TWO q–IDENTITIES FROM THE THEORY OF FOUNTAINS AND HISTOGRAMS PROVED WITH A TRI–DIAGONAL DETERMINANT

Helmut Prodinger[1]

The John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics, University of the Witwatersrand, P. O. Wits, 2050 Johannesburg, South Africa

helmut@maths.wits.ac.za

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Abstract

Two identities required in the theory of fountains and histograms are easily proved by expanding a tri–diagonal determinant (reminiscent of Schur's) in two different ways.

We consider the following infinite tri-diagonal determinant (elements not displayed are zero)

$$\mathsf{Schur}(x) := \begin{vmatrix} 1 & & \overbrace{0\dots0}^{p-2} & & xq^1 & & & & \dots \\ -1 & 1 & & 0\dots0 & & xq^2 & & & \dots \\ & -1 & 1 & & 0\dots0 & & xq^3 & & \dots \\ & & -1 & 1 & & 0\dots0 & & xq^4 & \dots \\ & & & \ddots & \ddots & & \ddots & & \end{vmatrix}.$$

Schur, when providing his proof of the Rogers–Ramanujan identities in 1917 [3] used a similar determinant; since I am advocating that Schur's work deserves to be better known, I use the name $\mathsf{Schur}(x)$. This short note shows that two identities that were required in the study of fountains and histograms [1] are most easily proved by expanding the determinant in two different ways.

Expanding the determinant with respect to the first column ("top–recursion") we get

$$\mathsf{Schur}(x) = \mathsf{Schur}(xq) + (-1)^p xq\ \mathsf{Schur}(xq^p).$$

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Setting

$$\mathsf{Schur}(x)=\sum_{n\ge 0}a_nx^n,$$

we get, upon comparing coefficients,

$$a_n=q^na_n+(-1)^pq^{1+p(n-1)}a_{n-1}=\frac{(-1)^pq^{1+p(n-1)}}{1-q^n}.$$

Since $a_0=1$, iteration leads to

$$a_n=\frac{q^{n+p\binom{n}{2}}(-1)^{pn}}{(1-q)(1-q^2)\dots(1-q^n)}.$$

Therefore

$$\begin{aligned}\mathsf{Schur}((-q)^{p-1})&=\sum_{n\ge 0}\frac{(-1)^nq^{n+p\binom{n}{2}}}{(1-q)(1-q^2)\dots(1-q^n)}q^{(p-1)n}\\&=\sum_{n\ge 0}\frac{(-1)^nq^{p\binom{n+1}{2}}}{(1-q)(1-q^2)\dots(1-q^n)}.\end{aligned}$$

Now consider the *finite* determinants $\mathsf{Schur}_n(x)$, obtained from $\mathsf{Schur}(x)$ by taking the first n rows and columns. Expanding this determinant with respect to the last row ("bottom–recursion") we get

$$\mathsf{Schur}_n(x)=\mathsf{Schur}_{n-1}(x)+(-1)^pxq^{n-p+1}\,\mathsf{Schur}_{n-p}(x).$$

In particular,

$$\mathsf{Schur}_n((-q)^{p-1})=\mathsf{Schur}_{n-1}((-q)^{p-1})-q^n\,\mathsf{Schur}_{n-p}((-q)^{p-1}),$$

and $\mathsf{Schur}_j((-q)^{p-1})=1$ for $j=0,\dots,p-1$. The quantities $\mathsf{Schur}_n((-q)^{p-1})$ were called E_n in [1] (with matching initial conditions $E_j=1$ for $j=0,\dots,p-1$). Whence we proved

$$\lim_{m\to\infty}E_m=\sum_{n\ge 0}\frac{(-1)^nq^{p\binom{n+1}{2}}}{(1-q)(1-q^2)\dots(1-q^n)}.$$

Merlini and Sprugnoli had asked for a direct proof, which was given in [2], by showing an explicit form for E_m. The present proof avoids this and is thus simpler.

A second (similar) formula was also requested, namely

$$\lim_{m\to\infty}D_m=\sum_{n\ge 0}\frac{(-1)^nq^{n+p\binom{n}{2}}}{(1-q)(1-q^2)\dots(1-q^n)}.$$

for $D_n=D_{n-1}-q^nD_{n-p}$ and (different) initial values $D_j=1-\sum_{i=1}^{j}q^i$ for $j=0,\dots,p-1$. This follows immediately by setting $D_n=\mathsf{Schur}_{n+p-1}((-1)^{p-1})$.

References

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