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# ON CONSECUTIVE INTEGER PAIRS WITH THE SAME SUM OF DISTINCT PRIME DIVISORS

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**Abstract**

We define the arithmetic function $P$ by $P(1) = 0$, and $P(n) = p_1 + p_2 + \cdots + p_k$ if $n$ has the unique prime factorization given by $n = \prod_{i=1}^{k} p_i^{a_i}$; we also define $\omega(n) = k$ and $\omega(1) = 0$. We study pairs $(n, n+1)$ of consecutive integers such that $P(n) = P(n+1)$. We prove that $(5, 6)$, $(24, 25)$, and $(49, 50)$ are the only such pairs $(n, n+1)$ where $\{\omega(n), \omega(n+1)\} = \{1, 2\}$. We also show how to generate certain pairs of the form $(2^{2n}pq, rs)$, with $p < q$, $r < s$ odd primes, and lend support to a conjecture that infinitely many such pairs exist.

**Keywords:** Ruth–Aaron pairs, cyclotomic polynomials, Pell sequences, primes

**Subject Class:** 11A25, 11Y55

## 1. Introduction

For positive integers $n$, we define the arithmetic function $P(n)$ by $P(1) = 0$, and, for a positive integer $n$ having as its unique prime factorization $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$,

$$P(n) = p_1 + p_2 + \cdots + p_k.$$

That is, $P(n)$ gives the sum of prime divisors of $n$ without multiplicity taken into account. The function is additive, in that $P(m) + P(n) = P(mn)$ if $(m, n) = 1$.

This function compares to the arithmetic function defined for positive integers $n$ by $S(1) = 0$ and $S(n) = \sum_{i=1}^{k} a_i p_i$ whenever $n = \prod_{i=1}^{k} p_i^{a_i}$; that is, $S(n)$ gives the sum of primes dividing $n$, taken with multiplicity. Then $S(n)$ is completely additive, in that $S(mn) = S(m) + S(n)$ for any two positive integers $m$ and $n$. A *Ruth–Aaron pair* is a pair $(n, n+1)$ of consecutive integers such that $S(n) = S(n+1)$. These were first discussed by Pomerance et. al. [4], and have been the subject of several articles (such as by Pomerance [6], Drost [2]) and numerous websites since.

However, in this article we are interested in finding pairs of consecutive positive integers

$(n, n+1)$, such that $P(n) = P(n+1)$. For the sake of easy reference, we may call these *Ruth–Aaron pairs of the second kind,* or RAP2s for short. Note, however, that a RAP2 is also an ordinary Ruth–Aaron pair if both members $n$ and $n+1$ are square-free.

Some observations regarding RAP2s are immediate. For example, the members $(n, n+1)$ of a RAP2 are of opposite parity, and are relatively prime. Let $n$ be a positive integer. If $n$ has the unique prime factorization $n = \prod_{i=1}^{k} p_i^{a_i}$, then the prime powers $p_i^{a_i}$, $1 \le i \le k$, are called the *components* of $n$, and we define $\omega(n) = k$, $\omega(1) = 0$ (thus $\omega$ counts the components of $n$). For any given RAP2 $(n, n+1)$, since 2 divides exactly one of the members (all other prime divisors of $n$ and $n+1$ being odd), we see that $\omega(n)$ and $\omega(n+1)$ are of opposite parity. In this article we shall completely determine all RAP2s $(n, n+1)$ whose members have one or two components. We will also investigate RAP2s of the form $(2^{2n}pq, rs)$, with $p < q$, $r < s$ odd primes.

## 2. Preliminaries

If $p$ is a prime and $a$, $m$, are positive integers we write $p^m \| a$ if $p^m \mid a$ and $p^{m+1} \nmid a$. In this case we say $p^m$ *exactly* divides $a$. For distinct primes $p$ and $q$ we write $e_p(q)$ to denote the exponent to which $q$ belongs modulo $p$.

For positive integers $n$, we denote the $n^{th}$ cyclotomic polynomial evaluated at $x$ by $\Phi_n(x)$. The cyclotomic polynomials (as shown by Niven [5], Ch. 3) may be defined inductively by

$$x^n - 1 = \prod_{d|n} \Phi_n(x). \tag{1}$$

By Theorems 94 and 95, Nagell [3], Ch. 5, we have

**Lemma 1.** *Let $p$ be and $q$ be odd primes and let $m$ be a positive integer. Let $h = e_p(q)$. Then $p \mid \Phi_m(q)$ if and only if $m = hp^j$ for some integer $j \ge 0$. If $j > 0$ then $p \| \Phi_{hp^j}(q)$.*

**Lemma 2.** *Let $q$ be an odd prime and let $m$ be a positive integer. Then $2 \mid \Phi_m(q)$ if and only if $m = 2^j$ for some integer $j \ge 0$. If $j > 1$ then $2 \| \Phi_{2^j}(q)$.*

Let $q$ be prime and let $m > 0$ be an integer. Since, by definition,

$$\Phi_m(q) = \prod_{\substack{k=1 \\ (k,m)=1}}^{m-1} (q - e^{2\pi ik/m}),$$

and since $\Phi_m(q) > 0$, we have

$$\Phi_m(q) = \prod_{\substack{k=1 \\ (k,m)=1}}^{m-1} \left| q - e^{2\pi ik/m} \right|,$$

and since $|q - e^{2\pi ik/m}| \ge q - 1$ for $1 \le k \le m-1$, we have

**Lemma 3.** *For a prime $q$ and an integer $m > 0$ we have $\Phi_m(q) \geq (q-1)^{\phi(m)}$.*

## 3. RAP2s of the form $(2^a p^b, q^c)$

The smallest numbers of components the members of a RAP2 can have are 1 and 2. In this instance, the members of the RAP2 have the form $2^a p^b$ and $q^c$ for positive integers $a$, $b$, and $c$, where $p$ and $q$ are necessarily twin (odd) primes (that is, $p+2=q$). We have two cases arising in this instance, those being $2^a p^b = q^c \pm 1$. In this section we consider the easier case of the two,

$$2^a p^b = q^c - 1. \tag{2}$$

Clearly $c > 1$, since $2^a p^b \geq 2(q-2) = q + (q-4) > q-1$. Thus, since $q = p+2$, (2) factors as

$$2^a p^b = (p+1)(q^{c-1} + q^{c-2} + \cdots + q + 1).$$

Since $(p, p+1) = 1$, it follows that $p+1 = 2^t$ for some positive integer $t$. Hence $p = 2^t - 1$, $q = 2^t + 1$, which is possible only if $t = 2$; that is, $p = 3$ and $q = 5$. Then (2) becomes

$$2^a 3^b = 5^c - 1.$$

Since $5^c \equiv 1 \pmod 3$, we have $2 \mid c$, so we write $c = 2\gamma$ for some positive integer $\gamma$. Thus

$$2^a 3^b = (5^\gamma - 1)(5^\gamma + 1).$$

Since $2 \| 5^\gamma + 1$, we must have $3 \mid 5^\gamma + 1$. Since $(5^\gamma + 1, 5^\gamma - 1) = 2$, we have $3 \nmid 5^\gamma - 1$. Hence

$$5^\gamma - 1 = 2^{a-1}, \qquad 5^\gamma + 1 = 2 \cdot 3^b.$$

Certainly $\gamma$ is odd (as $3 \nmid 5^\gamma - 1$). Suppose $\gamma > 1$. Then

$$5^\gamma - 1 = (5-1)(5^{\gamma-1} + 5^{\gamma-2} + \cdots + 5 + 1).$$

But the second factor is odd, and greater than 1; this contradicts $5^\gamma - 1 = 2^{a-1}$. Therefore $\gamma = 1$, and so $c = 2$. Hence (2) becomes $2^3 \cdot 3 = 5^2 - 1$; that is, $a = 3$, $b = 1$, and we have the RAP2 $(24, 25)$. Hence the only RAP2 of the form $(2^a p^b, q^c)$ is $(24, 25)$.

## 4. RAP2s of the form $(q^c, 2^a p^b)$

Suppose now that

$$2^a p^b = q^c + 1 \tag{3}$$

for positive integers $a$, $b$, and $c$, where $p$ and $q$ are primes such that $p + 2 = q$. This case is more difficult than that in Section 3.

By (1), we see that (3) is equivalent to

$$2^a p^b = \prod_{\substack{d \mid 2c \\ d \nmid c}} \Phi_d(q). \tag{4}$$

Let $h = e_p(q)$; we observe $h = e_p(2)$ as well (since $q = p + 2$). By Lemmas 1 and 2, each divisor $d \mid 2c$ such that $d \nmid c$ must either have the form $hp^j$ for some integer $j \geq 0$ or the form $2^k$ for some integer $k \geq 1$. Writing $c = 2^m s$ for some integer $m \geq 0$ and odd integer $s$, we see that $2^{m+1} \| d$ for all divisors $d \mid 2c$ such that $d \nmid c$. In particular $2^{m+1} \| h$.

Suppose $s$ is composite. Then $t \mid s$ for some odd integer $t$ such that $1 < t < s$. Then by (4), $\Phi_s(q)\Phi_t(q) \mid 2^a p^b$. This is impossible as $2 \nmid \Phi_s(q)\Phi_t(q)$ by Lemma 2, and, as $2 \mid h$, we have $h \nmid s$, and so $p \nmid \Phi_s(q)\Phi_t(q)$ by Lemma 1. Hence either $s$ is prime or $s = 1$.

Suppose $s$ is prime. Then $h = 2^{m+1}s$. For, if this were not the case then we would have $h = 2^{m+1}$; since $2 \nmid \Phi_{2c}(q)$ by Lemma 2, it follows that either $p \nmid \Phi_{2c}(q)$ (if $s \neq p$) or $p \| \Phi_{2c}(q)$ (if $s = p$) by Lemma 1. The former possibility clearly contradicts (4); the latter implies $\Phi_{2c}(q) = p$, which is impossible as $\Phi_{2c}(q) > p$ by Lemma 3.

Therefore, since $h = 2^{m+1}s$, we have

$$2^a p^b = \Phi_{2^{m+1}s}(q)\Phi_{2^{m+1}}(q) \tag{5}$$

by (4). This implies $m = 0$ because otherwise (5) is impossible since we have $p \nmid \Phi_{2^{m+1}}(q)$, $2 \| \Phi_{2^{m+1}}(q)$, and $\Phi_{2^{m+1}}(q) > 2$ by Lemmas 1, 2, and 3 respectively. Therefore $h = 2s$ and

$$2^a p^b = \Phi_2(q)\Phi_{2s}(q),$$

with $2^a = \Phi_2(q) = q+1$ and $p^b = \Phi_{2s}(q)$. But then $q = 2^a - 1$, so that $p = 2^a - 3$. It is clear that $a > 2$, hence $p \equiv 5 \pmod 8$. Thus 2 is a quadratic nonresidue of $p$, and hence $2^{(p-1)/2} \equiv -1 \pmod p$ by Euler's criterion. Since $2^2 \mid p - 1 = 2^a - 4$, it follows that $2^2 \mid e_p(2)$. But $e_p(2) = h$, and since $h = 2s$, we have $2 \| h$, a contradiction.

Therefore $s = 1$ and hence $c = 2^m$ for some integer $m \geq 0$. Thus (3) becomes

$$2^a p^b = q^{2^m} + 1. \tag{6}$$

First let us suppose that $m > 1$. Then $q^{2^m} \equiv 1 \pmod 4$ so that $a = 1$. Hence

$$2p^b = q^{2^m} + 1. \tag{7}$$

Since $p \mid q^{2^m} + 1 = \Phi_{2^{m+1}}(q)$, it follows from Lemma 1 that $h = 2^{m+1}$; recalling as well $h = e_p(2)$, it follows that $p \equiv 1 \pmod{2^{m+1}}$. Since $e_p(2) = 2^{m+1}$ and $\Phi_{2^{m+1}}(2) = 2^{2^m} + 1$, it follows from Lemma 1 that

$$p \mid 2^{2^m} + 1. \tag{8}$$

Suppose $p = 2^{m+1}t + 1$ for some odd integer $t$. Then, as $2^{2^m} \equiv -1 \pmod p$ by (8),

$$2^{(p-1)/2} = 2^{2^m t} = (2^{2^m})^t \equiv (-1)^t \equiv -1 \pmod p,$$

and hence $\left(\frac{2}{p}\right) = -1$ by Euler's criterion, where $\left(\frac{\cdot}{\cdot}\right)$ denotes the Legendre symbol. But, $p \equiv 1 \pmod 8$, which implies $\left(\frac{2}{p}\right) = +1$, a contradiction. Therefore

$$p \equiv 1 \pmod{2^{m+2}}. \tag{9}$$

Suppose $b > 2^m$. Then from (7) we have

$$2p^{b-2^m} = \left(1 + \frac{2}{p}\right)^{2^m} + \frac{1}{p^{2^m}}.$$

By (9), $p > 2^{m+2}$, and so

$$2p^{b-2^m} < \left(1 + \frac{1}{2^m}\right)^{2^m} + 1 < e + 1 < 4,$$

which implies $2p < 4$, a contradiction. On the other hand, suppose $b < 2^m$. Then by (6),

$$2 = \left(1 + \frac{2}{p}\right)^{b} (p+2)^{2^m - b} + \frac{1}{p^b} > (p+2)^{2^m - b} \geq p + 2,$$

a contradiction. Therefore we must have $b = 2^m$, so that (7) becomes

$$2p^{2^m} = q^{2^m} + 1. \tag{10}$$

Since $q = p + 2$, (10) becomes

$$\begin{aligned} p^{2^m} &= -p^{2^m} + (p+2)^{2^m} + 1 \\ &= \sum_{k=1}^{2^m} \binom{2^m}{k} p^{2^m - k} 2^k + 1. \qquad (11) \end{aligned}$$

Since by (9) $p > 2^{m+2}$, we have for each $k$ such that $1 \leq k \leq 2^m$,

$$\begin{aligned} \binom{2^m}{k} p^{2^m-k} 2^k &= \frac{2^m(2^m-1)(2^m-2)\cdots(2^m-k+1)}{k!} \cdot p^{2^m-k}2^k \\ &< \frac{2^{mk}}{k!} \cdot p^{2^m-k}2^k \\ &= \frac{1}{2^k} \cdot \frac{1}{k!} \cdot p^{2^m}. \end{aligned}$$

Hence by (11),

$$p^{2^m} < \sum_{k=1}^{2^m} \frac{1}{2^k} \cdot \frac{1}{k!} \cdot p^{2^m} + 1 < p^{2^m}\left(\sqrt{e} - 1 + \frac{1}{p^{2^m}}\right) < 0.8p^{2^m},$$

a contradiction.

Hence we have (6) with either $m = 0$ or $m = 1$. If $m = 0$ then (6) becomes

$$2^a p^b = q + 1 = p + 3,$$

implying $p \mid 3$, and hence $p = 3$, $q = 5$. Therefore $2^a 3^b = 6$, and so $a = b = 1$, and we have the RAP2 $(5, 6)$.

If $m = 1$ then (6) becomes

$$2^a p^b = q^2 + 1,$$

which implies $a = 1$ since $q^2 \equiv 1 \pmod 4$. Therefore

$$2p^b = q^2 + 1 = (p + 2)^2 + 1 = p^2 + 4p + 5,$$

implying $p \mid 5$, and hence $p = 5$, $q = 7$. Therefore $2 \cdot 5^b = 50$, and so $b = 2$, and we have the RAP2 $(49, 50)$. Therefore the only RAP2s of the form $(q^c, 2^a p^b)$ are $(5, 6)$ and $(49, 50)$.

We summarize our results from this and the previous section:

**Theorem 1.** *The only RAP2s* $(n, n+1)$ *such that* $\{\omega(n), \omega(n+1)\} = \{1, 2\}$ *are* $(5, 6)$, $(24, 25)$, *and* $(49, 50)$.

## 5. RAP2s of the form $(2^{2n}pq, rs)$

We now turn our attention to RAP2s $(n, n + 1)$ where $\{\omega(n), \omega(n + 1)\} = \{2, 3\}$. There are 88 such pairs less than $10^9$. Of these, 41 have the form $(4pq, rs)$, six have the form $(16pq, rs)$, and three have the form $(64pq, rs)$, with $p < q$, $r < s$ odd primes. Among the remaining 38 pairs, no discernable patterns emerged. These data led us to narrow our investigation to those pairs of the form $(2^{2n}pq, rs)$, $n \geq 1$.

Given such a pair, we have

$$2 + p + q = r + s, \tag{12}$$

$$2^{2n}pq + 1 = rs. \tag{13}$$

By (12) we have integers $x$, $y$, and $z$ such that

$$r = x - y, \qquad s = x + y, \tag{14}$$

$$p = x - 1 - z, \qquad q = x - 1 + z. \tag{15}$$

Substituting (14) and (15) into (13), and simplifying, gives us

$$((2^{2n} - 1)x - (2^{2n} + 1))(x - 1) = (2^n z - y)(2^n z + y),$$

which may be expressed as

$$\frac{(2^{2n} - 1)x - (2^{2n} + 1)}{2^n z - y} = \frac{2^n z + y}{x - 1} = \frac{a}{b}, \tag{16}$$

where $a/b$ represents the fractions in (16) in their lowest terms; thus $(a, b) = 1$. Separating the variables $x$, $y$, and $z$ in (16) gives us

$$(2^{2^n} - 1)bx + ay - 2^n az = (2^{2^n} + 1)b,$$
$$ax - by - 2^n bz = a,$$

which we solve for $x$, $y$, in terms of $z$:

$$(a^2 + (2^{2^n} - 1)b^2)x = 2^{n+1}abz + a^2 + (2^{2^n} + 1)b^2, \tag{17}$$

$$(a^2 + (2^{2^n} - 1)b^2)y = 2^n(a^2 - (2^{2^n} - 1)b^2)z + 2ab. \tag{18}$$

Our data of RAP2s less than $10^9$ revealed to us many different rational numbers for the quotient $a/b$ in (16), but some persisted more than others, especially 2/1 and 7/4 in the cases where $n = 1$. Recognizing these values as solutions to the Pell equation $a^2 - 3b^2 = 1$, we decided to assume that $a$, $b$, solved the Pell equation

$$a^2 - (2^{2^n} - 1)b^2 = 1 \tag{19}$$

in the general case for $n \geq 1$. Under this hypothesis, (17) and (18) simplify to

$$\begin{aligned} (2a^2 - 1)x &= 2^{n+1}abz + 2a^2 + 2b^2 - 1, \\ (2a^2 - 1)y &= 2^n z + 2ab. \end{aligned} \tag{20}$$

It is well known (e.g., as shown by Shockley [7], Ch. 12) that all positive solutions to (19) are given by

$$\begin{aligned} a_1 &= 2^n, & b_1 &= 1, \\ a_{j+1} &= 2^n a_j + (2^{2^n} - 1)b_j & &(j \geq 1), \\ b_{j+1} &= a_j + 2^n b_j & &(j \geq 1). \end{aligned} \tag{21}$$

One shows by induction that $2^n \mid a_j b_j$ for all $j \geq 1$. Hence we may parametrize $z$ from (20): since $y$ is an integer it follows that $2a^2 - 1$ divides $2^n z + 2ab$, and since $2a^2 - 1$ is odd we have

$$z \equiv -\frac{2ab}{2^n} \pmod{2a^2 - 1}.$$

Thus $z$ has the form given by

$$z = (2a^2 - 1)k + 2a^2 - 1 - \frac{2ab}{2^n} \tag{22}$$

for integers $k \geq 0$. Substituting (22) into (17) and (18) gives us

$$x = 2^{n+1}abk + 2^{n+1}ab - 2b^2 + 1, \tag{23}$$

$$y = 2^n k + 2^n. \tag{24}$$

Substituting (22), (23), and (24) into (14) and (15) gives us

**Theorem 2.** *Let integral $n \geq 1$ be given and let $a$, $b$, be solutions to the Pell equation (19). Then $(2^{2n}pq, rs)$ is a RAP2 if, for an integer $k \geq 0$, the following four quantities are all prime:*

$$\begin{aligned} p &= 2(2^{n+1}ab - 2a^2 + 1)k + \left(2^{n+1} - 2b^2 - 2a^2 + 1 + \frac{2ab}{2^n}\right), \\ q &= 2(2^{n+1}ab + 2a^2 - 1)k + \left(2^{n+1} - 2b^2 + 2a^2 - 1 - \frac{2ab}{2^n}\right), \\ r &= 2^{n+1}(2ab - 1)k + 2^n(2ab - 1) - 2b^2 + 1, \\ s &= 2^{n+1}(2ab + 1)k + 2^n(2ab + 1) - 2b^2 + 1. \end{aligned}$$

Note that we substituted $2k$ instead of $k$ to ensure $p$ and $q$ as given in Theorem 2 are odd. We also kept $2ab$ in the numerators above (rather than reduce to $ab/2^{n-1}$) since the Pell sequences (21) have the property $b_{2j} = 2a_jb_j$ (as well as $a_{2j} = 2a_j^2 - 1$). Moreover, one shows by induction that for all $n$ and $k$, if $a_{3j}$, $b_{3j}$ in (21) are the solutions used in applying Theorem 2, then at least one of $p$, $q$, $r$, and $s$ is divisible by 3 (hence no RAP2 is produced).

There are 149 RAP2s of the form $(2^{2n}pq, rs)$ whose elements are less than $2^{34}$. Of these, 116 correspond to $n = 1$, and 16 of these involve the solutions $a_1 = 2$, $b_1 = 1$ of the Pell equation $a^2 - 3b^2 = 1$, while an additional 3 involve $a_2 = 7$, $b_2 = 4$. Also, 16 such RAP2s correspond to $n = 2$, 3 of which involve the solutions $a_1 = 4$, $b_1 = 1$ of the equation $a^2 - 15b^2 = 1$, and 9 of the RAP2s correspond to $n = 3$, 3 of which involve $a_1 = 8$, $b_1 = 1$ $(a^2 - 63b^2 = 1)$. Finally, 3 of the RAP2s involve $n = 4$.

We had found the RAP2s less than $2^{34}$ by a straightforward computer search. Later on, we applied Theorem 2 to search for the RAP2s of the special form described in that theorem. We found literally thousands of them. We computed them on a PC, using the UBASIC software package. Primality of $p$, $q$, $r$, $s$, were verified by the APR primality test due to Adleman, Pomerance, and Rumely [1].

## 6. Concluding Remarks

It is unknown if there are infinitely many RAP2s. The question of infinitude also remains open for ordinary Ruth–Aaron pairs—see Pomerance [6] for a detailed history. In light of Theorem 2, fixing $n$ at say $n = 1$, if one could show that for each solution $a_j$, $b_j$, $3 \nmid j$, to (19), there exists at least one $k$ for which $p$, $q$, $r$, $s$, are all prime, then a proof of infinitely many RAP2s of the form $(4pq, rs)$ would be obtained. We have not been able to produce such a proof, but we conjecture the existence of infinitely many RAP2s nonetheless.

We have also considered RAP2s $(n, n+1)$ for which $\{\omega(n), \omega(n+1)\} = \{1, 4\}$. These would be obtained by finding distinct odd primes $p_1$, $p_2$, $p_3$, $q$, and positive integers $a$, $b_1$, $b_2$, $b_3$, $c$, such that

$$2 + p_1 + p_2 + p_3 = q, \tag{25}$$

and such that

$$2^a p_1^{b_1} p_2^{b_2} p_3^{b_3} = q^c \pm 1. \tag{26}$$

Let $h = [e_{p_1}(q), e_{p_2}(q), e_{p_3}(q)]$. Then $p_1$, $p_2$, $p_3$, all divide $q^c - 1$ only if $h \mid c$, in which case $q^h - 1$ divides $2^a p_1^{b_1} p_2^{b_2} p_3^{b_3}$. Thus if $q^h - 1$ is found to contain any prime factors other than 2, $p_1$, $p_2$, $p_3$, then a contradiction is obtained. Using modular arithmetic, we can find $\alpha$, $\beta_1$, $\beta_2$, $\beta_3$, such that $2^\alpha \| q^h - 1$ and $p_i^{\beta_i} \| q^h - 1$ $(1 \le i \le 3)$. A contradiction is obtained if $2^\alpha p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} < q^h - 1$.

In the case of $q^c + 1$, (26) becomes

$$2^a p_1^{b_1} p_2^{b_2} p_3^{b_3} = \prod_{\substack{d \mid 2c \\ d \nmid c}} \Phi_d(q)$$

by (1). By Lemma 1, the primes $p_1$, $p_2$, $p_3$, all divide $q^c + 1$ only if $e_{p_1}(q)$, $e_{p_2}(q)$, and $e_{p_3}(q)$ are all even such that each quantity is exactly divisible by the same power of 2. In this case we have $q^{h/2} + 1 \mid 2^a p_1^{b_1} p_2^{b_2} p_3^{b_3}$. Thus a contradiction is obtained if $q^{h/2} + 1$ contains any prime factors other than 2, $p_1$, $p_2$, or $p_3$.

For all odd primes $q < 20000$, we found all triples of odd primes $p_1 < p_2 < p_3$ satisfying (25), and then we disproved the possibility of (25) and (26) by computation. We conjecture the nonexistence of RAP2s $(n, n+1)$ for which $\{\omega(n), \omega(n+1)\} = \{1, 4\}$, although we have not yet obtained a proof.

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