ASYMPTOTIC ORDER OF THE SQUARE-FREE PART OF N!

Kevin A. Broughan
Department of Mathematics, University of Waikato, Hamilton, New Zealand
kab@waikato.ac.nz

Received: 4/27/02, Revised: 7/26/02, Accepted: 7/26/02, Published: 7/29/02

Abstract

The asymptotic order of the logarithm of the square-free part of $n!$ is shown to be $(\log 2)n$ with error $O(\sqrt{n})$.

1. Introduction

If the standard prime factorization of $n!$ is considered over a range of values of n then a number of patterns are apparent:

$$\begin{aligned} 10! &= 2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \\ 20! &= 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \\ 30! &= 2^{26} \cdot 3^{14} \cdot 5^7 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \\ 40! &= 2^{38} \cdot 3^{18} \cdot 5^9 \cdot 7^5 \cdot 11^3 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37. \end{aligned}$$

All the primes up to n appear. If p and q are primes appearing in the factorization with $p < q$ and α, β are the highest powers of p and q dividing $n!$ respectively, then $\alpha \geq \beta$, i.e. the smaller the prime, the larger the power. Even though sometimes a given power does not appear (the power 3 is missing from 20! even though the powers 2 and 4 appear), the power 1 always appears.

The square-free part of $n!$ is the number a, with no square factors, which appears in the factorization

$$n! = ab^2.$$

It is easy to see that a is exactly the product of each of the primes which appear to an odd power in the standard factorization, and in particular is divisible by the primes appearing to power 1 in that factorization.

Two natural questions arise: what is the size of the square-free part a of $n!$ and what proportion of a is the product of the primes which occur to power 1 ? In this note it will be shown that, asymptotically, the square-free part of $n!$ has order 2^n and that the proportion of primes to power 1 is about 72%.

2. Integer Square Roots

For each whole number n let the integer lower square root be defined by

$$r_-(n) = \prod_{p^\alpha || n} p^{\lfloor \frac{\alpha}{2} \rfloor}$$

and the integer upper square root by

$$r_+(n) = \prod_{p^\alpha || n} p^{\lceil \frac{\alpha}{2} \rceil}.$$

If $n = ab^2$ and $cn = d^2$ with a and c square-free, then

$$b = r_-(n), d = r_+(n), a = c = \frac{r_+(n)}{r_-(n)}.$$

This pair of functions $r_\pm$ is quite useful. They are multiplicative, can be generalized to integer k'th roots and are related to the integer conductor or square-free core. For examples and applications see [3, 4].

3. Computing the square-free part of n!

To obtain some idea of the behavior of the square-free part of $n!$, for large n, it pays to do some computations. However, for numbers of quite small size, say $n = 400$, $n!$ is a number with over 800 digits, so finding the square-free part should not be attempted directly. The following strategy was adopted:

For each $n \geq 1$, let θ_n be the square-free part of $n + 1$, i.e.,

$$\theta_n = r_+(n+1)/r_-(n+1).$$

Because $a_{n+1}b_{n+1}^2 = (n+1)n! = (n+1)a_n b_n^2$ and $n + 1 = \theta_n c^2$ for some integer c, we have $\theta_n a_n b_n^2 = a_{n+1}b_{n+1}^2$.

If a prime $p \mid (\theta_n, a_n)$, then p occurs as a factor in both θ_n and a_n, so must occur to an odd power in both $n!$ and $n + 1$, and therefore to an even power in $(n + 1)!$. Hence it does not occur in a_{n+1}. If a prime occurs in just one of θ_n and a_n, then it must occur in a_{n+1}. This leads directly to the formula:

(1) $$a_{n+1} = \frac{a_n \theta_n}{(a_n, \theta_n)^2}.$$

Note that this formula can be used to evaluate the sequence (a_n) recursively, so the values of $\log a_n$ can be plotted, revealing a nice approximately linear dependence on n. See Figure 1.

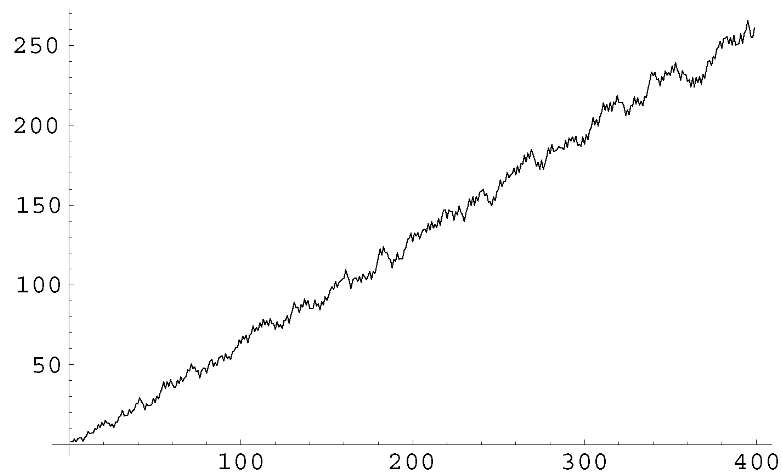

Figure 1. The sequence $\log a_n$ as a function of n.

4. Asymptotic orders

The result of these computations of the square-free part of $n!$ leads to two natural tasks: determining the slope of a line approximating the graph of $\log a_n$, and finding an upper bound for the error in this approximation. The completion of both tasks is summarized in the next theorem.

Theorem 1: For each $n \in \mathbb{N}$ let $n! = a_n b_n^2$ where $a_1 = b_1 = 1$ and where for all $n \geq 1$, a_n is square-free.

Then
$$\log a_n = n \log 2 + O(\sqrt{n}),$$
and
$$\log b_n = \frac{1}{2} n \log n - \frac{1 + \log 2}{2} n + O(\sqrt{n}).$$

Proof: Consider the central binomial coefficient $\binom{2n}{n} = t_n s_n^2$ where t_n is square-free. Then
$$b_{2n}^2 a_{2n} = (2n)! = (n!)^2 s_n^2 t_n$$
so $t_n = a_{2n}$ for all $n \in N$. By the main result in [7], there is a real strictly positive constant c such that for all $\epsilon > 0$ and all n sufficiently large
$$(c - \epsilon)\sqrt{n} < 2 \log s_n < (c + \epsilon)\sqrt{n}.$$
Therefore $\log s_n = O(\sqrt{n})$.

Stirling's approximation for $n!$ [8] is $n! \approx \sqrt{2\pi n}(n/e)^n$. It leads to the formula:

$$\log n! = n\log n - n + O(\log n).$$

Consequently:

$$\begin{aligned}\log a_{2n} &= \log\binom{2n}{n} - 2\log s_n \\ &= 2n\log 2n - 2n - 2n\log n + 2n + O(\sqrt{n}) \\ &= 2n\log 2 + O(\sqrt{n}).\end{aligned}$$

By equation (1)

$$\begin{aligned}\log a_{2n+1} &= \log a_{2n} + \log\theta_{2n} - 2\log(a_{2n},\theta_{2n}) \\ &= \log a_{2n} + O(\log n) \text{ since } \theta = O(n) \\ &= (2n+1)\log 2 + O(\sqrt{n})\end{aligned}$$

and therefore

$$\log a_n = n\log 2 + O(\sqrt{n}).$$

But, by Stirling's approximation again and this estimate for $\log a_n$:

$$\begin{aligned}2\log b_n &= n\log n - n - n\log 2 + O(\sqrt{n}) \\ &= n\log n - (1+\log 2)n + O(\sqrt{n})\end{aligned}$$

and therefore $\log b_n = \frac{1}{2}n\log n - \frac{1+\log 2}{2}n + O(\sqrt{n})$. This completes the proof of the theorem.

It follows also that the square-free part of $\binom{2n}{n}$, namely t_n, satisfies $\log t_n = 2n\log 2 + O(\sqrt{n})$, giving the asymptotic order. This relates to the solved conjecture of Erdős [5] that the binomial coefficient $\binom{2n}{n}$ is not square-free for $n > 4$. It relates also to the parity of the exponents of the prime factors of $n!$, [2].

5. Primes dividing n!

Lemma 1: Let $k \geq 1$ and let p be a prime integer. If $n \geq k(k+1)$ then $p^k \| n!$ if and only if $\frac{n}{k+1} < p \leq \frac{n}{k}$.

Proof: If $\frac{n}{k+1} < p \leq \frac{n}{k}$ then $k \leq \frac{n}{p} < k+1$, so therefore

$$k = \lfloor \frac{n}{p} \rfloor.$$

Since $k(k+1) \leq n$ we have $k \leq \frac{n}{k+1} < p$, so that

$$\lfloor \frac{n}{p^2} \rfloor < \frac{k+1}{p} \leq 1.$$

It follows that $\lfloor \frac{n}{p^2} \rfloor = 0$, by Legendre's formula

$$\alpha_p = \sum_{j=1}^{\infty} \lfloor \frac{n}{p^j} \rfloor = \lfloor \frac{n}{p} \rfloor = k.$$

Conversely, if $p^k \| n!$ then $k = \lfloor \frac{n}{p} \rfloor + \cdots$. Thus $\lfloor \frac{n}{p} \rfloor \leq k$, which implies $\frac{n}{k+1} < p$, so $k < p$. In addition $k < \frac{n}{k+1}$, therefore $\frac{n}{p^2} \leq \frac{k}{p} < 1$ so $\lfloor \frac{n}{p^2} \rfloor = 0$ and $k = \lfloor \frac{n}{p} \rfloor$, which shows $p \leq \frac{n}{k}$. This completes the proof of the lemma.

For $x > 0$ let

$$\theta(x) = \sum_{2 \leq p \leq x} \log p,$$

Chebyshev's function [1], where the sum is over all primes less than or equal to x. If $x \geq 563$ then $\theta(x)$ is close to x in that [6]

$$x\big(1 - \frac{1}{2\log x}\big) < \theta(x) < x\big(1 + \frac{1}{2\log x}\big).$$

If follows that if $n \geq n_k$

$$|\theta\big(\frac{n}{k}\big) - \theta\big(\frac{n}{k+1}\big) - \frac{n}{k(k+1)}| \leq \frac{n}{k \log \frac{n}{k}}.$$

By Lemma 1, the logarithm of the product of primes which appear in $n!$ to the k'th power is

$$\begin{aligned} \log \prod_{\frac{n}{k+1} < p \leq \frac{n}{k}} p &= \sum_{\frac{n}{k+1} < p \leq \frac{n}{k}} \log p \\ &= \theta\big(\frac{n}{k}\big) - \theta\big(\frac{n}{k+1}\big) \\ &= \frac{n}{k(k+1)} + O_k\big(\frac{n}{\log n}\big), \end{aligned}$$

so the asymptotic order of the product is $\frac{n}{k(k+1)}$ as $n \to \infty$.

Therefore, by Theorem 1, the asymptotic proportion of the square-free part of $n!$ due to primes appearing to powers $1, 3, \ldots, 2k-1$ is

$$\frac{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2k-1} - \frac{1}{2k}}{\log 2}.$$

For example, primes to power one contribute $\frac{1/2}{\log 2}$ or about 72%, and those to power one or three to $\frac{7/12}{\log 2}$, or about 84% of the square-free part.

Acknowledgments

This work was done in part while the author was on study leave at Columbia University. The support of the Department of Mathematics at Columbia University and the valuable discussions held with Patrick Gallagher are warmly acknowledged. The contributions of a referee who, in particular, supplied the general case for Lemma 1, are also gratefully acknowledged.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory,* New York, Berlin Heidelberg: Springer-Verlag, 1976.

[2] D. Berend, *On the parity of the exponents in the factorization of* $n!$*,* J. Number Theory **64** (1997), 13-19.

[3] K. A. Broughan, *Restricted divisor sums,* Acta Arithmetica, **101** (2002), 105-114.

[4] K. A. Broughan, *Relationships between the integer conductor and k'th root functions*, (preprint).

[5] A. Granville and O. Ramaré, *Explicit bounds on exponential sums and the scarcity of square-free binomial coefficients,* Mathematica **43** (1996), 73-107.

[6] J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers,* Illinois J. Math. **6** (1962), 64-94.

[7] A. Sárközy, *On divisors of binomial coefficients I*, J. Number Theory **20** (1985), 70-80.

[8] **http://mathworld.wolfram.com/StirlingsSeries.html**.