SYLVESTER: USHERING IN THE MODERN ERA OF RESEARCH ON ODD PERFECT NUMBERS

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Abstract

In 1888, James Joseph Sylvester (1814-1897) published a series of papers that he hoped would pave the way for a general proof of the nonexistence of an odd perfect number (OPN). Seemingly unaware that more than fifty years earlier Benjamin Peirce had proved that an odd perfect number must have at least four distinct prime divisors, Sylvester began his fundamental assault on the problem by establishing the same result. Later that same year, he strengthened his conclusion to five. These findings would help to mark the beginning of the modern era of research on odd perfect numbers. Sylvester's bound stood as the best demonstrated until Gradstein improved it by one in 1925. Today, we know that the number of distinct prime divisors that an odd perfect number can have is at least eight. This was demonstrated by Chein in 1979 in his doctoral thesis. However, he published nothing of it. A complete proof consisting of almost 200 manuscript pages was given independently by Hagis. An outline of it appeared in 1980.

What motivated Sylvester's sudden interest in odd perfect numbers? Moreover, we also ask what prompted this mathematician who was primarily noted for his work in algebra to periodically direct his attention to famous unsolved problems in number theory? The objective of this paper is to formulate a response to these questions, as well as to substantiate the assertion that much of the modern work done on the subject of odd perfect numbers has as it roots, the series of papers produced by Sylvester in 1888.

1. Introduction

A perfect number, first introduced in antiquity, is any positive integer that is equal to the sum of its proper divisors. Euclid (~ 300 B.C.) established that if $2^p - 1$ is prime

then the resulting integer $2^{p-1}(2^p - 1)$ is perfect. Twenty centuries later, Euler proved that all even perfect numbers are necessarily of Euclid's form. However, despite Euler's success in providing a defining characteristic for all even perfect numbers, we still do not know how many of them there are. Furthermore, whether an odd perfect number exists or not remains an unanswered question.

Having had its origin in mathematical thought more than 2000 years ago, perfect numbers did not generate a great deal of interest until the latter part of the 19th century. This point is underscored in [48, pg 93], when in reference to Euler's assertion that $2^{30}(2^{31} - 1)$ is perfect, Ore cites Peter Barlow's remark that it "is the greatest [perfect number] that will ever be discovered, for as they are merely curious without being useful, it is not likely that any person will attempt to find one beyond it." Time, of course, proved Barlow wrong. In 1876 Lucas demonstrated that $2^{126}(2^{127} - 1)$ is perfect. It was the tenth one to be discovered. Today, there are thirty-nine known even perfect numbers.

One of the goals of this paper is to develop the argument that the modern era of research on odd perfect numbers began with Sylvester.[1] To this end, we begin by stating that Euler established the first condition of existence when he proved that if n is an odd perfect number, then

$$n = p^{\alpha} q_1{}^{2\beta_1} q_2{}^{2\beta_2} \cdots q_k{}^{2\beta_k} \tag{1}$$

where, $p, q_1, q_2, \ldots, q_k$ are distinct odd primes and $p \equiv \alpha \equiv 1 \,(\mathrm{mod}\ 4)$. However, it was not until 1937 that Steuerwald extended this particular line of thought by showing that not all of the β_i's can equal one [61].

Studying existence criteria from a different perspective, Peirce provided the first known lower bound on the number of distinct prime divisors when in 1832 he proved that an OPN must have at least four such factors [51]. This important result seems to have been generally overlooked as even Dickson neglected to mention it in his magnum opus, *History of the Theory of Numbers*, [16].

More than a half century after Peirce's paper appeared, both Servais and Sylvester independently published proofs of the same theorem (See [60] and [67]). It is remarkable that Sylvester makes no mention of Peirce's ground-breaking discovery in any of his work for the two had known each other since at least the early 1840s. This is evidenced, for example, in [1], wherein Archibald shows a correspondence between Sylvester and Peirce that began shortly after the former departed Charlottesville in 1843. In [18, pg 18], it is also stated that Sylvester was a pall-bearer at Peirce's funeral in 1880.

Likely, this omission of credit is attributable to Sylvester's often demonstrated aversion for having been kept apprised of the works of others. For instance, in [20, pg 303],

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[1]In his doctoral dissertation [53], Pomerance credited Sylvester with beginning the modern work on OPNs.

Fabian Franklin, a former student and colleague of Sylvester's at the Johns Hopkins University, described [Sylvester's] *powers* as "being set in motion by two opposite kinds of stimulus; that of abundantly rewarding results, and that of the stubborn resistance of concentrated difficulty." He further adds, "that intermediate kind of effort which slowly and patiently builds up and improves and perfects one's own work, and which gives minute and prolonged study to the work of others, he did not command in any notable degree." Parshall also notes in [49, pg 40] that at "numerous times throughout his career, Sylvester found himself in the situation of having claimed as his own a previously published result."

Forthcoming, we shall describe the manner in which Sylvester's attention was called to odd perfect numbers. But first, we present a historical survey of the study of odd perfect numbers since the time of Peirce.

In contrast to the early part of the 1800s, interest in odd perfect numbers began to mount during the latter part of the 19th century with the general line of thought having mirrored that of Peirce. In addition, up to 1888 most of the work that was done on the subject appears to have been isolated efforts with the dissemination of findings sometimes missing those individuals that would turn out to be the key players. For instance, in 1863 Nocco proved that an odd perfect number is divisible by at least three distinct prime divisors. However, Servais, in [58] showed that an odd integer with either one or two prime factors cannot be perfect. The following year, Sylvester demonstrated the three factor case [66].

In 1888, both Sylvester and Servais independently published proofs that an odd perfect number is divisible by at least four different prime divisors (see [67] and [60]). Later that year, Sylvester advanced his bound to five [68]. He furthermore showed that no odd perfect number can be divisible by 105, as well as put into place a lower bound of eight distinct prime factors for an OPN not divisible by three [67]. Also in 1888, Servais, considering an odd perfect number with k distinct prime divisors established an upper bound of $k + 1$ on its least prime divisor [59]. Improvements or extensions of this result were later realized by Grün [23], Cohen and Hendy [13], and McDaniel [46].

In 1913 Dickson showed that for any k there can be only finitely many odd perfect numbers with exactly k components [17].[2] He proved this as a corollary to a similar result for odd primitive non-deficient numbers (which by definition must contain all the odd perfect numbers).[3] Thus, one may check for the existence of an odd perfect number with exactly k components by first delineating all of the finitely many primitive odd non-deficient numbers associated with that k-value and then determining which among them are equal to the sum of their proper divisors.

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[2]The number n in Eq(1), for example, has $k + 1$ *components*, namely, p^α, $q_1^{2\beta_1}$, ... , $q_k^{2\beta_k}$.

[3]Let n be any positive integer. Then $\sigma(n)$ is the sum of the positive divisors of n. A *deficient number* is any positive integer n such that $\sigma(n) < 2n$. Thus, n is *non-deficient* provided that $\sigma(n) \geq 2n$. Dickson called a number *primitive non-deficient* provided that it is not a multiple of a smaller non-deficient number.

Unfortunately, this approach is not feasible for most values of k. It is also questionable whether Dickson intended for his lists of primitive non-deficient numbers to be taken too seriously when searching for an odd perfect number. There is an extremely large number of them, say, with up to five distinct prime factors and they have never all been listed. Furthermore, Dickson's tables for those with up to four distinct prime factors contain many errors. An account of them is provided in [12].

In 1956, Dickson's theorem was generalized by Kanold to include any number n that satisfies $\sigma(n)/n = a/b$, where a, b are positive integers and $b \neq 0$ [40].

In 1949, Kanold [39] produced a proof of the same 4-fold result previously given by Peirce, Servais, Sylvester, and Dickson. However, the significance of his paper reached far beyond the stated result. Because approaching the OPN question from the standpoint of Dickson was considered impractical for most values of k, it therefore became necessary to seek out alternative methods for examining the possible structure of such a number. In 1974, Pomerance suggested the following class of theorems: *an OPN is divisible by j distinct primes $> N$* [53]. Interestingly enough, Kanold had demonstrated the case of $j = 1$ and $N = 60$ using elementary techniques in 1949 [39].

In 1973, by enlisting the use of computation, Hagis and McDaniel advanced Kanold's finding to $j = 1$ and $N = 11200$ [30]. Two years later, they improved N to 100110 [31]. The first proof for $j > 1$ was given by Pomerance in 1975 when he showed $j = 2$ and $N = 138$ [54]. In a paper that has recently appeared, Jenkins reports that the largest prime divisor of an odd perfect number exceeds 10 million [37]. It betters the previous bound of 1 million established by Hagis and Cohen in 1998 [27]. New estimates on the second and third largest prime divisors of an OPN were given by Iannucci in 1999 when he announced that they are greater than than 10000 and 100, respectively (See [34] and [35]).

The investigation of odd perfect numbers has also included several attempts at imposing a lower bound on its overall magnitude. In 1908, Turcaninov proved that an odd perfect number is necessary larger than $2 \cdot 10^6$. More recently, and by integrating the use of computers, Brent, Cohen, and te Riele tell us that it is greater than 10^{300} [4].

In terms of an upper bound on its overall size, Heath-Brown had shown in 1994 that if n is an odd number with $\sigma(n) = an$, then $n < (4d)^{4^k}$, where d is the denominator in a and k is equal to the number of distinct prime factors of n [33]. Specifically, for an OPN this means that $n < 4^{4^k}$ and it sharpened the previous estimate of $n < (4k)^{(4k)^{2^{k^2}}}$ given by Pomerance in 1977 [55]. In reference to his own result, Heath-Brown has noted that it is still too large to be of practical value. Nevertheless, we point out that if it is viewed in conjunction with the lower bound of 10^{300} provided by Brent et. al., then Sylvester's theorem that every odd perfect number has at least five distinct prime factors can be demonstrated by a footnote.[4] In 1999, Cook improved Heath-Brown's result for

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[4]$10^{300} < n < 4^{4^k}$ implies that $k > 4.48$.

an OPN with k components to $n < (2.124)^{4^k}$ [15]. In 2003, Nielsen refined Cook's bound to $n < 2^{4^k}$ [47]

A lower bound of 10^{20} on the largest component of an odd perfect number was established by Cohen in 1987 [11].

Addressing the OPN question from a different perspective, Steuerwald's analysis of allowable exponents continued in 1941 when Kanold discovered that it is neither possible for all of the β_i's to equal two nor for one of the β_i to be equal to two while all the rest are equal to one [38]. In 1972, Hagis and McDaniel determined that not all of the β_i's can be equal to three [29]. In 1985, Cohen and Williams eliminated possibilities for the β_i, assuming that either some or all of the β_i are the same. They also provided a summary of all previous work done in this area [14].

In a recently published paper, Iannucci and Sorli restrict the β_i in order to show that an odd perfect number cannot be divisible by three if, for all i, $\beta_i \equiv 1 \,(\mathrm{mod}\ 3)$ or $\beta_i \equiv 2 \,(\mathrm{mod}\ 5)$ [36]. In that article, they also provided a slightly different analysis by giving a lower bound of 37 on the total number of prime factors (counting multiplicities) that an odd perfect number can have.

We now summarize some of the immediate extensions of Sylvester's 1888 work on OPNs.

First, we note that unlike those that had studied the odd perfect number question before him, Sylvester seems to have sought to spotlight attention on both the problem itself, as well as on his own work on it. For instance, before he proves that an OPN cannot have less than four distinct prime divisors [66], he declares that "I am going to demonstrate that such a number does not exist, by means of a form of reasoning with which I have also provided a demonstration of the theorem that there does not exist a perfect number which contains fewer than six distinct prime factors." However, his proof was incorrect.

It was not until 1925 that Gradstein offered the first correct demonstration of the six-component case [22]. In 1949, and then in 1951, Kühnel [44] and Webber [75] independently published their own versions of this result. It would, however, take almost fifty years years from the appearance of Gradstein's paper for the bound of six to be improved. In 1972, Pomerance and Robbins independently showed that an odd perfect number has at least seven different prime divisors (see [53] and [57]). Several years later, in his Ph.D. thesis [7], Chein proved that an OPN has at least eight such prime factors but he published nothing of the result. In 1980, Hagis published an outline of a proof that consisted of almost 200 pages [25]. Hagis has also stated his belief that an extension to nine distinct prime factors is possible but that it would require an prohibitive amount of effort and computer time.

In terms of the distinct prime divisors of an odd perfect number not divisible by three,

the first improvement over Sylvester's lower bound of eight came in 1957 when Kanold proved that there must be at least nine such factors [41]. The best result known today is eleven. It was produced independently in 1983 by both Kishore [42] and Hagis [28].

In retrospect, we note that it was Euler who rendered the first significant finding pertaining to the structure of an odd perfect number. He may also very well have been the first to comment on the caliber of the problem when in [19, pg 355], he remarked that "whether ... there are any odd perfect numbers is a most difficult question." More than one hundred years later, Sylvester echoed the same sentiment in [66], albeit more descriptively, when he announced that " ... a prolonged meditation on the subject has satisfied me that the existence of any one such — its escape, so to say, from the complex web of conditions which hem it in on all sides — would be little short of a miracle. Thus then there seems to be every reason to believe that Euclid's perfect numbers are the only perfect numbers which exist!" Having thus proclaimed the problem's worthiness, as well as its ancient roots, Sylvester also declared in that paper that he had "found a method for determining what (if any) odd perfect numbers exist of any specified order of manifoldness."

Perhaps it was exactly this sort of exposition that Sylvester was so often inclined to include in his work that served to create a greater awareness of the odd perfect number problem. It may also be possible that the wider readership of France's *Comptes Rendus*, over say, the limited purview of the *New York Mathematical Diary* had also helped to disseminate some of Sylvester's 1888 findings on OPNs more effectively than Peirce was able to do in 1832. In any case, Sylvester deserves the credit for having drawn enduring attention to the problem, as well as for having approached the problem in a manner, which with some variations, is still manifest today.

While the wait continues for Sylvester's conjecture that an odd perfect number does not exist to reach its definitive end, we paraphrase Guy [24, pg 45] when we say that perhaps it is due to frustration over not being able to settle the existence question of an odd perfect number that mathematicians have introduced many similarly defined numerical concepts, as well as many problems associated with them to study. Alas, most of them as Guy has suggested appear to be no more tractable than the original.

For example, we may define a *multiperfect* number to be a positive integer n, where $\sigma(n) = kn$ and $k \geq 1$. Such numbers include all the perfect numbers, as well as the number 1 (which is the only known odd example). However, a nontrivial odd specimen or a proof of its nonexistence has yet to be found.

In spite of this, theory has been developed for odd multiperfect numbers which in many ways has reflected the spirit of approach that has come to characterize the study of OPNs. For example, in 1906, Carmichael showed that an odd "multiply perfect" number has at least four distinct prime divisors [6]. In 1987, Kishore proved that if n is odd and $k = 3$ then odd *triperfect* numbers must have at least twelve distinct prime factors [43]. An alternative demonstration of this result was later given by Hagis [26] with the first

proof of its kind having come in 1983 at the hand of Reidlinger [56]. We remark that in 2001, a multiperfect number with index $k = 11$ was found. Although even, it was a noteworthy discovery. As recently as early 1992 no such number with $k > 8$ was known.

We now wish to offer a few comments on *amicable* numbers, which are represented by pairs of positive integers, n and m, satisfying $\sigma(m) = m + n = \sigma(n)$. It follows that a number is perfect if and only if it is amicable with itself. There are over four million known pairs of amicable numbers [21]. However, the question, say, of their infinitude remains an unanswered one. Furthermore, although odd pairs of such numbers do exist (e.g. 12285 and 14595) we do not know if there are any of opposite parity. Also open is the question of whether or not there can be a relatively prime pair of amicable numbers. An interesting speculation by Wall [74, pg 68] asks, "is it true that odd amicable numbers are always incongruent modulo 4?" An affirmative answer, says Wall, would imply the nonexistence of an odd perfect number.[5]

Another example of a number connected by definition to a perfect number is a *quasi-perfect* number; that is, any positive integer n such that $\sigma(n) = 2n + 1$. Although we know that such numbers must be odd squares, we do not know if there are any [24, pg 45].

On the other hand, a numerical curiosity that has proved rather easy to find is any positive integer equal to the sum of some of its divisors. Among other things, such numbers have been called *pseudoperfect*. However, any integer that is abundant[6] but not pseudoperfect has been labeled as *weird*. We have yet to learn if an odd weird number exists.

Before we conclude, we would be remiss if we did not provide some opinion contrary to the assertion that the modern interest in OPNs began with Sylvester. For this purpose, we cite McCarthy, who in 1957 credited the modern revival of interest in the odd perfect number question to Steuerwald [45].

McCarthy, after having made reference to Sylvester's lower bounds on the number of distinct prime factors of an odd perfect number (considering both the case when the OPN is divisible by 3, as well as when it is not), declares in [45] that, "after this, progress in the [odd perfect number] problem was at a relative standstill until recent times." Upon introducing cyclotomic polynomials and referring the reader back to discussions on them in secondary sources dated from 1950, he then goes on to credit Steuerwald with initiating the modern revival of interest in odd perfect numbers.

It would appear that in making his statement, McCarthy failed to recognize certain

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[5]This conjecture also appears in [24, pg 57]. We remark that [24, pg 56] has given te Riele's 33-digit example of an odd amicable pair not divisible by 3. Although the illustration would appear to answer Wall's question in the negative, we have learned that there is a misprint. Therefore, Wall's conjecture remains alive. Our thanks goes to Dr. Iannucci for an astute observation, as well as for his follow-up inquiry to the author of [24].

[6]n is *abundant* if $\sigma(n) > 2n$.

significant achievements that occurred during the first half of the twentieth century. Some examples include Gradstein's 1925 direct extension of Sylvester's 1888 lower bound of five on the number of distinct prime divisors of an OPN, as well as Dickson's 1913 theorem stating that there can be only a finite number of odd perfect numbers having a given number of distinct prime factors. In fact, Sylvester may have actually anticipated the result that we now ascribe to Dickson when in [66], he remarked that, "I have found a method for determining what (if any) odd perfect numbers exist of any specified order of manifoldness." Moreover, it was Dickson's paper that ultimately motivated attempts in the 1970s at placing lower bounds on the magnitudes of the prime divisors of an OPN. The first result of this kind was, in fact, realized by Kanold in 1949 prior to McCarthy having made his statement. Also not considered was Turcaninov's 1908 lower bound on the magnitude of an odd perfect number which has led to today's best estimate of 10^{300}.

We recognize that there are indeed some modern efforts that reflect the ideas advanced by Steuerwald. Nevertheless, it is our contention that the overall focus of today's present study of odd perfect numbers is more closely related to the original work of Sylvester, than say, to that of any other pioneer in the field.

2. Sylvester's Motivation for Studying Odd Perfect Numbers

We shall begin this section by stating that Sylvester's study of odd perfect numbers appears to have been quite sudden and not related in any obvious way to his research immediately prior.

We also acknowledge that the term *motivation* may be a multifaceted one insomuch as it applies to ascertaining Sylvester's reason for embarking on the odd perfect number problem. In one sense, motivation may suggest the historical occurrence that has led to the endeavor; that is, the specific event that put the question in Sylvester's mind. To this end, we shall argue that the subject of odd perfect numbers was initially brought to his attention by Mr. Robert W. D. Christie.

In a second sense, motivation may also denote the place that Sylvester thought the problem occupied in the broader scheme of mathematics, as well as in intellectual history as a whole. More specifically, what did he see himself as doing if he were able to solve this problem? In [66], Sylvester notes that the odd perfect number question is an ancient problem possessing roots in classical Greece. While Euclid's work on perfect numbers had been known for centuries, Sylvester seems to have sought to give the topic increased significance by also referring it back to the works of Aristotle, Plato, and Pythagoras.

Thirdly, we may also interpret motivation as a manifestation of a particular quality in Sylvester's disposition that may have caused him to be interested in exactly this sort of a problem once he had entertained its notion. We shall suggest that Sylvester's interest

in the question of odd perfect numbers, as well as in other famous questions in number theory was part of a pattern he exhibited during unsettled periods in his professional life. Three such periods will be identified that were brought on, respectively, by the occurrence of the following events:

1. An abrupt voluntary departure from the University of Virginia (1842)

2. A forced retirement from the Royal Military Academy, Woolwich (1870)

3. An inauspicious resignation from the Johns Hopkins University in order to accept the Savilian Chair of Geometry at Oxford University (1883)

In order to more fully elucidate our arguments, we shall partition our discussion into three subsections, each addressing one of the previously stated questions of motivation.

At the Urging of Mr. Christie

The principle source for answering the first question of motivation (i.e., the event that first put the question in Sylvester's mind), is a paper that Sylvester published in *Nature* entitled, *Note on a Proposed Addition to the Vocabulary of Ordinary Arithmetic* [66]. In a footnote contained therein, Sylvester reveals that "my particular attention was called to perfect numbers by a letter from Mr. Christie, dated from 'Carlton Selby,' containing some inquiries relative to the subject." Unfortunately, there were several gentlemen with the surname of Christie whom Sylvester was likely to have been acquainted.

Since the first name of Christie is not disclosed, perhaps Sylvester felt that the identity of this individual would be obvious to his readers. Perhaps also, he may not have considered it a very important detail. However, the proper identification of this person will help to produce an answer as to how and why a specific individual would have been able to influence the research direction of Sylvester during a professionally isolated time.

A difficulty that arises in making this determination is that other than the commentaries provided by Sylvester on the subject in his published articles, details that directly link his interest to odd perfect numbers are scarce. A perusal of both [1] and [49] furthermore reveals no correspondence of Sylvester's that specifically addresses the subject of OPNs, save a possible allusion to them made in letter dated Feb. 26, 1888 to Daniel Coit Gilman, President of the Johns Hopkins University [49, pg 269]. In particular, Sylvester writes that "I have not been quite idle since my accident occurred and have recently fired off a few papers for Nature and the Comptes Rendus." Other than this, we are aware of no other extant communique of Sylvester that makes even the faintest reference to an odd perfect number.

Nonetheless, we have been able to infer that from among the several Christies that Sylvester had either met or likely corresponded with, there is one that appears to have

had a demonstrated connection to number theory, perfect numbers, and Sylvester in the vicinity of 1888. It is thus our contention that the identity of the person to whom Sylvester refers in [66] is Mr. *Robert William Dougall Christie.*

In order to substantiate our assertion, we begin by noting that this Christie was elected to the London Mathematical Society in 1888, a time in which Sylvester was one of its distinguished members. We furthermore add that this individual had been cited on several occasions in Dickson's, *History of the Theory of Numbers*, [16] for work done on perfect numbers, as well as in other areas of number theory. Moreover, this Christie was also a frequent contributor to the *Mathematical Questions* column of the *Educational Times and Journal of the College of Preceptors*, a periodical that published reader's solutions to questions posed by prominent mathematicians.[7] In fact, in [9] Christie had posed the question: "Show that the tenth perfect number is $P_{10} = 2^{40}(2^{41} - 1) = 2,417,851,639,228,158,837,784,576$." Although there is an apparent oversight as $2^{41} - 1 = 13367 \cdot 164511353$ is composite, this problem nevertheless serves to establish Christie's interest in perfect numbers around the time of 1888.[8]

An even stronger piece of evidence is found in [8], *A Note on Perfect Numbers*, that Christie contributed to the *Mathematical Questions* in 1888. It began with, "these numbers [OPNs] have engaged the attention of mathematicians from very early times, but there are still several problems connected with them requiring a solution before the subject can be said to be fully elucidated." Also included is an excerpt from a letter written by Descartes to an undisclosed individual on December 20, 1638. Upon translation, it reads, " ... and I don't know why you judge that this means the invention of a true [odd] perfect number will not be successful; if you have a demonstration, I hold that it is beyond my capability and I would hold it in extremely high regard; as for me, I judge that one can find real odd perfect numbers." This citation thus provides a link between Christie and the existence question of an *odd* perfect number.

Perhaps the strongest evidence of all is obtained from the paper, "A Theorem in Combinations", [10], which Christie published in the 1889 volume of the *Proceedings of the London Mathematical Society*. In particular, it concerns Euler's ϕ-function. This is exactly the place that Sylvester began his discussion in [66] on OPNs. Moreover, excerpts from the footnote that Sylvester included just before his mention of Christie states, "Euler's function $\phi(n)$, which means the number of numbers not exceeding n and prime to it, I call the *totient* of n ... I am in the habit of representing the totient of n by the symbol τn, τ (taken from the initial of the word it denotes) being a less hackneyed letter than Euler's ϕ, which has no claim to preference over any other letter of the Greek alphabet, but rather the reverse." Hence, we see Sylvester providing here an explanation

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[7]Sylvester contributed problems to every volume of this journal from its inception up to and including the seventieth.

[8]A reader, in fact, did respond with a "solution" to the stated problem citing that "The tenth perfect number is given by Mr. Carvallo in his work 'Theory of Perfect Numbers'". The reader then proceeded to give the "proof". We add that Dickson separately points out in [16, pgs 22 and 24] that Carvallo had twice erroneously announced having had a proof of the nonexistence of an odd perfect number.

of the nomenclature of the topic of Christie's paper, immediately prior to acknowledging him for bringing to his attention some inquiries related to the subject of odd perfect numbers.

Sylvester's Invocation of Ancient Luminaries

The second question of Sylvester's motivation asks, "What did Sylvester see himself as doing if he were to solve this problem?" That is, where exactly did Sylvester see this problem fitting into the broader picture of number theory, or indeed, into intellectual history as a whole? Again, we get a picture of this in [66], where in the second footnote, the issue of perfect numbers is taken up more generally.

After mentioning the inquiry of Christie, Sylvester then defines a perfect number and praises Euclid's ingenuity for showing in the ninth book of the *Elements* that a number m is even perfect provided that $m = 2^n(2^{n+1} - 1)$, where $2^{n+1} - 1$ is prime. Sylvester proceeds to give a brief sketch of Euler's proof of the converse of Euclid's theorem, upon which he offers the comment, "It is remarkable that Euler makes no reference to Euclid in proving his own theorem. It must always stand to the credit of the Greek geometers that they discovered a class of perfect numbers which in all probability are the only numbers which are perfect." Having thus duly chastised Euler for not citing his Greek predecessors, Sylvester seeks to imbed the problem in its full pre-Euclidean context when he adds, "Reference is made to *so-called* perfect numbers in Plato's 'Republic' H, 546B, and also by Aristotle, 'Probl.' I E 3 and 'Metaph.' A5."

We now qualify Sylvester's references by noting that in all cases, the term "perfect" [$\tau\epsilon\lambda\iota o\nu$] does modify "number" but not in the Euclidean sense. For example, in the cited sections of both the *Problemata* [3] and *Metaphysics* [2], Aristotle considers what he understands to be the claim of the number ten as "perfect"; that is, one which plays the role of a container for all categories of number in the Pythagorean numerological metaphysic. Also, in the passage from book VIII of the *Republic* [52], Plato derives the so-called nuptial number. He first speaks of a cycle that is comprehended by a "perfect number" that governs the birth of divine creatures and then proceeds to give a numerological derivation of the period of this cycle for humans. Thus, in none of the citations provided by Sylvester does it appear that the term "perfect" is used in its Euclidean context.

A possible explanation as to why Sylvester may have chosen to include them is found in [20]. From this first-hand account of Franklin, we learn that Sylvester had a special interest in problems that were traceable back to the "great masters". In particular, Franklin recounts that "any crucial problem, especially one that was associated with the name of one of the great masters, if once it attracted Sylvester's attention, fastened itself upon his mind with a grip that seemed never to slacken its tenacity."

Unfortunate Circumstances and Celebrated Problems in Number Theory

Primarily noted for his work in algebra, Sylvester occasionally demonstrated an affinity for problems in number theory. In fact, he engaged in several such efforts during the early 1860s while teaching at the Royal Military Academy in Woolwich. Later, he generated a bevy of results while directing the graduate program in mathematics at Johns Hopkins. However, Sylvester's pursuit of longstanding, notably difficult unsolved problems in the field appears to have been isolated to short-lived efforts conducted during unsettled points in his career.

By 1888, the year in which Sylvester published his entire collection of papers on odd perfect numbers, we find the seventy-three year old mathematician coping with the predominantly teaching-oriented duties of the Savilian Chair of Geometry at Oxford. Having recently given up a satisfying position at the newly founded Johns Hopkins University where his most recent discoveries often became the subject of his next graduate lecture, Sylvester arrived at Oxford in 1884 ready to teach a predominantly undergraduate student population primarily interested in only doing well on the university examinations. Not surprisingly, the students were generally not very receptive to his unconventional approach to pedagogy.

To provide some insight as to how Sylvester's teaching techniques may have clashed with the undergraduate environment at Oxford, we cite the following testimony given by E. W. Davis, a student of Sylvester's at Hopkins: [9] "Sylvester's Methods! He had none. 'Three lectures will be delivered on a New Universal Algebra,' he would say; then, 'The course must be extended to twelve.' It did last all the rest of the year. The following year the course was to be *Substitutions-Theorie*, by Netto. We all got the text. He lectured about three times, following the text closely and stopping sharp at the end of the hour. Then he began to think about matrices again. 'I must give one lecture a week on those,' he said. He could not confine himself to the hour, nor to the one lecture a week. Two weeks were passed, and Netto was forgotten entirely and never mentioned again." A historian's critique of his classroom style is also found in [50, pg 81] where it is concluded that "The Englishman was completely incapable of satisfying the student who wished to take away a notebook containing a crystallization of some mathematical topic."

Although Sylvester had proved successful from time to time in attracting certain individuals at Oxford to take an interest in his scholarly pursuits, the highly charged research environment that Sylvester had helped to cultivate in Baltimore was conspicuously absent at his new academic home. It was in part because of this that Sylvester lamented his frustration on March 11, 1887 to Gilman and inquired about his possible return to the Johns Hopkins [49, pg 264]: "*Entre nous* this University except as a school of taste and elegant light literature is a magnificent sham. It seems to me that Mathematical

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[9]Other examples can also be found in [5]

science is doomed and must eventually fall off like a withered branch from a Tree which derives no nutriment from its roots. ... I am out of heart in regard of my Professorial work at this University in which all the real powers of influencing the studies of the place lies in the hands of the College Tutors ... The mathematical school here is at a very low ebb and the number of Mathematical students here continually diminishing ... It depends exclusively on the Tutors whether a Professor can get undergraduates to attend his lectures ... Under these circumstances I am induced to inquire from you whether you think that there is an opening for me to return to the Johns Hopkins ... I chafe under the sense of enforced inactivity at a time when notwithstanding the weight of my years I feel in the plentitude of my powers physical and intellectual."

Unfortunately, another voyage across the Atlantic never came to pass.

It was under these inauspicious circumstances that the aged Sylvester began his fundamental assault on the existence question of an odd perfect number. This was not unlike two other earlier periods in his life when a significant career disruption shifted his thoughts to problems related to either Fermat's Last Theorem or Goldbach's conjecture.

We shall now briefly describe the three periods of obscurity that we alluded to earlier, for they all led to Sylvester taking an active interest in a "great master" problem in number theory.

Post-Virginia

Forty-one years before his findings on odd perfect numbers were published, Sylvester wrote three papers [62], [63], and [64] related to the cubic diophantine equation $Ax^3 + By^3 + Cz^3 = Dxyz$. In the first of these [62, pg 189], he remarked that "I venture to flatter myself that as opening out a new field in connexion with Fermat's renowned Last Theorem, and as breaking new ground in the solution of equations of the third degree, these results will be generally allowed to constitute an important and substantial accession to our knowledge of the Theory of Numbers". This trilogy of papers appeared at a time when Sylvester was attempting to regain his footing as a mathematician. Due to an unpleasant and short-lived experience at the University of Virginia, it had been several years since he had been active in mathematics.[10]

In [49, pg 3], Parshall notes that "the years from 1842 to 1847 had been mathematically barren but by 1847, number theory, and in particular, a problem no less formidable than Fermat's Last Theorem, had begun to refocus his mathematical energies." She adds on page 19, that "in hunting for mathematical research problems in 1847, Sylvester had big game — Fermat's Last Theorem — in his sights. Significant progress on such a famous open problem would certainly have established Sylvester quickly as a mathematician of note."

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[10]An exaggerated but entertaining version of Sylvester's ordeal in Virginia is offered by Sylvester's ex-student at Hopkins and popularizer of mathematics, Halsted [32]. A more factual account is, however, provided by Yates in [76].

By the time Sylvester had published his 1847 set of papers, he had been employed as an actuary. Later, he met Cayley while both were studying law. As a result, his enthusiasm for mathematics was restored. He later re-entered academe in 1855 at the Royal Military Academy, Woolwich.

The Royal Military Academy was not a research institution. It was also considered to be inferior to its French counterpart, *l'Ècole Polytechnique.* In addition, Sylvester would occasionally engage in squabbles with the military governorship over matters related to his teaching duties. Nevertheless, he subsisted and remained relatively comfortable there for fifteen years. By 1870, a change in military regulations forced him into retirement at the age of fifty-five. The hiatus would prove to be temporary.

Post-Woolwich

After a nearly one-year ordeal to secure a pension, Sylvester devoted a considerable amount of time and energy to his avocations of poetry and singing. His first book-length study on versification, *The Laws of Verse* [70], was published in 1871. It proved to be a source of great pride for him. He followed this up with a private printing of *Fliegende Blätter, Supplement to the Laws of Verse*, [71]. During this period of professional repose, his mathematical output would consist of eight short articles. Among them, however, was [65]. It addressed the famous conjecture of Goldbach.

Sylvester also provided a short discussion of this famous problem in *The Laws of Verse* [70, pg 123]. He admitted that it was not due to "Euler's correspondence with Goldbach" that caused him to become aware of the problem's existence, but rather, Sylvester claims to have "re-discovered" the problem in connection with a theory of his regarding cubic forms. This claimed accidental discovery also bears some resemblance to his later account of how he first encountered the odd perfect number question.

Regardless of the exact manner of his introduction to Goldbach's conjecture, Sylvester chose not to spend a great deal of time on trying to prove it. Rather, he opted to describe a plan that he hoped would ultimately lead to a demonstration that the probability of the conjecture being true can be made to be as close to unity as one pleases.

By the fall of 1875, Gilman, who had been appointed president of the newly founded Johns Hopkins University arrived in London on a faculty finding expedition. His objective was to recruit internationally renowned scholars to help bring to fruition a vision of an institution committed to teaching, research, and the training of future researchers. Among other things, Gilman was looking for a world-class mathematician capable of inspiring and directing the scholarly pursuits of the mathematics graduate students in a research-oriented university. In other words, the position that Gilman sought to fill was perfectly suited for the style and temperament of Sylvester.

Initially, Gilman was a bit skeptical of Sylvester [49, pg 73] for setting research credentials aside, it had been a while since he had been active in academe. In addition,

Sylvester was not regarded as one who had honed superior teaching skills. Moreover, the tenacity that he was capable of demonstrating in matters of personal importance was widely known.

In spite of it all, perhaps the most accurate testimony of how well Sylvester might fit into the plans of the new university was provided in a recommendation letter written on his behalf by Peirce [50, pg 73]: "If you inquire about him, you will hear his genius universally recognized but his power of teaching will probably said to be quite deficient. ... as the barn yard fowl cannot understand the flight of the eagle, so it is the eaglet only who will be nourished by his instruction. ... among your pupils, sooner or later, there must be one, who has a genius for geometry. He will be Sylvester's special pupil — the one pupil who will derive from the master, knowledge and enthusiasm — and that one pupil will give more reputation to your institution than the ten thousand, who will complain of the obscurity of Sylvester, and for whom you will provide another class of teachers."

Sylvester was ultimately offered the job. However, his steadfastness in negotiating remuneration almost caused the appointment not to materialize.[11] Sylvester arrived in Baltimore in 1876 with a renewed vigor for mathematics and a strong desire to make good on the responsibilities that were being entrusted to him. Moreover, he was about to occupy the most satisfying academic appointment he would ever hold. It therefore remains somewhat of a mystery as to why Sylvester would decide to resign seven years later in order to accept the Savilian Chair of Geometry at Oxford.

Oxford

Perhaps it was the allure of a most distinguished university in his own homeland. Perhaps it also was the prestige associated with being the twelfth occupant of the oldest Chair (founded in 1619) of any British university and previously held by the likes of Briggs, Wallis, and Halley. Perhaps, it may even have been a matter of poetic justice to him that he be offered and accept academic rank at an institution that would have certainly rejected him years earlier on religious grounds.[12] Whatever the reason may have been, an optimistic Sylvester arrived on the grounds of Oxford in 1884.

At first, Sylvester found Oxford agreeable. He was able to continue his research momentum on reciprocants (differential invariants).[13] He also attempted to augment

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[11]The final agreement called for an annual salary and housing allowance of $5,000 and $1,000, respectively, with both amounts to be paid in gold.

[12]Because he was Jewish, Sylvester as a student at Cambridge refused to swear allegiance to the thirty-nine Articles of Faith of the Church of England. This not only precluded him from receiving the degree that he had earned but it also denied him potential fellowships and professorships at all Anglican institutions. In 1871, the Universities Test Act was repealed. This made non-Anglicans, such as Sylvester, eligible to hold positions at schools like Oxford.

[13]Sylvester described them as being "a great and unlooked for *revelation* which will alter the whole face of Analytical Geometry and also produce no less effect in the theory of Differential Equations and Transcendental Functions." [49, pg 259]

his teaching duties with graduate-level lectures geared toward the College Tutors and Lecturers with some positive response but it did not last long [49, pg 238]. His attempts to institute a learning environment similar to the one that he had enjoyed in Baltimore had failed to materialize at Oxford.

By 1888, and well into his seventies, Sylvester felt capable of producing more than the diminished expectations that Oxford had placed on him. It was under these circumstances that his attention redirected itself to placing a definitive answer on the question of the existence of an odd perfect number.

We point out that it seems to have been either a stimulating academic environment or the want of one that dictated the nature of Sylvester's mathematical interests. For example, although Woolwich in the 1850s and 1860s was not a haven for research, Cayley was nearby. As a result, Sylvester made what some have considered to be his chief contribution to the development of the mathematical sciences — Invariant Theory.[14] Conversely, when left professionally isolated, such as during the years that immediately followed his departures from Virginia and Woolwich, or during his latter years at Oxford, Sylvester was at a disadvantage. Franklin tells us in [20] that, "those who knew him cannot fail to be convinced that, eminent as were his actual achievements, they do not afford a true measure of his mathematical powers, in comparison to his great contemporaries. For he was at once less advantageously circumstanced than they, and in an exceptional degree subject to the influence of his surroundings."

Thus far, we have tried to point to Sylvester's propensity to consider celebrated problems in number theory during periods of isolation and uncertainty. We shall now take this opportunity to remark that during his career, Sylvester also cultivated an interest in longstanding questions in other areas of mathematics. Moreover, his work on these problems was not solely relegated to turbulent professional times. In fact, perhaps Sylvester's single most important achievement while at Woolwich was his proof of Newton's Rule for locating the imaginary roots of a polynomial equation. This was a two-hundred year old problem that had previously eluded a rigorous demonstration by Newton himself, Maclaurin, Waring, Euler, Lagrange, and others.

Before we conclude our findings, we offer the following translation (from the French) of Sylvester's proof that an odd perfect number contains at least five distinct prime divisors [68].

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[14]Franklin contended in 1897 that Sylvester and Cayley's development of invariant theory marked one of the greatest contributions of British thought to pure mathematics since the days of Newton [20].

Sur L'Impossibilité De L'Existence D'un Nombre Parfait Impair Qui Ne Contient Pas Au Moins 5 Diviseurs Premiers Distincts

James Joseph Sylvester

We have shown in an earlier note that an odd perfect number with less than seven factors must be divisible by 3, but in any case, no [odd] perfect number is divisible by 105.

We now add that because

$$\frac{3}{2} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} = \frac{151(\frac{15}{16})}{80} < 2$$

and that in substituting 11, 13, 17 for the other elements, one can diminish this product so that it encroaches upon either 5 or 7, thus it follows that the element 3 must be associated with 7 or with 5 in a perfect number with four elements.

Let us therefore suppose that such a number N exists.

(1)   Let 3 and 7 be two of its elements. The third element in order of size cannot exceed 13; because

$$\frac{3}{2} \cdot \frac{7}{6} \cdot \frac{17}{16} \cdot \frac{19}{18} = \frac{119}{64}(1 + \frac{1}{18}) < \frac{126}{64} < 2.$$

(α)   Let 11 be the third element; because

$$\frac{3}{2} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{29}{28} = \frac{77}{40}(1 + \frac{1}{28}) < 2,$$

one can see that the fourth element must be from among the numbers 13, 17, 19, or 23.

But among the elements, one must be of the form $4x + 1$.

Moreover, we have shown in a previous note that no perfect number can contain the number 17 without at the same time containing an element smaller than 67. Therefore the four elements will be 3, 7, 11, 13.

The divisor-sum [1] of 7 cannot contain the algebraic factor $7^9 - 1$, for else $\frac{1}{3} \cdot \frac{7^3 - 1}{7 - 1}$, $\frac{1}{3} \cdot \frac{7^9 - 1}{7^3 - 1}$ will be divisors other than 3 and 7 of this prime sum, and further contain 13 because

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[1] If p is an element and p^i is a component of a number N, we call p^i the component of p, and $\frac{p^{i+1} - 1}{p - 1}$ the divisor-sum of p.

13 is neither a unilinear [2] of q nor a divisor of $7^3 - 1$. Thus on this supposition, there will be at least five distinct elements. Therefore the divisor-sum of 7 cannot contain 9, but the component of 3 necessarily contains 3^2; consequently, because the divisor-sum of 11 (ordinary elements and not of the form $3x + 1$) cannot contain 3, the divisor-sum of 13 carries with it an algebraic factor of the form $\frac{13^3 - 1}{13 - 1}$ which is equal to $169 + 13 + 1$. Therefore 61 will be an element greater than 3, 7, 11, 13 and that is contrary to the hypothesis.

(1)   (β)   Let 13 be the third element.

Because $\frac{3}{2} \cdot \frac{7}{6} \cdot \frac{13}{12} \cdot \frac{23}{22} = \frac{91}{48}(1 + \frac{1}{22}) < 2$, the fourth element will necessarily be less than 23 and the system of elements will be 3, 7, 13, 19, because 17 is excluded.

The divisor-sums, either of 13 or 19, cannot contain 3; because they necessarily contain the factors $\frac{13^2 - 1}{13 - 1}$ and $\frac{19^3 - 1}{19 - 1}$, and therefore $\frac{1 + 13 + 13^2}{3}$, that is to say 61, and $\frac{1 + 19 + 19^2}{3}$, that is to say 127.

Therefore the divisor-sum of 7 will algebraically contain the factors $\frac{1}{3} \cdot \frac{7^9 - 1}{7^3 - 1}$, $\frac{1}{3} \cdot \frac{7^3 - 1}{7 - 1}$; the last is equal to 19; the first is necessarily prime to 3, 7, 19 and, for the reason already given, to 13.

We have therefore demonstrated that 7 cannot be an element of N.

(2)   Suppose that 3 and 5 are two of the elements.

2.   A.   Let 5 be the special element.

2.   A(α).   If the index of the element 3 is 2, then, because $1 + 3 + 3^2 = 13$, we have the elements $3, 5, 13$; therefore the divisor-sum of 13 will contain 3, and consequently, algebraically contain the factor $\frac{13^2 + 13 + 1}{3}$, that is to say, 61.

Hence we have the elements 3, 5, 13, 61.

But $\frac{1 + 3 + 3^2}{9} \cdot \frac{1 + 5}{5} \cdot \frac{13}{12} \cdot \frac{61}{60} < 2$, which is inadmissible.

2.   A(β).   We therefore suppose that the index of the component 3 is at least 4.

Let 3, 5, p be the three elements; the index of the divisor-sum of p cannot be 9 for then we shall have at least two other elements greater than 3, 5, p and prime to 3, 5, p.

Let q be the fourth element; the same thing will be true of the divisor-sum of q.

Therefore the product of the divisor-sums of 3, 5, p, q cannot contain a power of 3 greater than 3^3; but it must contain one less than 3^4.

Thus the hypothesis that 5 is a special element is inadmissible.

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[2]It is very convenient in this type of research to use the phrase "unilinear function of x" to signify $kx + 1$.

2. B.   Moving on to the hypothesis that 5 is an ordinary element.

We remark that $\frac{3}{2} \cdot \frac{5}{4} \cdot \frac{31}{30} \cdot \frac{37}{36} < 1.992 < 2$.

Consequently, there is at least one element, call it p, that does not exceed 29: I say that p cannot be contained in the divisor-sum of 5, for if that were the case, the index of this sum would necessarily be an odd divisor in excess of some prime number less than 31, that is to say 3, 5, 7, 9, or 11, of which the last four correspond to the prime numbers 11, 29, 19, and 23.

It cannot be 3, for $\frac{5^3-1}{5-1} = 31$; nor 5, for $\frac{5^5-1}{5-1} = 11 \cdot 71$ (and we have the combination of elements 3, 5, 11, 71; which is inadmissible since 5 is, by hypothesis, not special, and other elements are of the form $4x + 3$).

It cannot be 7 because it is easily shown that $5^7 - 1$ contains neither 29 nor 9; for although it is true that (5 being a quadratic residue of 19) $5^9 - 1$ contains 19, it contains at the same time $5^3 - 1$, and we have the combination 3, 5, 19, 31, which is forbidden for the same reason as 3, 5, 11, 71.

We are left only with 11, but $5^{11} - 1$ does not contain 23, because 5 is not a quadratic residue of 23.

Hence the element 5 cannot *beget* (by means of the divisor-sum to which it responds) an element which is not outside the limit 29.

The divisor-sum of such an element (if it is 11 and only if in this case) may contain 5, but not 5^2; for if it contains 5^2, we will then have at least two divisors of this prime sum between it and 3, 5, 11.

We remark that the component of the special element cannot be the power (to the exponent $4j + 1$) of a number; for, if $j > 0$, $q^{4j+2} - 1$ necessarily contains two distinct prime factors in addition to 3, 5, and p; therefore $j = 0$; hence we see that $q + 1$ must contain the powers of 3 and 5 contained in $3^2 \cdot 5^2$, that are not contained in the divisor-sum of the other undetermined element, which can easily be shown not to contain 3 or 5 and not 3^2, $3 \cdot 5$, or 5^2; for on the first or last of these hypotheses, the number of elements will be greater than four, and on the remaining hypotheses even greater than 5. Therefore we augment the special element to be either of the form $2k \cdot 3^2 \cdot 5 - 1$ or $2k \cdot 3^2 \cdot 5^2 - 1$: consequently, its value cannot exceed 89; this proves nonetheless that the p of which we have spoken is not a special element.

Let q be this element, we have

$$q = 30\lambda - 1.$$

But the divisor-sum of 5 contains neither 3 nor p.

We therefore inevitably get

$$\frac{5^x - 1}{5 - 1} = q = 30\lambda - 1,$$

that is to say $5^x - 120\lambda + 3 = 0$, which is impossible.

This demonstrates that hypothesis 2. B is inadmissible, and finally the result is achieved that there do not exist odd perfect numbers that are divisible by fewer than 5 prime factors; the case of multiplicity $3, 2, 1$ has already been demonstrated for this theorem.

We now add a few words on perfect numbers with five elements.

Here,

$$\frac{3}{2} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{23}{22} < 1.986,$$

but

$$\frac{3}{2} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{18} > 2.004.$$

We see that for a perfect number with five elements, where 5 and 7 are missing, such elements cannot be the numerals 3, 11, 13, 17, 19.

But 17 (a cyclotomic number of Gauss) cannot exist without a companion number of the form $17k \pm 1$. Therefore a perfect number with five elements, if it exists, must necessarily have either the elements 3, 5 or the elements 3, 7.

I have succeeded in demonstrating each of these hypotheses; but the proof is too long to be included here.

□

The parallel between the last line of [68] and Fermat is rather striking. Furthermore, the result to which Sylvester makes reference does not seem to have been explicitly demonstrated by him in any of his articles. However, we do find in [9] displayed prominently above Christie's erroneous question on the perfection of $2^{40}(2^{41} - 1)$, the following problem posed by Sylvester: "If there exist any perfect number divisible by a prime number p of the form $2^n + 1$, show that it must be divisible by another prime number of the form $px \pm 1$."

The proof by reader, W. S. Foster, appears to be correct.

3. Conclusion

In March of 1897, Sylvester had a paralytic seizure after having leaned over to pick up a pen that he dropped while working in his rooms. After that, he never spoke again. Sylvester died on March 15 of that year. His last paper was published posthumously in 1898 [73]. It was only the second article of his to appear after resigning from Oxford for health reasons. Both papers were number theory efforts.

Sylvester was primarily known as an algebraist. In that field, he made significant contributions to invariant theory, matrix theory, determinant theory, and the theory of equations. Although throughout his career he had studied problems from other areas of mathematics, it would appear that Sylvester manifested a special interest in notable unanswered questions in the higher arithmetic during unsettled periods in his career.

In 1847, Sylvester sought to connect his trilogy of papers on cubic diophantine equations to Fermat's Last Theorem. This marked his first set of publications after a three-year absence from mathematics that was brought on by an abrupt decision to leave the University of Virginia. Shortly after being forced into retirement from Woolwich in 1870 for being superannuated, Sylvester would consider another problem of the same genre — Goldbach's conjecture. In 1888, after having made a regrettable decision to forfeit the favorable research surroundings of the Johns Hopkins University, Sylvester took up the existence question of an odd perfect number while at Oxford. His intention was to prove that such numbers do not exist.

Environment was not the only factor that motivated his research in 1888. It was, as Sylvester claimed, some inquires related to the subject and posed to him by 'Mr. Christie' that initially put the idea in his mind. Evidence seems to suggest that the person to whom Sylvester alluded was Mr. Robert William Dougall Christie, a number theorist and frequent contributor of problems to the *Mathematical Questions* column of the *Educational Times*.

Sylvester began his assault on the odd perfect number problem in 1888 by first proving what Peirce had demonstrated more than fifty years earlier — that an OPN necessarily has at least four distinct prime factors. Later that year, he extended his result to five. He also established that an odd perfect number cannot be divisible by 105 and showed that any such number not divisible by three must have at least eight distinct prime factors. Unlike Peirce, Sylvester was successful in disseminating his results to a wide audience. In turn, this contributed to an increased level of interest in the problem that has lasted to this day.

Finally, we know not whether an odd perfect number exists nor can we be completely sure of the question's decidability. However, should an answer someday be provided, there will exist a tall pyramid of contributors.

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