RELATIONS AMONG FOURIER COEFFICIENTS OF CERTAIN ETA PRODUCTS

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Abstract

Certain arithmetic relations for the coefficients in the expansions of $(q)_\infty^r$, $(q)_\infty^r(q^t)_\infty^s$, $t = 2, 3, 4$, were studied by M. Newman, S. Cooper, M. D. Hirschhorn, R. Lewis, S. Ahlgren and R. Chapman. In this work, we prove similar identities for certain multi-product expansions using an elementary method.

1. Introduction

For an integer r, let

$$(q)_\infty^r = \prod_{n=1}^{\infty}(1-q^n)^r = \sum_{n\geq 0} a_r(n)q^n, \tag{1.1}$$

where $q = e^{2\pi i z}$ and $\mathrm{Im}(z) > 0$ and let

$$f_j(q) = (q)_\infty^{r_j}(q^2)_\infty^{s_j}(q^4)_\infty^{t_j}, \tag{1.2}$$

where r_j, s_j, t_j are certain specific integers (see theorem). In this article we consider the following products

$$f_i(q^\ell)f_k(q^m) = \sum_{n=0}^{\infty} a(n)q^n \quad (l, m > 0),$$

and prove certain identities involving the Fourier coefficients $a(n)$ by elementary arguments. Similar identities for eta powers and products of two eta functions were earlier obtained by several authors [1, 3, 4, 5].

2. Statement of theorem

Let q be a complex number satisfying $|q| < 1$. It is readily checked that $(-q)_\infty = \dfrac{(q^2)^3_\infty}{(q)_\infty (q^4)_\infty}$. Let

$$\begin{aligned}
f_1(q) &= (q)_\infty, & f_7(q) &= f_1(-q) = \frac{(q^2)^3_\infty}{(q)_\infty (q^4)_\infty},\\
f_2(q) &= (q)^3_\infty, & f_8(q) &= f_2(-q) = \frac{(q^2)^9_\infty}{(q)^3_\infty (q^4)^3_\infty},\\
f_3(q) &= \frac{(q)^2_\infty}{(q^2)_\infty}, & f_9(q) &= f_3(-q) = \frac{(q^2)^5_\infty}{(q)^2_\infty (q^4)^2_\infty},\\
f_4(q) &= \frac{(q^2)^2_\infty}{(q)_\infty}, & f_{10}(q) &= f_4(-q) = \frac{(q)_\infty (q^4)_\infty}{(q^2)_\infty},\\
f_5(q) &= \frac{(q)^5_\infty}{(q^2)^2_\infty}, & f_{11}(q) &= f_5(-q) = \frac{(q^2)^{13}_\infty}{(q)^5_\infty (q^4)^5_\infty},\\
f_6(q) &= \frac{(q^2)^5_\infty}{(q)^2_\infty}, & f_{12}(q) &= f_6(-q) = \frac{(q)^2_\infty (q^4)^2_\infty}{(q^2)_\infty}.
\end{aligned}$$

Observe that each function $f_i(q)$ has the form $f_i(q) = (q)^{r_i}_\infty (q^2)^{s_i}_\infty (q^4)^{t_i}_\infty$, for certain integers r_i, s_i, t_i. By the triple product and quintuple product identities, we have [2, pp. 64–65 and 306–307], [4]

$$\begin{aligned}
f_1(q) &= \sum_{\alpha \equiv 1 \pmod 6} (-1)^{(\alpha-1)/6} q^{(\alpha^2-1)/24},\\
f_2(q) &= \sum_{\alpha \equiv 1 \pmod 4} \alpha q^{(\alpha^2-1)/8},\\
f_3(q) &= \sum_{\alpha} (-1)^{\alpha} q^{\alpha^2},\\
f_4(q) &= \sum_{\alpha \equiv 1 \pmod 4} q^{(\alpha^2-1)/8},\\
f_5(q) &= \sum_{\alpha \equiv 1 \pmod 6} \alpha q^{(\alpha^2-1)/24},\\
f_6(q) &= \sum_{\alpha \equiv 1 \pmod 3} (-1)^{\alpha-1} \alpha q^{(\alpha^2-1)/3}, \qquad (2.1)
\end{aligned}$$

where in each case the sum is over all integers α, positive and negative, satisfying the given congruence.

For $1 \le i \le 12$, let $d_i = r_i + 2s_i + 4t_i$ and $\lambda_i = \left\lceil \frac{1}{2}(r_i + s_i + t_i) \right\rceil - 1$. Let

$$\begin{aligned}(e_1, e_2, \cdots, e_{12}) &= (1,3,24,3,1,8,1,3,24,3,1,8),\\ (n_1, n_2, \cdots, n_{12}) &= (6,4,1,4,6,3,6,4,1,4,6,3).\end{aligned}$$

Observe that $d_i = e_i$ unless $i = 3$ or 9, in which case $d_3 = d_9 = 0$. For $1 \le i \le 12$ and p an odd prime, define $\epsilon_i(p) = \left(\dfrac{a_i}{p}\right)$, where $(a_1, \ldots, a_{12}) = (3,-1,1,1,-3,-3,6,-2,1,2,-6,-3)$. The main purpose of this article is to prove the following result.

Theorem. *Let ℓ and m be positive integers, and let $1 \le j,k \le 12$. Let $p > 3$ be any prime satisfying $\left(\frac{-e_je_k\ell m}{p}\right) = -1$ and put $\Delta = \frac{p^2-1}{24}$. Let $f_j(q^\ell)f_k(q^m) = \sum_{n=0}^{\infty} a(n)q^n$. Then the coefficients $a(n)$ satisfy*

$$a(pn + (\ell d_j + m d_k)\Delta) = \epsilon_j(p)\epsilon_k(p)p^{\lambda_j+\lambda_k} a\left(\frac{n}{p}\right).$$

Example. $(j = 5,\ k = 10)$ We have $f_5(q) = (q)_\infty^5(q^2)_\infty^{-2}$ and $f_{10}(q) = (q)_\infty(q^2)_\infty^{-1}(q^4)_\infty$, so $(r_5, s_5, t_5) = (5,-2,0)$, $(r_{10}, s_{10}, t_{10}) = (1,-1,1), d_5 = r_5 + 2s_5 + 4t_5 = 1$, $d_{10} = r_{10} + 2s_{10} + 4t_{10} = 3, \lambda_5 = \left\lceil \frac{1}{2}(r_5 + s_5 + t_5) \right\rceil - 1 = 1$, $\lambda_{10} = \left\lceil \frac{1}{2}(r_5 + s_5 + t_5) \right\rceil - 1 = 0. e_5 = 1$, $e_{10} = 3, \epsilon_5(p) = \left(\frac{-3}{p}\right)$, and $\epsilon_{10}(p) = \left(\frac{2}{p}\right)$. Let p be any prime satisfying $\left(\dfrac{-e_5e_{10}\ell m}{p}\right) = -1$, i.e., $\left(\dfrac{-3\ell m}{p}\right) = -1$. Let $f_5(q^\ell)f_{10}(q^m) = \sum_{n=0}^{\infty} a(n)q^n$. Then the Theorem implies

$$a(pn + (\ell + 3m)\Delta) = \left(\frac{-3}{p}\right)\left(\frac{2}{p}\right) p\, a\left(\frac{n}{p}\right),$$

i.e.,

$$a(pn + (\ell + 3m)\Delta) = \left(\frac{-6}{p}\right) p\, a\left(\frac{n}{p}\right).$$

3. Proofs

We shall require the following elementary lemma, which we state without further comment.

Lemma. *Let ℓ and m be positive integers and let p be an odd prime satisfying $\left(\dfrac{-\ell m}{p}\right) = -1$. Let α and β be integers satisfying $\ell\alpha^2 + m\beta^2 \equiv 0 \pmod{p}$. Then $\alpha \equiv 0 \pmod{p}$ and $\beta \equiv 0 \pmod{p}$.*

In order to illustrate the technique, we first prove the example, before proving the general statement of the theorem.

Proof of example. We have

$$f_5(q^\ell)f_{10}(q^m) = \sum_{\alpha\equiv 1 \pmod 6} \alpha q^{\ell(\alpha^2-1)/24} \sum_{\beta\equiv 1 \pmod 4} (-q^m)^{(\beta^2-1)/8},$$

so

$$\begin{aligned} a(n) &= \sum_{\substack{\alpha\equiv 1 \pmod 6,\ \beta\equiv 1 \pmod 4 \\ \ell(\alpha^2-1)/24+m(\beta^2-1)/8=n}} \alpha(-1)^{(\beta^2-1)/8} \\ &= \sum_{\substack{\alpha\equiv 1 \pmod 6,\ \beta\equiv 1 \pmod 4 \\ \ell\alpha^2+3m\beta^2=24n+\ell+3m}} \alpha(-1)^{(\beta^2-1)/8}. \end{aligned}$$

Therefore

$$a(pn+(\ell+3m)\Delta) = \sum_{\substack{\alpha\equiv 1 \pmod 6,\ \beta\equiv 1 \pmod 4 \\ \ell\alpha^2+3m\beta^2=24pn+(\ell+3m)p^2}} \alpha(-1)^{(\beta^2-1)/8}. \tag{3.1}$$

Now $\ell\alpha^2+3m\beta^2 \equiv 0 \pmod p$, and the lemma implies $p \mid \alpha$, $p \mid \beta$. Let

$$\alpha = \left(\frac{-3}{p}\right) p\alpha', \quad \beta = \left(\frac{-1}{p}\right) p\beta'. \tag{3.2}$$

Then $\alpha' \equiv 1 \pmod 6$ and $\beta' \equiv 1 \pmod 4$. Also, modulo 2,

$$\begin{aligned} \frac{\beta^2-1}{8} - \frac{\beta'^2-1}{8} &= \frac{\beta^2-\beta'^2}{8} \\ &= \frac{(p^2-1)\beta'^2}{8} \\ &\equiv \frac{p^2-1}{8} \\ &\equiv \begin{cases} 0 & \text{if } p\equiv 1 \text{ or } 7 \pmod 8 \\ 1 & \text{if } p\equiv 3 \text{ or } 5 \pmod 8 \end{cases} \end{aligned}$$

Therefore

$$(-1)^{(\beta^2-1)/8} = \left(\frac{2}{p}\right)(-1)^{(\beta'^2-1)/8}. \tag{3.3}$$

Substituting (3.2) and (3.3) into (3.1) we get

$$\begin{aligned} a(pn+(\ell+3m)\Delta) &= \sum_{\substack{\alpha'\equiv 1 \pmod 6,\ \beta'\equiv 1 \pmod 4 \\ \ell\alpha'^2+3m\beta'^2=24n/p+\ell+3m}} \left(\frac{-3}{p}\right) p\alpha' \left(\frac{2}{p}\right)(-1)^{(\beta'^2-1)/8} \\ &= \left(\frac{-6}{p}\right) p\, a\left(\frac{n}{p}\right). \end{aligned}$$

This completes the proof of the example. □

Proof of Theorem. Writing

$$f_j(q^\ell)f_k(q^m) = \sum_{n=0}^{\infty} a(n)q^n,$$

we have, using (2.1),

$$a(n) = \sum_{\substack{\alpha\equiv 1 \pmod{n_j},\ \beta\equiv 1 \pmod{n_k} \\ e_j\ell\alpha^2+e_k m\beta^2=24n+d_j\ell+d_k m}} \phi_j(\alpha)\phi_k(\beta),$$

where

$$\begin{array}{ll} \phi_1(\alpha)=(-1)^{(\alpha-1)/6}, & \phi_7(\alpha)=(-1)^{(\alpha-1)/6+(\alpha^2-1)/24}, \\ \phi_2(\alpha)=\alpha, & \phi_8(\alpha)=\alpha(-1)^{(\alpha^2-1)/8}, \\ \phi_3(\alpha)=(-1)^{\alpha}, & \phi_9(\alpha)=1, \\ \phi_4(\alpha)=1, & \phi_{10}(\alpha)=(-1)^{(\alpha^2-1)/8}, \\ \phi_5(\alpha)=\alpha, & \phi_{11}(\alpha)=\alpha(-1)^{(\alpha^2-1)/24}, \\ \phi_6(\alpha)=(-1)^{\alpha-1}\alpha, & \phi_{12}(\alpha)=(-1)^{\alpha-1+(\alpha^2-1)/3}\alpha. \end{array}$$

Therefore

$$a(pn+(\ell d_j+md_k)\Delta) = \sum_{\substack{\alpha\equiv 1 \pmod{n_j},\ \beta\equiv 1 \pmod{n_k} \\ e_j\ell\alpha^2+e_k m\beta^2=24pn+p^2(d_j\ell+d_k m)}} \phi_j(\alpha)\phi_k(\beta).$$

Observe that $e_j\ell\alpha^2+e_km\beta^2\equiv 0 \pmod{p}$. The Lemma implies $p\mid\alpha$, $p\mid\beta$. Let

$$\alpha = \begin{cases} \left(\frac{-3}{p}\right)p\alpha' & \text{if } j=1,\ 5,\ 6,\ 7,\ 11 \text{ or } 12 \\ \left(\frac{-1}{p}\right)p\alpha' & \text{if } j=2,\ 4,\ 8 \text{ or } 10 \\ p\alpha' & \text{if } j=3 \text{ or } 9, \end{cases}$$

$$\beta = \begin{cases} \left(\frac{-3}{p}\right)p\beta' & \text{if } k=1,\ 5,\ 6,\ 7,\ 11 \text{ or } 12 \\ \left(\frac{-1}{p}\right)p\beta' & \text{if } k=2,\ 4,\ 8 \text{ or } 10 \\ p\beta' & \text{if } k=3 \text{ or } 9. \end{cases}$$

Then it is easily verified that $\alpha'\equiv 1 \pmod{n_j}$ and $\beta'\equiv 1 \pmod{n_k}$, and that $\phi_j(\alpha) = \epsilon_j(p)\,p^{\lambda_j}\,\phi_j(\alpha')$ and $\phi_k(\beta)=\epsilon_k(p)\,p^{\lambda_k}\,\phi_k(\beta')$. Consequently,

$$\begin{aligned} a(pn+(\ell d_j+md_k)\Delta) &= \sum_{\substack{\alpha'\equiv 1 \pmod{n_j},\ \beta'\equiv 1 \pmod{n_k} \\ e_j\ell\alpha'^2+e_k m\beta'^2=24n/p+d_j\ell+d_k m}} \epsilon_j(p)\epsilon_k(p)p^{\lambda_j+\lambda_k}\phi_j(\alpha')\phi_k(\beta') \\ &= \epsilon_j(p)\epsilon_k(p)p^{\lambda_j+\lambda_k}\,a\left(\tfrac{n}{p}\right). \end{aligned}$$

This completes the proof of the theorem. □

Remark. Though our theorem can be proved using the theory of lacunary modular forms, we prefer to present an elementary proof for its simplicity.

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