# ON A VARIATION OF THE COIN EXCHANGE PROBLEM FOR ARITHMETIC PROGRESSIONS

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**Abstract**

Let $a_1, a_2, \ldots, a_k$ be relatively prime, positive integers arranged in increasing order. Let $\Gamma^\star$ denote the positive integers in the set $\{\, a_1x_1 + a_2x_2 + \cdots + a_kx_k : x_j \geq 0 \,\}$. Let

$$\mathcal{S}^\star(a_1, a_2, \ldots, a_k) \doteq \{\, n \notin \Gamma^\star : n + \Gamma^\star \subseteq \Gamma^\star \,\}.$$

We determine $\mathcal{S}^\star(a_1, a_2, \ldots, a_k)$ in the case where the $a_j$'s are in *arithmetic progression*. In particular, this determines $g(a_1, a_2, \ldots, a_k)$ in this particular case.

## 1. Introduction

Let $a_1, a_2, \ldots, a_k$ be relatively prime, positive integers arranged in increasing order. Let $\Gamma$ denote $\{\, a_1x_1 + a_2x_2 + \cdots + a_kx_k : x_j \geq 0 \,\}$, and let $\Gamma^\star \doteq \Gamma \setminus \{0\}$. It is well known and easy to show that $\Gamma^c \doteq I\!N \setminus \Gamma$ is a *finite* set. We use the classical notation $g(a_1, a_2, \ldots, a_k)$ to denote the *largest* number in $\Gamma^c$. *J.J. Sylvester* [15] showed that $g(a_1, a_2) = a_1a_2 - a_1 - a_2$. In later years, the number of elements in $\Gamma^c$, denoted by $n(a_1, a_2, \ldots, a_k)$, was also studied, and it was shown that $n(a_1, a_2) = (a_1 - 1)(a_2 - 1)/2$. Another function related to this is the function $s(a_1, a_2, \ldots, a_k)$ that denotes the sum of elements in $\Gamma^c$. Introduced in [4], it was shown that $s(a_1, a_2) = (a_1 - 1)(a_2 - 1)(2a_1a_2 - a_1 - a_2 - 1)/12$.

There is a neat formula for each of the functions $g$ and $n$ when the $a_j$'s are in *arithmetic progression* ([1],[5],[9],[16]), but other results obtained are mostly partial results ([2],[3],[6],[7],[10],[11], [12],[13],[14]) and often not as neat. Due to an obvious connection with making change given money of different denominations, this problem is also known as the *Coin Exchange Problem*.

## 2. Main Result

We study a variation of the Coin Exchange Problem in this note. We denote by $\mathcal{S}^\star(a_1, a_2, \ldots, a_k)$ the set of all $n \in \Gamma^c$ such that

$$n + \Gamma^\star \subseteq \Gamma^\star,$$

and let $g^\star(a_1, a_2, \ldots, a_k)$ (respectively, $n^\star(a_1, a_2, \ldots, a_k)$ and $s^\star(a_1, a_2, \ldots, a_k)$) denote the *least* (respectively, the *number* and *sum* of) elements in $\mathcal{S}^\star$. Since $g(a_1, a_2, \ldots, a_k)$ is the *largest* element in $S^\star$,

$$g^\star(a_1, a_2, \ldots, a_k) \leq g(a_1, a_2, \ldots, a_k),$$

and $n^\star(a_1, a_2, \ldots, a_k) \geq 1$, with equality if and only if $g^\star = g$. This problem arises from looking at the generators for the Derivation modules of certain curves [8], and has been extensively studied.

For each $j$, $1 \leq j \leq a_1 - 1$, let $m_j$ denote the *least* number in $\Gamma$ congruent to $j$ (mod $a_1$). Then $m_j - a_1$ is the largest number in $\Gamma^c$ congruent to $j$ (mod $a_1$), and no number less than this in this residue class can be in $\mathcal{S}^\star$, for they would differ by a multiple of $a_1$, an element in $\Gamma^\star$. Therefore,

$$\mathcal{S}^\star(a_1, a_2, \ldots, a_k) \subseteq \{\, m_j - a_1 : 1 \leq j \leq a_1 - 1 \,\}, \tag{1}$$

$$g^\star(a_1, a_2, \ldots, a_k) \leq \left( \max_{1 \leq j \leq a_1 - 1} m_j \right) - a_1 = g(a_1, a_2, \ldots, a_k), \tag{2}$$

$$n^\star(a_1, a_2, \ldots, a_k) \leq a_1 - 1, \tag{3}$$

and

$$s^\star(a_1, a_2, \ldots, a_k) \leq \sum_{j=1}^{a_1 - 1} m_j - a_1(a_1 - 1). \tag{4}$$

More precisely,

$$m_j - a_1 \in \mathcal{S}^\star(a_1, a_2, \ldots, a_k) \iff (m_j - a_1) + m_i \geq m_{j+i} \ \text{ for } \ 1 \leq i \leq a_1 - 1. \tag{5}$$

We shall explicitly evaluate the set $\mathcal{S}^\star$, and as a consequence, the functions $g$, $g^\star$, $n^\star$ and $s^\star$, when the $a_j$'s are in *arithmetic progression*. We write $a_j = a + (j-1)d$ for $1 \leq j \leq k$, and assume $\gcd(a, d) = 1$. In this case, we denote the functions $g$, $g^\star$, $n^\star$ and $s^\star$ by $g(a, d; k)$, $g^\star(a, d; k)$, $n^\star(a, d; k)$ and $s^\star(a, d; k)$, respectively. To determine $\mathcal{S}^\star(a, d; k)$, we recall Lemma 2 from [16].

**Lemma:** For each $t$, $1 \leq t \leq a - 1$, the least integer in $\Gamma^\star$ congruent to $dt$ (mod $a$)

is given by $a(1+[\frac{t-1}{k-1}])+dt$.

**Theorem:** Let $a, d$ be relatively prime, positive integers, and let $k \geq 2$. If $a-1 = q(k-1)+r$, with $1 \leq r \leq k-1$, then

$$\mathcal{S}^\star(a,d;k) = \left\{ a\left[\frac{x-1}{k-1}\right] + dx : a-r \leq x \leq a-1 \right\}.$$

Proof: Fix $k \geq 2$. Throughout this proof, and elsewhere, by $x \bmod m$ we mean $x - x[\frac{x}{m}]$. By (1) and Lemma,

$$\mathcal{S}^\star(a,d;k) \subseteq \left\{ a\left[\frac{x-1}{k-1}\right] + dx : 1 \leq x \leq a-1 \right\}.$$

From (5), $n = a[\frac{x-1}{k-1}] + dx \in \mathcal{S}^\star$ if and only if for each $y$ with $1 \leq y \leq a-1$,

$$a\left(1+\left[\frac{((x+y) \bmod a)-1}{k-1}\right]\right) + d((x+y) \bmod a) \leq \left\{a\left[\frac{x-1}{k-1}\right]+dx\right\} + \left\{a\left(1+\left[\frac{y-1}{k-1}\right]\right)+dy\right\},$$

or,

$$a\left[\frac{((x+y) \bmod a)-1}{k-1}\right] + d((x+y) \bmod a) \leq a\left\{\left[\frac{x-1}{k-1}\right]+\left[\frac{y-1}{k-1}\right]\right\} + d(x+y). \quad (6)$$

Suppose $2 \leq k \leq a-1$. Let $a-1 = q(k-1)+r$, with $1 \leq r \leq k-1$. Unless $x = a-1$, $x+y \leq a-1$ for at least one $y$, for such a $y$, (6) reduces to proving the inequality

$$\left[\frac{x+y-1}{k-1}\right] \leq \left[\frac{x-1}{k-1}\right] + \left[\frac{y-1}{k-1}\right].$$

If we now write $x = q_1(k-1)+r_1$, $y = q_2(k-1)+r_2$, with $1 \leq r_1, r_2 \leq k-1$, the reduced inequality above fails to hold precisely when $r_1 + r_2 \geq k$. Given $x$, and hence $r_1$, the choice $y = r_2 = k - r_1$ will thus ensure that (6) fails to hold provided $x+y \leq a-1$. However, such a choice for $y$ is not possible precisely when $x \geq q(k-1)+1 = a-r$, so that (6) always holds in only these cases. Finally, it is easy to verify that (6) holds if $x = a-1$. This shows $\mathcal{S}^\star = \{\, a[\frac{x-1}{k-1}] + dx : a-r \leq x \leq a-1 \,\}$ if $2 \leq k \leq a-1$.

If $k \geq a$, (6) reduces to $d((x+y) \bmod a) \leq d(x+y)$. Thus, $\mathcal{S}^\star = \{\, dx : 1 \leq x \leq a-1 \,\}$, as claimed, since $r = a-1$ and $[\frac{x-1}{k-1}] = 0$ in this case. This completes the proof. □

**Corollary:** If $a, d$ be relatively prime, positive integers, $k \geq 2$, and $a-1 = q(k-1)+r$, with $1 \leq r \leq k-1$, then

$$g(a,d;k) = aq + d(a-1),$$

$$g^\star(a,d;k) = aq + d(a-r),$$
$$n^\star(a,d;k) = r,$$

and

$$s^\star(a,d;k) = aqr + \frac{1}{2}dr(2a-r-1).$$

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## References

[1] Bateman, P.T., Remark on a Recent Note on Linear Forms, *American Mathematical Monthly* **65** (1958), 517-518.

[2] Brauer, A., On a Problem of Partitions, *American Journal of Mathematics* **64** (1942), 299-312.

[3] Brauer, A. and Shockley, J.E., On a problem of Frobenius, *Crelle* **211** (1962), 215-220.

[4] Brown, T.C. and Shiue, P.J., A remark related to the Frobenius problem, *Fibonacci Quarterly*, **31** (1993), 31-36.

[5] Grant, D.D., On linear forms whose coefficients are in Arithmetic progression, *Israel Journal of Mathematics* **15** (1973), 204-209.

[6] Hofmeister, G.R., Zu einem Problem von Frobenius, *Norske Videnskabers Selskabs Skrifter* **5** (1966), 1-37.

[7] Nijenhuis, A. and Wilf, H.S., Representations of integers by linear forms in non negative integers, *Journal of Number Theory* **4** (1972), 98-106.

[8] Patil, D.P. and Singh, Balwant, Generators for the derivation modules and the relation ideals of certain curves, *Manuscripta Mathematica* **68** (1990), 327-335.

[9] Roberts, J.B., Note on Linear Forms, *Proceedings of the American Mathematical Society* **7** (1956), 465-469.

[10] Roberts, J.B., On a Diophantine problem, *Canadian Journal of Mathematics* **9** (1957), 219-222.

[11] Rödseth, Ö.J., On a linear Diophantine problem of Frobenius, *Crelle* **301** (1978), 171-178.

[12] Rödseth, Ö.J., On a linear Diophantine problem of Frobenius II, *Crelle* **307/308** (1979), 431-440.

[13] Selmer, E.S., On the linear Diophantine problem of Frobenius, *Crelle* **293/294** (1977), 1-17.

[14] Selmer, E.S. and Beyer, Ö., On the linear Diophantine problem of Frobenius in three variables, *Crelle* **301** (1978), 161-170.

[15] Sylvester, J.J., Mathematical questions with their solutions, *The Educational Times* **41** (1884), 21.

[16] Tripathi, A., The Coin Exchange Problem for Arithmetic Progressions, *American Mathematical Monthly* **101** (1994), no. 10, 779-781.