (2.7) we have

$$5\left\{\frac{\theta_1''}{\theta_1}\left(\frac{\pi}{5}\Big|q\right)-\frac{\theta_1''}{\theta_1}\left(\frac{2\pi}{5}\Big|q\right)\right\}-4\left\{\left(\frac{\theta_1'}{\theta_1}\right)'\left(\frac{\pi}{5}\Big|q\right)-\left(\frac{\theta_1'}{\theta_1}\right)'\left(\frac{2\pi}{5}\Big|q\right)\right\}=\frac{2C(q)\theta_1'(q)^2}{5\theta_1^5(\frac{\pi}{5}|q)\theta_1^5(\frac{2\pi}{5}|q)}. \quad (6.3)$$

Substituting (2.11), (2.12), (2.23) and (2.29) into the above equation we obtain

$$40\sqrt{5}\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2}-\frac{16}{\sqrt{5}}\left\{1-5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n}\right\}=\frac{8q^{-1}C(q)(q;q)_\infty}{125\sqrt{5}(q^5;q^5)_\infty^5}. \quad (6.4)$$

Substituting (1.3) and (1.8) into the above equation we obtain

$$C(q)=250q(q;q)_\infty^4(q^5;q^5)_\infty^4+3125q^2\frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^2}, \quad (6.5)$$

thereby completing the proof of (1.10). Taking $x=\frac{\pi}{5}$ in (1.10) and using (2.29) we obtain (1.12).

7. The Proofs of (1.14) and (1.16)

Taking $f_1(z)=e^{2iz}\theta_1^5(z+\frac{\pi\tau}{5}|q)$ and $f_2(z)=e^{4iz}\theta_1^5(z+\frac{2\pi\tau}{5}|q)$ in Theorem 1, we find that

$$\begin{aligned}C(q)\theta_1(x|q)\theta_1(2x|q)&=\theta_1^5(\frac{2\pi\tau}{5}|q)\left\{e^{2ix}\theta_1^5(x+\frac{\pi\tau}{5}|q)-e^{-2ix}\theta_1^5(x-\frac{\pi\tau}{5}|q)\right\}\\&-\theta_1^5(\frac{\pi\tau}{5}|q)\left\{e^{4ix}\theta_1^5(x+\frac{2\pi\tau}{5}|q)-e^{-4ix}\theta_1^5(x-\frac{2\pi\tau}{5}|q)\right\}. \quad (7.1)\end{aligned}$$

Replacing q by q^5, the above identity becomes

$$\begin{aligned}C(q^5)\theta_1(x|q^5)\theta_1(2x|q^5)&=\theta_1^5(2\pi\tau|q^5)\left\{e^{2ix}\theta_1^5(x+\pi\tau|q^5)-e^{-2ix}\theta_1^5(x-\pi\tau|q^5)\right\}\\&-\theta_1^5(\pi\tau|q^5)\left\{e^{4ix}\theta_1^5(x+2\pi\tau|q^5)-e^{-4ix}\theta_1^5(x-2\pi\tau|q^5)\right\}. \quad (7.2)\end{aligned}$$

Differentiating both sides of (7.2), twice, with respect to x, setting $x=0$, we find that

$$\begin{aligned}&12+20i\left\{\frac{\theta_1'}{\theta_1}(\pi\tau|q^5)-2\frac{\theta_1'}{\theta_1}(2\pi\tau|q^5)\right\}+5\left\{\frac{\theta_1''}{\theta_1}(\pi\tau|q^5)-\frac{\theta_1''}{\theta_1}(2\pi\tau|q^5)\right\}\\&+20\left\{\left(\frac{\theta_1'}{\theta_1}\right)^2(\pi\tau|q^5)-\left(\frac{\theta_1'}{\theta_1}\right)^2(2\pi\tau|q^5)\right\}=\frac{2C(q^5)\theta_1'(q^5)^2}{\theta_1^5(\pi\tau|q^5)\theta_1^5(2\pi\tau|q^5)}. \quad (7.3)\end{aligned}$$

Using (2.7) we have

$$\begin{aligned}&12+20i\left\{\frac{\theta_1'}{\theta_1}(\pi\tau|q^5)-2\frac{\theta_1'}{\theta_1}(2\pi\tau|q^5)\right\}+25\left\{\frac{\theta_1''}{\theta_1}(\pi\tau|q^5)-\frac{\theta_1''}{\theta_1}(2\pi\tau|q^5)\right\}\\&-20\left\{\left(\frac{\theta_1'}{\theta_1}\right)'(\pi\tau|q^5)-\left(\frac{\theta_1'}{\theta_1}\right)'(2\pi\tau|q^5)\right\}=\frac{2C(q^5)\theta_1'(q^5)^2}{\theta_1^5(\pi\tau|q^5)\theta_1^5(2\pi\tau|q^5)}. \quad (7.4)\end{aligned}$$

Substituting (2.15), (2.22), (2.23) and (2.26) into the above equation we obtain

$$-8\left\{1-5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n}\right\}-80\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2}=-\frac{8q^{\frac{5}{2}}C(q^5)(q^5;q^5)_\infty}{(q;q)_\infty^5}. \qquad (7.5)$$

Using (1.3) and (1.8) in the above equation we obtain

$$q^{\frac{5}{2}}C(q^5)=10q(q;q)_\infty^4(q^5;q^5)_\infty^4+q\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty}, \qquad (7.6)$$

thereby completing the proof of (1.14). Taking $x=\pi\tau$ in (1.14) and using (2.25) we obtain (1.16).

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