POSET GAME PERIODICITY

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Received: 3/26/03, Revised: 10/15/03, Accepted: 11/20/03, Published: 11/21/03

Abstract

Poset games are two-player impartial combinatorial games, with normal play convention. Starting with any poset, the players take turns picking an element of the poset, and removing that and all larger elements from the poset. Examples of poset games include Chomp, Nim, Hackendot, Subset-Takeaway, and others. We prove a general theorem about poset games, which we call the Poset Game Perioidicity Theorem: as a poset expands along two chains, positions of the associated poset games with any fixed g-value have a regular, periodic structure. We also prove several corollaries, including applications to Chomp, and results concerning the computational complexity of calculating g-values in poset games.

1 Introduction

1.1 Posets

A partially-ordered set (poset) is a set and a relation among its elements satisfying irreflexivity, transitivity, and antisymmetry. In the sequel, the relation will always be denoted $<$ ("less than"). For any set S, we will abbreviate the poset $(S, <)$ as just S (it will be clear from context whether we are talking about poset or the set). We define $>$, $\leq$, and $\geq$ as would be expected (in terms of $<$). We say $a \parallel b$ ("a is incomparable to b") if and only if neither $a \leq b$ nor $a \geq b$. A *chain* in a poset is a subset containing no pair of incomparable elements. For convenience later on, we will use the following notation to compare elements of X to subsets of X: for $a \in X$ and $S \subseteq X$, $a < S$ if and only if $a < x$ for each $x \in S$; $a > S$ and $a \parallel S$ are defined similarly.

[1]This paper was written and submitted while the author was a senior at the Roxbury Latin School, West Roxbury, MA, USA, and revised for publication while a freshman at Harvard College, Cambridge, MA, USA.

1.2 How to Play a Poset Game

In order to play a poset game, given any poset A, two players take turns making moves. On each move, a player picks an element $x \in A$, and removes all the elements of A greater than or equal to x from the set A, to form a smaller poset A'. This becomes the new A, then the other player picks an element, and so on. The player unable to move (when $A = \emptyset$) loses. For the purposes of this paper, we will only consider poset games on finite posets.

1.3 Poset Game Background

Over the past century, many types of poset games have been named and studied. All poset game literature and research has been directed at the study of these specific games, each in isolation from the others — before this paper, no major theorem was known that applied to all poset games. We will now discuss some of these specific results.

For some specific poset games, efficient (polynomial-time) winning strategies have been found. Such games include Nim [1], Von Neumann's Hackendot [2], and impartial Hackenbush on trees [3]. For each of these three poset games, the poset is N-free (that is, it has no four elements a, b, c, d satisfying $a \parallel b$, $a < c$, $a < d$, $b \parallel c$, $b < d$, and $c \parallel d$). In fact, a polynomial-time winning strategy for any poset game on an N-free poset is given in [4].

However, other well-known poset games (which, of course, are not N-free) have remained unsolved for as many as fifty years, with neither a known polynomial-time winning strategy, nor a demonstration that none exists.

One such unsolved poset game is called Chomp, proposed in 1974 by D. Gale [5], and named later by M. Gardner [6]. An $m \times n$ bar of chocolate is divided into unit squares, and the top-left square is poisoned. On each turn, a player bites off a square, along with all the squares directly below it, directly to the right of it, and below and to the right of it. Eventually, one player is forced to eat the poisoned square, thus losing the game. If we remove the poisoned square from the set and say that one chocolate square is greater than or equal to another if the former is below and/or to the right of the latter, then we see that this is in fact a poset game.

Two other well-known unsolved poset games are Schuh's Game of Divisors and Subset Takeaway (also called the superset game). The former starts with the poset consisting of the positive divisors of a fixed integer n, excluding 1, partially ordered by divisibility, and was proposed in 1952 by F. Schuh [7]. The latter starts with the poset consisting of all subsets of a given set, excluding the null set, partially ordered by set inclusion, and was proposed in 1982 by D. Gale [8]. Interestingly, both Subset Takeaway and Chomp are isomorphic to special cases of Schuh's Game of Divisors, where the integer n is square-free or has at most two prime divisors, respectively.

In Chomp, two special cases of the Poset Game Periodicity Theorem have already been stated as conjectures. Based on previous work by D. Zeilberger [9], X. Sun wrote a Maple program that calculated $\mathcal{P}$-positions in Chomp by searching for periodic patterns with all but the top two rows fixed [10]. The success of that algorithm led him to conjecture that, when all but the top two rows are fixed, the difference in length between the top two rows in $\mathcal{P}$-positions is eventually periodic with respect to the length of the second row. A. Brouwer [11] verified that conjecture by computer in the special case of 3-rowed Chomp positions with at most $90,000$ squares in the bottom row. Later, X. Sun wrote another Maple program, which calculated g-values for Chomp positions [12]. In the data generated, he found a periodic relation between the length of the top row and the g-value, when all other rows are fixed, and conjectured that this always holds. We will see that both of these conjectures are corollaries of the Poset Game Periodicity Theorem, proposed and proved in this paper.

Finally, it is known that, if a poset X has a largest or smallest element, the associated poset game is in an $\mathcal{N}$-position. The proof is short but clever. If X has a smallest element, taking that would be the winning move. If X has a largest element a, the first player can take a on the first move. Either that leaves a $\mathcal{P}$-position, or else the second player has a winning response b. The first player therefore always has a winning opening move – a in the first case, b in the second – so X is always an $\mathcal{N}$-position. In particular, Schuh's Game of Divisors, Chomp, and Subset-Takeaway are all first-player wins (since they have largest elements), but the nonconstructive nature of the proof says nothing about what the strategy is.

1.4 Game Theory Background

Since poset games are finite, impartial, combinatorial games, they are described by the Sprague-Grundy Theory of Games. The essential property of a position is its *g-value* (also called grundy-value, nim-value, or Sprague-Grundy function), and understanding the structure of the g-values of a game's positions is essential for understanding that game.

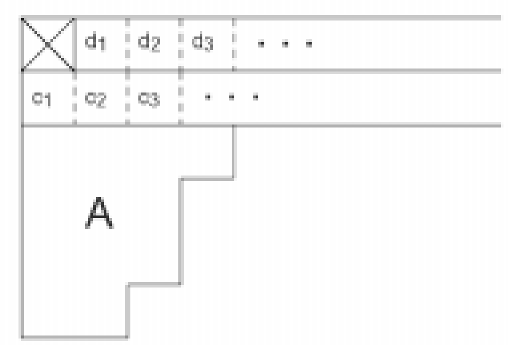
Let the *mex* ("minimal excluded value") of any set be the smallest nonnegative integer not in the set. The g-value of any position in a game is recursively defined as the mex of the set of g-values of all game-positions that can remain after exactly one move. (In particular, $\mathcal{P}$-positions are those with g-value 0.) In other words, g-values have two properties: First, if we start from a position with g-value k, then, for any integer n, $0 \leq n \leq k-1$, there is some move that leaves a position with g-value n. Second, there is no move from a position with g-value k to another with g-value k. (For a more complete explanation of g-values, see [3].) For any finite poset P, let $g(P)$ be the g-value of the position P in a poset game.

2 Statement of the Periodicity Theorem

In an infinite poset X, suppose we have two infinite chains C ($c_1 < c_2 < \cdots$) and D ($d_1 < d_2 < \cdots$), and a finite subset A, all pairwise disjoint, and assume that no element of C is less than an element of D (Figure 2 displays an example). Let

$$A_{m,n} = A \cup C \cup D - \{x \in X \mid x \geq c_{m+1}\} - \{x \in X \mid x \geq d_{n+1}\}$$

(that is, $A_{m,n}$ is the position that results from starting with the poset $A \cup C \cup D$, then making the two moves c_{m+1} and d_{n+1}). Let k be a nonnegative integer. Then the **Poset Game Periodicity Theorem** states that either: (1) there are only finitely many different $A_{m,n}$ with g-value k; or (2) we can find a positive integer p such that, for large enough n, $g(A_{m,n}) = k$ if and only if $g(A_{m+p,n+p}) = k$. Thus, as the poset A expands along the chains C and D, positions with any fixed g-value have a regular structure.



An example of a poset game (Chomp) to which the Periodicity Theorem applies

The paper will proceed as follows: in Section 3, we define terms and prove nine lemmas; in Section 4, we use these tools to prove the periodicity theorem; in Section 5, we provide some corollaries and implications of the theorem; and in Section 6, we suggest possible avenues for future work.

3 Preliminaries for the Periodicity Theorem

For the rest of the paper, we will assume the following:

$$\begin{gathered}(X,<) \text{ is an infinite poset, containing two infinite chains } C \text{ and } D \text{, and a} \\ \text{finite subset } A \text{, all pairwise disjoint; } C = \{c_1, c_2, \ldots,\} \text{ with } c_1 < c_2 < \cdots; \\ D = \{d_1, d_2, \ldots,\} \text{ with } d_1 < d_2 < \cdots; \text{ and if } c \in C \text{ and } d \in D \text{, then } c \not< d.\end{gathered} \tag{1}$$

For $m, n \in \mathbb{N}_0$, let

$$A_{m,n} = A \cup C \cup D - \{x \in X \mid x \geq c_{m+1}\} - \{x \in X \mid x \geq d_{n+1}\}$$

and let

$Q(A) = \{k \in \mathbb{N}_0 \mid \text{only finitely many different positions of the form } A_{m,n} \text{ have g-value } k\}.$

Note that this is not the same as saying that there are only finitely many solutions (m,n) to $g(A_{m,n}) = k$. For example, if $c_2 > d_2$, then $A_{1,1}$, $A_{2,1}$, $A_{3,1}$, etc. are all the same position.

Lemma 1. *Suppose* $(m_1, n_1) \neq (m_2, n_2)$. *Then* $A_{m_1,n_1} = A_{m_2,n_2}$ *if and only if* $n_1 = n_2$, $c_{m_1+1} > d_{n_1+1}$, *and* $c_{m_2+1} > d_{n_1+1}$.

Proof. Suppose $(m_1, n_1) \neq (m_2, n_2)$ and $A_{m_1,n_1} = A_{m_2,n_2}$. Since no element of D is greater than any element of C, $A_{m,n} \cap D = \emptyset$ if and only if $n = 0$, and $\max(A_{m,n} \cap D) = d_i$ if and only if $i = n$. Hence, we can recover n from the position $A_{m,n}$, so if $A_{m_1,n_1} = A_{m_2,n_2}$ then $n_1 = n_2 = n$. Since $(m_1, n) \neq (m_2, n)$, $m_1 \neq m_2$, so without loss of generality, assume $m_1 > m_2$. If $c_{m_2+1} \not> d_{n+1}$ then c_{m_2+1} would be in $A_{m_1,n}$ but not $A_{m_2,n}$, contradicting the fact that they are equal. Hence, $c_{m_2+1} > d_{n_1+1}$ and $c_{m_1+1} > c_{m_2+1} > d_{n_1+1}$. Conversely, suppose $n_1 = n_2 = n$, $c_{m_1+1} > d_{n+1}$, and $c_{m_2+1} > d_{n+1}$. Then

$$A_{m_1,n_1} = A_{m_1,n} = A \cup C \cup D - \{x \in X \mid x \geq d_{n+1}\} = A_{m_2,n} = A_{m_2,n_2}.$$

□

Lemma 2. *If* $n_1 \neq n_2$, *then* $g(A_{m,n_1}) \neq g(A_{m,n_2})$.

Proof. Without loss of generality, assume $n_1 < n_2$. Since $d_{n_1+1} \in A_{m,n_2}$, if we start with A_{m,n_2} and make the move d_{n_1+1}, the resulting position is

$$\begin{aligned} &A \cup C \cup D - \{x \in X \mid x \geq c_{m+1}\} - \{x \in X \mid x \geq d_{n_2+1}\} - \{x \in X \mid x \geq d_{n_1+1}\} \\ &= A \cup C \cup D - \{x \in X \mid x \geq c_{m+1}\} - \{x \in X \mid x \geq d_{n_1+1}\} \\ &= A_{m,n_1}. \end{aligned}$$

Since we can get from A_{m,n_2} to A_{m,n_1} in one move, their g-values must differ. □

Lemma 3. *If* $c_{m_1+1} \not> d_{n+1}$ *and* $m_1 \neq m_2$, *then* $g(A_{m_1,n}) \neq g(A_{m_2,n})$.

Proof. This proof is similar to the proof of Lemma 2. If $m_2 < m_1$, then, since $c_{m_2+1} < c_{m_1+1}$, $c_{m_2+1} \not> d_{n+1}$ also. Hence, without loss of generality, assume $m_1 < m_2$. Since $c_{m_1+1} \not> d_{n+1}$, $c_{m_1+1} \in A_{m_2,n}$. Thus, if we start with $A_{m_2,n}$, we can make the move c_{m_1+1} and get $A_{m_1,n}$. Since there is a move from $A_{m_2,n}$ to $A_{m_1,n}$, their g-values must differ. □

Lemma 4. $k \in Q(A)$ *if and only if one of the following is true:* (i) *There exists* $m, n \in \mathbb{N}_0$ *such that* $c_{m+1} > d_{n+1}$ *and* $g(A_{m,n}) = k$, *or* (ii) *there exists an* $a \in A$, $m, n \in \mathbb{N}_0$ *with* $a < c_{m+1}$, $a < d_{n+1}$, *and* $g(A_{m,n} - \{x \in X \mid x \geq a\}) = k$.

Proof. First, suppose $c_{m+1} > d_{n+1}$ and $g(A_{m,n}) = k$. Let $P = A_{m,n}$. By Lemma 1, for any $i \geq m$, $P = A_{i,n}$. By Lemma 2, for each $i \geq m$, $P = A_{i,n}$ is the only position $A_{i,y}$ that has g-value k. Also by Lemma 2, there are at most m positions of the form $A_{i,y}$

with g-value k for $0 \le i \le m-1$ (at most one for each such i). Thus, in total, there are at most $m+1$ different positions of the form $A_{x,y}$ with g-value k, which is finite, so $k \in Q(A)$.

Second, suppose (ii) holds. If we make the move a from the position $A_{m',n'}$ for $m' > m$ and $n' > n$, we will get a position with g-value k. Hence, if $m' > m$ and $n' > n$, then $g(A_{m',n'}) \neq k$. For each $m' \le m$, by Lemma 2, there is at most one position $A_{m',n'}$ with g-value k. For each $n' \le n$, by Lemmas 1 and 3, there is at most one position of the form $A_{m',n'}$ with g-value k. Hence, there are at most $m+n+2$ positions of the form $A_{m',n'}$ with g-value k, so $k \in Q(A)$.

Finally, suppose that $k \in Q(A)$. Since any $A_{m,n}$ is finite, we can find an M and N such that, if $c_M \in A_{m,n}$ or if $d_N \in A_{m,n}$, then $g(A_{m,n}) \neq k$. By Lemma 2, we can find a $y > N$ such that $g(A_{M,y}) > k$. Let z be a move that takes $A_{M,y}$ to a position with g-value k. Since d_N and c_M are not in the resulting position, $z < d_N$, and either $d_{y+1} < c_M$ or $z < c_M$. Since $d_N < d_{y+1}$, we get $z < d_N$ and $z < c_M$. Since $z < d_N$, either $z \in D$, in which case (i) holds, or $z \in A$, in which case (ii) holds. □

If $k \in Q(A)$, whether (i) or (ii) occurs, there is a number $T(A,k) \in \mathbb{N}_0$ such that, if $n \ge T(A,k)$, then there is a move from $A_{m,n}$ to a position with g-value k, no matter what the value of m. If (i) holds, then $c_{m'+1} > d_{n'+1}$ with $g(A_{m',n'}) = k$, and if (ii) holds, then, for some $a \in A$, $a < c_{m'+1}$, $a < d_{n'+1}$, and $g(A_{m',n'} - \{x \in X \mid x \ge a\}) = k$. For each m, by Lemma 2, there exists an n_m such that, if $n \ge n_m$, then $g(A_{m,n}) > k$. If we let $T(A,k) = \max(n_0, n_1, \ldots, n_{m'}, n'+1)$, it will have the desired property. Now, let

$$W(A,k) = \max(\{T(B,j) \mid B \subseteq A,\ j \le k,\ j \in Q(B)\})$$

with $\max(\emptyset)$ interpreted as 0. We will use this function later.

Lemma 5. *If $g(A_{m,n}) = k$, then $n - m \le |A| + k$. Also, if $k \notin Q(A)$, or if $c_{m+1} \not> d_{n+1}$, then $|n-m| \le |A| + k$.*

Proof. Suppose $g(A_{m,n}) = k$. Let

$$S = \{n' \in \mathbb{N}_0 \mid 0 \le n' \le n-1,\ g(A_{m,n'}) > k\}.$$

For each $n' \in S$, there is at least one move from $A_{m,n'}$ to a position with g-value k. Pick one such move, and call that move $f(n')$. If $f(n') < d_{n'+1}$, then we could have made the move $f(n')$ from $A_{m,n}$ directly, which would be a move from a position with g-value k to another with g-value k, which is impossible. Hence,

$$f(n') \not< d_{n'+1}. \tag{2}$$

Suppose that $f(n'_1) = f(n'_2) = x$, for $n'_1 > n'_2$. Since $d_{n'_2+1} \not> x$ by (2), we can start with A_{m,n'_1} and make the move x, leaving a position with g-value k, then make the move

$d_{n'_2+1}$, leaving another position with g-value k, a contradiction. Hence, if we view f as a function $f : S \to X$, then f is injective. From (2), $f(n') \notin D$, so the image of f is in $A \cup \{c_1, \ldots, c_m\}$. Since f is injective, $|S| \le |A| + m$. By Lemma 2, at most k values of $n' < n$ satisfy $g(A_{m,n'}) < k$, and none satisfy $g(A_{m,n'}) = k$. Hence, $|S| \ge n - k$, so $n - m \le |A| + k$.

If $k \notin Q(A)$, then, by Lemma 4, $c_{m+1} \not> d_{n+1}$. Thus, to finish the proof, we only need to show that, if $c_{m+1} \not> d_{n+1}$, then $m - n \le |A| + k$. The proof runs exactly like the one above, but switching the roles of C and D, and using Lemma 3 instead of Lemma 2. □

We will now add another assumption to (1), which will apply to the rest of this section:

$$\text{For each } a \in A, \text{ either } a < C \text{ or } a \parallel C, \text{ and either } a < D \text{ or } a \parallel D. \tag{3}$$

Lemma 6. *Assume (3). If $k \notin Q(A)$, and if $m \ge |A| + k$, then there exists a unique $n \in \mathbb{N}_0$ such that $g(A_{m,n}) = k$.*

Proof. Suppose that $k \notin Q(A)$ and $m \ge |A| + k$. By Lemma 5, $g(A_{m,m+|A|+k+1}) > k$. Hence, there is a move $z \in A_{m,m+|A|+k+1}$ that takes $A_{m,m+|A|+k+1}$ to a position with g-value k. We have six cases:

- $z \in A$, $z < C$, $z < D$: This is impossible, by Lemma 4, since $k \notin Q(A)$.
- $z \in A$, $z < C$, $z \parallel D$: This is impossible, since the resulting position is $B_{0,m+|A|+k+1}$, for some $B \subset A$. By Lemma 5, this has g-value greater than k.
- $z \in A$, $z \parallel C$, $z < D$: This is impossible, since the resulting position is $B_{m,0}$, for some $B \subset A$. Since $c_{m+1} \parallel z < d_1$, $c_{m+1} \not> d_1$, so, by Lemma 5, since $m \ge |A| + k > |B| + k$, $g(B_{m,0}) > k$.
- $z \in A$, $z \parallel C$, $z \parallel D$: This is impossible, since the resulting position is $B_{m,m+|A|+k+1}$, for some $B \subset A$. By Lemma 5, this has g-value greater than k.
- $z \in C$: This is impossible, since the resulting position is $A_{m',m+|A|+k+1}$, for some $0 \le m' \le m - 1$. By Lemma 5, this has g-value greater than k.
- $z \in D$: In this case, $z = d_{n+1}$ for some $n \in \mathbb{N}_0$, so $g(A_{m,n}) = k$.

By process of elimination, the last case must occur, so, for any $m \ge |A| + k$, there exists an $n \in \mathbb{N}_0$ such that $g(A_{m,n}) = k$. Uniqueness follows from Lemma 2. □

Let $f_{A,k}(m)$ be the unique (by Lemma 2) value satisfying $g(A_{m,f_{A,k}(m)}) = k$, whenever it exists. We will abuse notation slightly in order to handle the case that $f_{A,k}(m)$ does not exist. If $\alpha(B, j, m)$ is some set of conditions on B, j, and m, we will let

$$\{f_{B,j}(m) \mid \alpha(B, j, m)\} = \{i \mid \exists B, j, m \text{ with } \alpha(B, j, m) \text{ and } f_{B,j}(m) = i\}.$$

This allows the set on the left to still be defined, even if $f_{B,j}(m)$ does not always exist.

Finally, let $T = \{a \in A \mid a \parallel C,\, a \parallel D\}$, and let

$$H = \{A - \{x \in A \mid x \geq a\} \mid a \in T\}.$$

Lemma 7. *Assume (3). For $j \in \mathbb{N}_0$, $j \notin Q(A)$, $m \geq |A| + j$, and $n \geq \max(|A| + j, W(A, j))$, there is a move from $A_{m,n}$ to a position with g-value j if and only if*

$$n \in \{f_{A,j}(i) \mid 0 \leq i \leq m-1\} \cup \{f_{B,j}(m) \mid B \in H,\, j \notin Q(B)\} \cup \{i \mid i > f_{A,j}(m)\}. \quad (4)$$

Proof. From $A_{m,n}$, for $m \geq |A| + j$ and $n \geq \max(|A| + j, W(A, j))$, suppose we make move z. Then z will satisfy exactly one of the following: (I) $z \in A$ and $z < C$, $z < D$; (II) $z \in A$ and $z < C$, $z \parallel D$; (III) $z \in A$ and $z \parallel C$, $z < D$; (IV) $z \in A$ and $z \parallel C$, $z \parallel D$; (V) $z \in C$; or (VI) $z \in D$.

After a move of type (I), by Lemma 4, since $j \notin Q(A)$, we can never be in a position with g-value j.

After a move of type (II), $A_{m,n}$ becomes $B_{0,n}$, for some $B \subset A$. Since $n \geq |A| + j > |B| + j$, Lemma 5 gives us $g(B_{0,n}) \neq j$.

After a move of type (III), $A_{m,n}$ becomes $B_{m,0}$, for some $B \subset A$. Since $z \parallel c_{m+1}$ and $z < d_1$, $c_{m+1} \not> d_1$. Since $m \geq |A| + j > |B| + j$, Lemma 5 gives us $g(B_{m,0}) \neq j$.

After a move of type (IV), $A_{m,n}$ can become $B_{m,n}$, for any $B \subset A$, $B \in H$. Since $n \geq W(A, j)$, if $j \in Q(B)$ then $g(B_{m,n}) \neq j$. Hence, we need only look at sets B with $j \notin Q(B)$. Thus, there is a move of type (IV) from $A_{m,n}$ that leaves a position with g-value j if and only if $n \in \{f_{B,j}(m) \mid B \in H,\, j \notin Q(B)\}$.

After a move of type (V), $A_{m,n}$ can become $A_{i,n}$ for any $0 \leq i \leq m - 1$. Hence, there is a move of type (V) from $A_{m,n}$ that leaves a position with g-value j if and only if $n \in \{f_{A,j}(i) \mid 0 \leq i \leq m - 1\}$.

After a move of type (VI), $A_{m,n}$ can become $A_{m,i}$ for any $0 \leq i \leq n - 1$. Hence, there is a move of type (VI) from $A_{m,n}$ that leaves a position with g-value j if and only if $n > f_{A,j}(m)$ ($f_{A,j}(m)$ exists by Lemma 6), or, equivalently, $n \in \{i \mid i > f_{A,j}(m)\}$.

Combining these results, we see that there is a move from $A_{m,n}$ to a position with g-value j if and only if (4) holds. $\square$

Lemma 8. *Assume (3). For $k \notin Q(A)$ and $m \geq |A| + k + \max(|A| + k, W(A,k))$,*

$$\begin{aligned} f_{A,k}(m) - m = \min\Bigg(&\{-|A| - k, -|A| - k + 1, \ldots, |A| + k\} \cap \\ &\bigcap_{\substack{0 \leq j \leq k-1 \\ j \notin Q(A)}} \Bigg(\{f_{A,j}(i) - m \mid m - 2|A| - 2k \leq i \leq m - 1\} \\ &\cup \{f_{B,j}(m) - m \mid B \in H,\ j \notin Q(B)\} \cup \{i \mid i > f_{A,j}(m) - m\} \Bigg) \\ &- \{f_{A,k}(i) - m \mid 0 \leq i \leq m - 1,\ f_{A,k}(i) - m \geq -|A| - k\} \\ &- \{f_{B,k}(m) - m \mid B \in H,\ k \notin Q(B)\} \Bigg). \end{aligned} \tag{5}$$

Proof. By Lemma 6, $f_{A,k}(m)$ exists, and Lemma 5 constrains its possible values to the set $\{m-|A|-k, m-|A|-k+1, \ldots, m+|A|+k\}$. For each $n \in \{m-|A|-k, \ldots, m+|A|+k\}$, we need to check if $A_{m,n}$ can become a position with g-value j in one move for each j with $0 \leq j \leq k-1$, and we need to check if $A_{m,n}$ can become a position with g-value k in one move. The n which satisfies the first condition but not the second is the value of $f_{A,k}(m)$. Since $m \geq |A| + k + \max(|A| + k, W(A,k))$, $n \geq \max(|A| + k, W(A,k))$. Since $n \geq W(A,k)$, $A_{m,n}$ can, in one move, become a position with g-value j for any $j \in Q(A)$. Thus, we need only check the values of j for which $0 \leq j \leq k-1$ and $j \notin Q(A)$. Since $n \geq \max(|A|+k, W(A,k)) \geq \max(|A|+j, W(A,j))$, we can apply Lemma 7. The values of n (in the appropriate range) such that, in one move, $A_{m,n}$ can become a position with g-value j for every $0 \leq j \leq k-1$ are:

$$\begin{aligned} &\{m - |A| - k, m - |A| - k + 1, \ldots, m + |A| + k\} \cap \\ &\bigcap_{\substack{0 \leq j \leq k-1 \\ j \notin Q(A)}} \Bigg(\{f_{A,j}(i) \mid 0 \leq i \leq m-1\} \cup \{f_{B,j}(m) \mid B \in H,\ j \notin Q(B)\} \cup \{i \mid i > f_{A,j}(m)\} \Bigg) \end{aligned} \tag{6}$$

For $j < k$ and $j \notin Q(A)$, if $i \leq m - 2|A| - 2k$, then, by Lemma 5,

$$f_{A,j}(i) \leq i + |A| + j \leq (m - 2|A| - 2k) + |A| + (k-1) = m - |A| - k - 1,$$

so $f_{A,j}(i) \notin \{m - |A| - k, m - |A| - k + 1, \ldots, m + |A| + k\}$. Thus, we can add the restriction $i \geq m - 2|A| - 2k + 1$ to the set $\{f_{A,j}(i) \mid 0 \leq i \leq m-1\}$ without affecting the set (6). Since $m - 2|A| - 2k + 1 \geq 0$, we see that the set (6) is equal to:

$$\{m - |A| - k, m - |A| - k + 1, \dots, m + |A| + k\} \cap \bigcap_{\substack{0 \leq j \leq k-1 \\ j \notin Q(A)}} \Big(\{f_{A,j}(i) \mid m - 2|A| - 2k + 1 \leq i \leq m - 1\} \cup \{f_{B,j}(m) \mid B \in H,\, j \notin Q(B)\} \cup \{i \mid i > f_{A,j}(m)\} \Big) \quad (7)$$

Next, we need to eliminate the n for which there is a move from $A_{m,n}$ to a position with g-value k. Since $k \notin Q(A)$, $m \geq |A| + k$, and $n \geq \max(|A| + k, W(A, k))$, Lemma 7 says that there is a move from $A_{m,n}$ to a position with g-value k if and only if

$$n \in \{f_{A,k}(i) \mid 0 \leq i \leq m - 1\} \cup \{f_{B,k}(m) \mid B \in H,\, k \notin Q(B)\} \cup \{i \mid i > f_{A,k}(m)\}. \quad (8)$$

Combining (7) with (8), $f_{A,k}(m)$ is the single element of

$$\{m - |A| - k, m - |A| - k + 1, \dots, m + |A| + k\} \cap \bigcap_{\substack{0 \leq j \leq k-1 \\ j \notin Q(A)}} \Big(\{f_{A,j}(i) \mid m - 2|A| - 2k + 1 \leq i \leq m - 1\} \cup \{f_{B,j}(m) \mid B \in H,\, j \notin Q(B)\} \cup \{i \mid i > f_{A,j}(m)\} \Big) - \{f_{A,k}(i) \mid 0 \leq i \leq m - 1\} - \{f_{B,k}(m) \mid B \in H,\, k \notin Q(B)\} - \{i \mid i > f_{A,k}(m)\}$$

(this set consists of all n such that $g(A_{m,n}) = k$, and thus has the one element $f_{A,k}(m)$). Equivalently, $f_{A,k}(m)$ is the smallest element of

$$\{m - |A| - k, m - |A| - k + 1, \dots, m + |A| + k\} \cap \bigcap_{\substack{0 \leq j \leq k-1 \\ j \notin Q(A)}} \Big(\{f_{A,j}(i) \mid m - 2|A| - 2k + 1 \leq i \leq m - 1\} \cup \{f_{B,j}(m) \mid B \in H,\, j \notin Q(B)\} \cup \{i \mid i > f_{A,j}(m)\} \Big) - \{f_{A,k}(i) \mid 0 \leq i \leq m - 1\} - \{f_{B,k}(m) \mid B \in H,\, k \notin Q(B)\}.$$

This set is clearly the same as

$$\{m-|A|-k, m-|A|-k+1,\dots,m+|A|+k\} \cap \bigcap_{\substack{0\le j\le k-1\\ j\notin Q(A)}} \Big(\{f_{A,j}(i) \mid m-2|A|-2k+1 \le i \le m-1\} \cup \{f_{B,j}(m) \mid B\in H,\ j\notin Q(B)\} \cup \{i \mid i > f_{A,j}(m)\}\Big) - \{f_{A,k}(i) \mid 0\le i\le m-1,\ f_{A,k}(i) \ge m-|A|-k\} - \{f_{B,k}(m) \mid B\in H,\ k\notin Q(B)\}.$$

Subtracting m gives equation (5). □

Lemma 9. *Assume (3). For any A and k, with $k \notin Q(A)$, there exists $N_{A,k} \in \mathbb{N}_0$, $p_{A,k} \in \mathbb{N}$ such that, if $m \ge N_{A,k}$, then $f_{A,k}(m) - m = f_{A,k}(m+p_{A,k}) - (m+p_{A,k})$.*

Proof. We will prove this by strong double-induction on $|A|$ and k—for some A and k, assume that the lemma holds for all pairs (B,j) with $B \subset A$ (in particular, for $B \in H$), $j \le k$, and $j \notin Q(B)$, and also that it holds for all pairs (A,j) with $j<k$ and $j \notin Q(A)$. In the base case for the induction, these assumptions are vacuously true. Let

$$p = \operatorname{lcm}(\{p_{B,j} \mid B\in H,\ j\le k,\ j\notin Q(B)\} \cup \{p_{A,j} \mid j<k,\ j\notin Q(A)\})$$

with $\operatorname{lcm}(\emptyset)$ interpreted as 1, and let

$$N = \max(\{N_{B,j} \mid B\in H,\ j\le k,\ j\notin Q(B)\} \cup \{N_{A,j} \mid j<k,\ j\notin Q(A)\}) + |A| + k + \max(|A|+k, W(A,k))$$

with $\max(\emptyset)$ interpreted as 0. For $m \ge N$, from Lemma 8, we have the recursion (5).

Let $S(m) = \{f_{A,k}(i) - m \mid 0\le i\le m-1,\ f_{A,k}(i)-m \ge -|A|-k\}$. From Lemma 5, if $i \le m-1$, then $f_{A,k}(i) - m \le f_{A,k}(i) - i - 1 \le |A|+k-1$. Hence, $S(m) \subseteq \{-|A|-k,\dots,|A|+k-1\}$. This means that there are at most $2^{2|A|+2k} = 4^{|A|+k}$ possibilities for $S(m)$. Let m_p be the smallest nonnegative residue of $m \pmod p$. There are clearly p possibilities for m_p, so there are at most $4^{|A|+k}p$ possible pairs $(S(m), m_p)$. By the pigeonhole principle, there are two different numbers m_1, m_2 with $N \le m_1 < m_2 \le N + 4^{|A|+k}p$ such that $S(m_1) = S(m_2)$ and $(m_1)_p = (m_2)_p$.

By the construction of p and N, for $m \ge N$, all the sets on the right side of equation (5) other than $S(m)$, when viewed as functions of m, repeat with period p. Since $(m_1)_p = (m_2)_p$ and $S(m_1) = S(m_2)$, equation (5) implies that $f_{A,k}(m_1) - m_1 = f_{A,k}(m_2) - m_2 = r$. We also have $S(m_1+1) = S(m_2+1)$, since they can both be computed by inserting the element r into $S(m_1) = S(m_2)$, then subtracting 1 from every element of the resulting set, then eliminating from the set any element less than $(-|A|-k)$. Furthermore, clearly, $(m_1+1)_p = (m_2+1)_p$. Again, equation 5 implies that $f_{A,k}(m_1+1) - (m_1+1) = f_{A,k}(m_2+1) - (m_2+1)$. Repeating this argument, we get $f_{A,k}(m_1+i) - (m_1+i) = f_{A,k}(m_2+i) - (m_2+i)$ for all $i \ge 0$. Letting $N_{A,k} = m_1$ and $p_{A,k} = m_2 - m_1$, we get $f_{A,k}(m) - m = f_{A,k}(m+p_{A,k}) - (m+p_{A,k})$ for all $m \ge N_{A,k}$. □

4 Periodicity Theorem

Now that we have proven Lemma 9, we will drop the assumption (3) and prove our general theorem.

Poset Game Periodicity Theorem. *Assume (1). For any $k \in \mathbb{N}_0$, either there are only finitely many positions of the form $A_{m,n}$ with g-value k, or else there exists $N \in \mathbb{N}_0$, $p \in \mathbb{N}$ such that, for $m \geq N$, $f_{A,k}(m) - m = f_{A,k}(m+p) - (m+p)$.*

Proof. Note that the theorem holds for the sets C, D, A if and only if it holds for the sets $C' = C - \{c_1, \dots, c_\alpha\}$, $D' = D - \{d_1, \dots, d_\beta\}$, and $A' = A \cup \{c_1, \dots, c_\alpha\} \cup \{d_1, \dots d_\beta\}$. This occurs because, for $n \geq \alpha$, $f_{A,k}(n) - n = (f_{A',k}(n-\alpha) - (n-\alpha)) + (\beta - \alpha)$, so the left side is eventually periodic if and only if the right side is too; and, by Lemmas 1–3, there are only finitely many positions of the form $A_{m,n}$ with g-value k if and only if there are only finitely many positions of the form $A_{m,n}$ with g-value k and with $m > \alpha$, $n > \beta$, so $Q(A) = Q(A')$. By moving elements of C and D into A, we will show that we can assume (3), so that the theorem follows from Lemma 9.

Assume that $k \notin Q(A)$. If, for some i, $c_i > D$, then by Lemmas 2 and 4, $k \in Q(A)$, a contradiction. If, for any $a \in A$, $a > C$ or $a > D$, then $a \notin A_{m,n}$ for any m, n. Hence, removing a from A will have no effect on the truth or falsity or the theorem, so we can assume that no element of A is greater than all of C or all of D. We may thus assume that every element of $A \cup C \cup D$ is greater than only a finite number of other elements of $A \cup C \cup D$.

We have four steps. In the first step, we move all of the elements of $C \cup D$ that are less than an element of A into A, to get the new sets $A^{(1)} \supseteq A$, $C^{(1)} \subseteq C$, $D^{(1)} \subseteq D$. For each $a \in A^{(1)}$, either $a \parallel D$, or $a < d_i$ for all sufficiently large i. Thus, in the second step, we can move elements of $D^{(1)}$ into $A^{(1)}$ (making $A^{(2)} \supseteq A^{(1)}$, $C^{(2)} = C^{(1)}$, and $D^{(2)} \subseteq D^{(1)}$) so that, for each $a \in A^{(1)}$, $a \parallel D^{(2)}$ or $a < D^{(2)}$. By the same reasoning, in the third step, we can move elements of $C^{(2)}$ into $A^{(2)}$ so that, for each $a \in A^{(2)}$, $a < C^{(3)}$ or $a \parallel C^{(3)}$. Finally, in the fourth step, we move all elements of $D^{(3)}$ that are less than an element of $(C^{(2)} - C^{(3)})$ into $A^{(3)}$, to get $A^{(4)}$, $C^{(4)}$, $D^{(4)}$. Note that each of these steps moves only a finite number of elements.

From any $a \in A^{(1)}$, $a < C^{(4)}$ or $a \parallel C^{(4)}$ (from the third step), and $a < D^{(4)}$ or $a \parallel D^{(4)}$ (from the second step). For any $a \in (A^{(2)} - A^{(1)}) = (D^{(1)} - D^{(2)})$, $a < C^{(4)}$ or $a \parallel C^{(4)}$ (from the third step), and $a < D^{(4)}$. For any $a \in (A^{(3)} - A^{(2)}) = (C^{(2)} - C^{(3)})$, $a < C^{(4)}$, and $a \parallel D^{(4)}$ (from the fourth step and the fact that $c_i \not> d_j$). For any $a \in (A^{(4)} - A^{(3)}) = (D^{(3)} - D^{(4)})$, $a < C^{(4)}$ and $a < D^{(4)}$. Hence, the assumption (3) is satisfied by $A^{(4)}$, $C^{(4)}$, $D^{(4)}$. As noted above, replacing A, C, D by $A^{(4)}$, $C^{(4)}$, $D^{(4)}$ gives an equivalent problem, and by Lemma 9, we are done. $\square$

5 Consequences of the Periodicity Theorem

In the special case where $k = 0$, D is the top row in Chomp, and C is the second-to-top row in Chomp, this theorem resolves X. Sun's conjecture about the periodic behavior of $\mathcal{P}$-positions in Chomp [10].

By Lemma 4 and the Poset Game Periodicity Theorem, for any A and k, we can find an $M \in \mathbb{N}_0$ such that one of the following is true: (I) $f_{A,k}(m)$ does not exist for any $m \geq M$; (II) $f_{A,k}(m)$ is constant for $m \geq M$; or (III) $f_{A,k}(m) - m$ is periodic for $m \geq M$, with some period p.

We will say we have *solved* $f_{A,k}$ if we, in case (I), calculate $f_{A,k}(m)$ for every m where $f_{A,k}(m)$ is defined; in case (II), calculate M and $f_{A,k}(m)$ for all $m \leq M$ for which $f_{A,k}(m)$ is defined; or in case (III), calculate M, p, and $f_{A,k}(m)$ for all $m < M + p$ for which $f_{A,k}(m)$ is defined.

Lemma 10. *Given A, C, $D \subset X$ satisfying (1) and given $k \in \mathbb{N}_0$, we can solve $f_{A,k}$ in a finite amount of time.*

Proof. We will proceed by strong double-induction on $|A|$ and k. Assume (vacuously in the base case, and by the induction hypothesis otherwise) that we can solve $f_{B,j}$ in a finite amount of time for all $B \subseteq A$ and $j \leq k$ (besides when $B = A$ and $j = k$). Any solution to $f_{B,j}$ can be represented by a finite set of integers. Hence, we can test a solution to $f_{A,k}$ in a finite amount of time (by symbolically checking the recursive definition of g-value given in Section 1.4). If we systematically try the countable number of possible solutions to $f_{A,k}$ (of types (I), (II), and (III)), we will eventually find one that works, in a finite amount of time. $\square$

Corollary 1. *Given A, C, $D \subset X$ satisfying (1) and given $k \in \mathbb{N}_0$, we can calculate $f_{A,k}(m)$ (or show that it does not exist) in $O(\log m)$ time.*

Proof. By Lemma 10, after some finite amount of time independent of m, we can solve $f_{A,k}$. After that, if $f_{A,k}$ is type (I) or (II), $f_{A,k}(m)$ can be trivially calculated in $O(1)$ time. If $f_{A,k}$ is type (III), for large m, we can reduce $(m - M) \pmod{p}$ in $O(\log m)$ time (cf. [13]), and then calculate $f_{A,k}(m) = f_{A,k}(M + (m - M)_p)$ in $O(1)$ time. Thus, in any case, $f_{A,k}(m)$ can be calculated (or shown not to exist) in $O(\log m)$ time. $\square$

Corollary 2. *Given A, C, $D \subset X$ satisfying (1) and given $k \in \mathbb{N}_0$, and letting m and n vary, we can check whether $g(A_{m,n}) = k$ in $O(\log m)$ time.*

Proof. By Corollary 1, we can calculate $f_{A,k}(m)$ in $O(\log m)$ time. By Lemma 5, if it exists, $f_{A,k}(m) \leq m+|A|+k$. Hence, we can check if $n = f_{A,k}(m)$ in $O(\log(m+|A|+k)) = O(\log m)$ time. In total, then, we can check in $O(\log m)$ time. $\square$

Corollary 3. *Given A, C, $D \subset X$ satisfying (1), the poset game starting with $A_{m,n}$ for any m, $n \in \mathbb{N}_0$ has a polynomial-time winning strategy.*

Proof. Any game position in such a game can be written as $B_{i,j}$ for $B \subseteq A$ and i, $j \in \mathbb{N}_0$. Hence, the size of the input of a position $B_{i,j}$ is $(2^{|A|} + \log i + \log j)$.

By basic properties of g-values, in order to win, we must ensure that, after each move we make, the new position has g-value 0. Suppose we are in a position $B_{i,j}$ with $g(B_{i,j}) > 0$. We must find a winning move x, in $O(\log ij)$ time, that leaves a position with g-value 0. Before we start, we solve $f_{A',0}$ for each $A' \subseteq A$. By Lemma 10, this takes a fixed, finite amount of time. Next, we calculate $t = f_{B,0}(i)$. If $t < j$, then $x = d_{t+1}$. Otherwise, calculate $f_{A,0}(m)$ for $j - |A| \leq m \leq j + |A|$. If we get $f_{A,0}(m) = j$ and $c_{m+1} \in B_{i,j}$, then $x = c_{m+1}$. There are at most $|A|$ more possible winning moves x, and each leaves a position $A'_{i',j'}$ for $A' \subset A$, $i' \leq i$, $j' \leq j$. We check if any of these has g-value 0 to find a winning move x. By Corollary 2, we have taken $(3|A| + 2)O(\log i) = O(\log i)$ time, which is polynomial with respect to the input, so this is a polynomial-time winning strategy. □

Corollary 4. *In an infinite poset X, suppose there is an infinite chain C ($c_1 < c_2 < \cdots$), and a finite subset A disjoint from C. For $n \in \mathbb{N}_0$, let $A_n = A \cup C - \{x \in X \mid x \geq c_{n+1}\}$. For large enough n, $g(A_n) - n$ is periodic with respect to n.*

Proof. Create a new poset X' as the disjoint union of X with an infinite set $\{d_1, d_2, \dots\}$. In X', any element of D is incomparable to any element of X, the elements of X are ordered as they were before, and the elements of D form a chain $d_1 < d_2 < \cdots$. The poset game on a finite subset of X' is the disjoint sum of two games: the game with elements of D (a game of Nim), and the game with elements of $A \cup C$. We know that $g(\{d_1, \dots, d_n\}) = n$, since the g-value of a nim-heap of size n is n. Hence, by elementary properties of g-values, $g(A_m) = n$ if and only if $g(A_{m,n}) = 0$ if and only if $f_{A,0}(m) = n$, which means $g(A_m) = f_{A,0}(m)$. The corollary now follows directly from the Poset Game Periodicity Theorem. □

Note that in the case where the chain C is the top row in Chomp, this corollary proves a conjecture stated by X. Sun in [12].

Corollary 5. *Given A, $C \subset X$, we can calculate $g(A_n)$ in $O(\log n)$ time.*

Proof. From the proof of Corollary 4, calculating $g(A_n)$ is the same as calculating $f_{A,0}(n)$ in a different poset X'. Hence, this corollary follows from Corollary 1. □

6 Future Work

The Poset Game Periodicity Theorem creates several avenues for possible future work:

First, the realm of poset games contains several long-unsolved, well-studied games. In this paper, we mentioned some applications of the theorem to one such game, Chomp. We expect that the theorem will have applications to other such games as well, like Subset-Takeaway [8] or Schuh's Game of Divisors [7], and also infinite poset games such as Transfinite Chomp [14].

Second, we proved, in Corollary 2, that, given A, C, D, and k, one can check if $g(A_{m,n}) = k$ in polynomial time. A possible extension of this result, which would shed additional light on poset games, would be to show that the problem of calculating $g(A_{m,n})$ is in the complexity class NP (nondeterministic polynomial), by establishing a polynomial time limit that holds for all k at once.

Third, from the proof of the Poset Game Periodicity Theorem, explicit bounds could likely be calculated for both the starting point of periodic behavior and the period length, in any particular case. This would likely be useful both for theoretical and computational studies of these games.

Fourth, X. Sun's algorithm to calculate g-values in Chomp [12] could be made more efficient by implementing Corollary 4. Also, by the Poset Game Periodicity Theorem, his algorithm for calculating $\mathcal{P}$-positions in Chomp [10] could be altered to calculate positions with any fixed, small g-value, since these also have periodic patterns. Furthermore, if explicit bounds are calculated (as mentioned above), these might also aid in such algorithms.

Fifth, allowing n chains instead of only two adds a degree of complexity to the set of positions that result. Nevertheless, it is plausible that an analogue of the Poset Game Periodicity Theorem could hold in this case as well. If so, several poset games, like Schuh's Game of Divisors (which has been unsolved for half a century), could finally be completely solved, with a general polynomial-time winning strategy.

Finally, and most importantly, the Poset Game Periodicity Theorem shows that poset games are connected not only by common rules, but also by common structure, turning these seemingly unrelated problems into a unified field. Using this theorem as a foundation and starting point, we anticipate further study of the commonalities of poset games, extending the power and reach of this new field of combinatorial game theory.

Acknowledgments

I am grateful to my friends, parents, and teachers who helped me edit this paper, especially Gabriel Carroll and Dr. Jonathan Byrnes, and the anonymous referee who gave me valuable suggestions. I also would like to thank the Research Science Institute for sponsoring my exploration of Chomp in the first phase of this research, including Edward Early, Dr. John Rickert, and Dr. Hartley Rogers. Finally, I am grateful to Dr. Aviezri Fraenkel for his help with background information.

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