A COMMON GENERALIZATION OF SOME IDENTITIES[1]

Zhizheng Zhang
Department of Mathematics, Luoyang Teachers' College, Luoyang 471022, P. R. China
College of Mathematics and Information Science, Henan University, Kaifeng 475001, P. R. China
zhzhzhang-yang@163.com

Jun Wang
Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, P. R. China
junwang@dlut.edu.cn

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Abstract

By means of partial fraction decomposition, this paper provides an algebraic identity from which a lot of interesting identities follow.

1. Notation and Introduction

In [2], by the WZ method, the following identity was confirmed successfully.

$$\sum_{k=1}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \{1 + 2kH_{n+k} + 2kH_{n-k} - 4kH_k\} = 0, \tag{1}$$

where $H_0 = 0$ and $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Ahlgren and Ono [1] have shown that Beukers' conjecture is implied by this beautiful binomial identity. See [3] or [1, Theorem 7] for Beukers' conjecture. In [4] and [6], by means of partial fraction decomposition, W.-C. Chu gave beautiful proofs of (1) and a number of interesting combinatorial identities. At the same time, Identity (1) was also extended.

The purpose of this paper is to obtain an algebraic identity by means of partial fraction decomposition. As applications, we show a number of interesting identities.

Throughout the paper, $f(x)$ is known for an arbitrary polynomial of degree $< \lambda n$ and the function $A(T_1,\ T_2,\ \ldots,\ T_n)$ is defined by

$$A(T_1,\ T_2,\ \ldots,\ T_n) = \begin{cases} 1, & if\ n = 0, \\ \sum \frac{n!}{k_1!k_2!\ldots k_n!} \left(\frac{T_1}{1}\right)^{k_1} \left(\frac{T_2}{2}\right)^{k_2} \ldots \left(\frac{T_n}{n}\right)^{k_n}, & if\ n > 0, \end{cases}$$

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where the sum is taken over all nonnegative integers k_i's such that $k_1 + 2k_2 + \ldots + nk_n = n$. For example, $A(T_1,\ T_2,\ \ldots,\ T_n)$ for $n \leq 4$ are:

$$\begin{aligned} A(T_1) &= T_1 \\ A(T_1,\ T_2) &= T_1^2 + T_2 \\ A(T_1,\ T_2,\ T_3) &= T_1^3 + 3T_1T_2 + 2T_3 \\ A(T_1,\ T_2,\ T_3,\ T_4) &= T_1^4 + 8T_1T_3 + 6T_1^2T_2 + 3T_2^2 + 6T_4. \end{aligned}$$

2. Main Result

Theorem 2.1 *Let $a_1,\ a_2,\ \ldots,\ a_n$ be a real sequence with $a_i \neq a_j$ $(i \neq j)$ and $f(a_i) \neq 0$, $(i = 1,\ \ldots,\ n)$. Suppose that λ and r are two integers with $\lambda \geq 1$ and $r \geq 0$. Then*

$$\begin{aligned} &\sum_{k=1}^{n} \frac{1}{\prod_{\substack{j=1 \\ j \neq k}}^{n} (a_j - a_k)^{\lambda}} \sum_{\ell=0}^{\lambda-1} \frac{1}{\ell!} \sum_{s=0}^{\ell} (-1)^s \binom{\ell}{s} f^{(\ell-s)}(-a_k) \\ &\qquad \times A\left(\lambda T_1(x),\ \lambda T_2(x),\ \ldots,\ \lambda T_s(x)\right) \frac{\binom{\lambda-\ell+r-1}{r}}{(x+a_k)^{\lambda-\ell+r}} \\ &= \frac{(-1)^r}{r!(x+a_1)^{\lambda}(x+a_2)^{\lambda}\ldots(x+a_n)^{\lambda}} \sum_{j=0}^{r} (-1)^j \binom{r}{j} f^{(r-j)}(x) \\ &\qquad \times A\left(\lambda S_1(x),\ \lambda S_2(x),\ \ldots,\ \lambda S_j(x)\right), \end{aligned} \tag{2}$$

where

$$T_m(x) = \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{(x+a_i)^m},\ (1 \leq m \leq \lambda),$$

and

$$S_m(x) = \sum_{i=1}^{n} \frac{1}{(x+a_i)^m}, \quad (1 \leq m \leq r).$$

Proof. Applying the partial fraction decomposition, let

$$\frac{f(x)}{(x+a_1)^{\lambda}(x+a_2)^{\lambda}\ldots(x+a_n)^{\lambda}} = \sum_{k=1}^{n}\sum_{\ell=0}^{\lambda-1} \frac{G(k,\ \ell)}{(x+a_k)^{\lambda-\ell}}.$$

Multiply both sides in the preceding equation by $(x + a_k)^{\lambda-\ell}$ and take $x \to a_k$. Then

$$G(k,\ \ell) = \lim_{x \to -a_k} (x+a_k)^{\lambda-\ell} \left\{ \frac{f(x)}{(x+a_1)^{\lambda}(x+a_2)^{\lambda}\ldots(x+a_n)^{\lambda}} - \sum_{i=0}^{\ell-1} \frac{G(k,\ i)}{(x+a_k)^{\lambda-i}} \right\}$$

$$= \lim_{x\to -a_k}\left\{\frac{f(x)}{(x+a_k)^\ell \prod\limits_{\substack{j=1\\ j\neq k}}^{n}(x+a_j)^\lambda} - \sum_{i=0}^{\ell-1}\frac{G(k,\ i)(x+a_k)^i}{(x+a_k)^\ell}\right\}$$

$$= \lim_{x\to -a_k}\frac{\frac{f(x)}{\prod\limits_{\substack{j=1\\ j\neq k}}^{n}(x+a_j)^\lambda} - \sum\limits_{i=0}^{\ell-1}G(k,\ i)(x+a_k)^i}{(x+a_k)^\ell}.$$

Repeatedly applying L'Hôspital's rule we obtain the coefficient as follows:

$$\begin{aligned} G(k,\ \ell) &= \lim_{x\to -a_k}\frac{1}{\ell!}\frac{d^\ell}{dx^\ell}\frac{f(x)}{\prod\limits_{\substack{j=1\\ j\neq k}}^{n}(x+a_j)^\lambda} \\ &= \frac{1}{\ell!}\lim_{x\to -a_k}\sum_{r=0}^{\ell}\binom{\ell}{r}f^{(\ell-r)}(x)\left(\frac{1}{\prod\limits_{\substack{j=1\\ j\neq k}}^{n}(x+a_j)^\lambda}\right)^{(r)}. \end{aligned}$$

By using differentiation of composite functions (see [12]), we have

$$\left(\frac{1}{\prod\limits_{\substack{j=1\\ j\neq k}}^{n}(x+a_j)^\lambda}\right)^{(r)} = \frac{(-1)^r}{\prod\limits_{\substack{j=1\\ j\neq k}}^{n}(x+a_j)^\lambda}A\left(\lambda T_1(x),\ \lambda T_2(x),\ \ldots,\ \lambda T_r(x)\right),$$

where

$$T_m(x) = \sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{(x+a_i)^m},\ (1\le m\le \ell).$$

Hence,

$$G(k,\ell) = \frac{1}{\ell!\prod\limits_{\substack{j=1\\ j\neq k}}^{n}(a_j-a_k)^\lambda}\sum_{r=0}^{\ell}(-1)^r\binom{\ell}{r}f^{(\ell-r)}(-a_k)A\left(\lambda T_1(x),\ \lambda T_2(x),\ \ldots,\ \lambda T_r(x)\right),$$

where

$$T_m(x) = \sum_{\substack{i=1 \\ i\neq k}}^{n} \frac{1}{(a_i - a_k)^m}, \ (1 \le m \le \ell).$$

So we get

$$\frac{f(x)}{(x+a_1)^\lambda (x+a_2)^\lambda \dots (x+a_n)^\lambda} = \sum_{k=1}^{n} \frac{1}{\prod\limits_{\substack{j=1 \\ j\neq k}}^{n} (a_j - a_k)^\lambda} \sum_{\ell=0}^{\lambda-1} \frac{1}{\ell!} \sum_{s=0}^{\ell} (-1)^s \binom{\ell}{s}$$

$$\times f^{(\ell-s)}(-a_k) A\left(\lambda T_1(x),\ \lambda T_2(x),\ \dots,\ \lambda T_s(x)\right) \frac{1}{(x+a_k)^{\lambda-\ell}}.$$

By differentiating the above equation r times with respect to x, we have

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{1}{\prod\limits_{\substack{j=1 \\ j\neq k}}^{n} (a_j - a_k)^\lambda} \sum_{\ell=0}^{\lambda-1} \frac{1}{\ell!} \sum_{s=0}^{\ell} (-1)^s \binom{\ell}{s} f^{(\ell-s)}(-a_k) A\left(\lambda T_1(x),\ \lambda T_2(x),\ \dots,\ \lambda T_s(x)\right) \\
&\qquad \times \frac{(-1)^r (\lambda - \ell + r - 1)_r}{(x+a_k)^{\lambda-\ell+r}} \\
= \ & \frac{d^r}{dx^r}\left(\frac{f(x)}{(x+a_1)^\lambda (x+a_2)^\lambda \dots (x+a_n)^\lambda}\right) \\
= \ & \sum_{j=0}^{r} \binom{r}{j} f^{(r-j)}(x) \left(\frac{1}{(x+a_1)^\lambda (x+a_2)^\lambda \dots (x+a_n)^\lambda}\right)^{(j)} \\
= \ & \frac{1}{(x+a_1)^\lambda (x+a_2)^\lambda \dots (x+a_n)^\lambda} \sum_{j=0}^{r} (-1)^j \binom{r}{j} f^{(r-j)}(x) A\left(\lambda S_1(x),\ \lambda S_2(x),\ \dots,\ \lambda S_j(x)\right),
\end{aligned}$$

where

$$S_m(x) = \sum_{i=1}^{n} \frac{1}{(x+a_i)^m}, \quad (1 \le m \le r).$$

This completes the proof. □

3. Some Corollaries

Corollary 3.2

$$\sum_{k=1}^{n} \frac{f(-a_k)}{(x+a_k)^{r+1} \prod\limits_{\substack{j=1 \\ j\neq k}}^{n} (a_j - a_k)}$$

$$= \frac{(-1)^r}{r!(x+a_1)(x+a_2)\dots(x+a_n)} \sum_{j=0}^{r} (-1)^j \binom{r}{j} f^{(r-j)}(x) A\left(S_1(x),\ S_2(x),\ \dots,\ S_j(x)\right), \tag{3}$$

$$\sum_{k=1}^{n} \frac{f(-a_k)}{(x+a_k)^{r+1} \prod\limits_{\substack{j=1 \\ j\neq k}}^{n} (a_j-a_k)^2} \left\{ \frac{r+1}{x+a_k} + \frac{f'(-a_k)}{f(-a_k)} - 2 \sum_{\substack{i=1 \\ i\neq k}}^{n} \frac{1}{x+a_i} \right\}$$
$$= \frac{(-1)^r}{r!(x+a_1)^2(x+a_2)^2\dots(x+a_n)^2} \sum_{j=0}^{r} (-1)^j \binom{r}{j} f^{(r-j)}(x) A\left(2S_1(x),\ 2S_2(x),\ \dots,\ 2S_j(x)\right) \tag{4}$$

and

$$\sum_{k=1}^{n} \frac{f(-a_k)}{(x+a_k)^{r+1} \prod\limits_{\substack{j=1 \\ j\neq k}}^{n} (a_j-a_k)^3} \left\{ \frac{(r+2)(r+1)}{(x+a_k)^2} + 2(r+1) \frac{\frac{f'(-a_k)}{f(-a_k)} - 3 \sum\limits_{\substack{i=1 \\ i\neq k}}^{n} \frac{1}{x+a_i}}{x+a_k} \right.$$
$$\left. + \frac{f''(-a_k)}{f(-a_k)} - 6\frac{f'(-a_k)}{f(-a_k)} \sum_{\substack{i=1 \\ i\neq k}}^{n} \frac{1}{x+a_i} + 9 \left(\sum_{\substack{i=1 \\ i\neq k}}^{n} \frac{1}{x+a_i} \right)^2 + 3 \sum_{\substack{i=1 \\ i\neq k}}^{n} \frac{1}{(x+a_i)^2} \right\}$$
$$= \frac{2(-1)^r}{r!(x+a_1)^3(x+a_2)^3\dots(x+a_n)^3} \sum_{j=0}^{r} (-1)^j \binom{r}{j} f^{(r-j)}(x) A\left(3S_1(x),\ 3S_2(x),\ \dots,\ 3S_j(x)\right). \tag{5}$$

Proof. In Theorem 2.1, take $\lambda = 1,\ 2,\ 3.$ □

Note. The first formula (3) was given in [14] and an extension of (3) was obtained in [15].

Corollary 3.3

$$\sum_{k=1}^{n} \frac{1}{\prod\limits_{\substack{j=1 \\ j\neq k}}^{n} (a_j-a_k)^\lambda} \sum_{\ell=0}^{\lambda-1} \frac{1}{\ell!} \sum_{s=0}^{\ell} (-1)^s \binom{\ell}{s} f^{(\ell-s)}(-a_k)$$
$$\times A\left(\lambda T_1(x),\ \lambda T_2(x),\ \dots,\ \lambda T_s(x)\right) \frac{1}{(x+a_k)^{\lambda-\ell}}$$
$$= \frac{f(x)}{(x+a_1)^\lambda (x+a_2)^\lambda \dots (x+a_n)^\lambda}, \tag{6}$$

$$\sum_{k=1}^{n} \frac{1}{\prod_{\substack{j=1\\ j\neq k}}^{n} (a_j-a_k)^{\lambda}} \sum_{\ell=0}^{\lambda-1} \frac{1}{\ell!} \sum_{s=0}^{\ell} (-1)^s \binom{\ell}{s} f^{(\ell-s)}(-a_k)$$

$$\times A\left(\lambda T_1(x),\ \lambda T_2(x),\ \ldots,\ \lambda T_s(x)\right) \frac{\lambda-\ell}{(x+a_k)^{\lambda-\ell+1}}$$

$$= -\frac{1}{(x+a_1)^{\lambda}(x+a_2)^{\lambda}\ldots(x+a_n)^{\lambda}} \left\{ f'(x) - \lambda f(x) \sum_{i=1}^{n} \frac{1}{x+a_i} \right\} \tag{7}$$

and

$$\sum_{k=1}^{n} \frac{1}{\prod_{\substack{j=1\\ j\neq k}}^{n} (a_j-a_k)^{\lambda}} \sum_{\ell=0}^{\lambda-1} \frac{1}{\ell!} \sum_{s=0}^{\ell} (-1)^s \binom{\ell}{s} f^{(\ell-s)}(-a_k)$$

$$\times A\left(\lambda T_1(x),\ \lambda T_2(x),\ \ldots,\ \lambda T_s(x)\right) \frac{(\lambda-\ell+1)(\lambda-\ell)}{2(x+a_k)^{\lambda-\ell+2}}$$

$$= \frac{1}{2(x+a_1)^{\lambda}(x+a_2)^{\lambda}\ldots(x+a_n)^{\lambda}} \left\{ f''(x) - 2\lambda f'(x) \sum_{i=1}^{n} \frac{1}{x+a_i} \right.$$

$$\left. + \lambda^2 f(x) \left(\sum_{i=1}^{n} \frac{1}{x+a_i} \right)^2 + \lambda f(x) \sum_{i=1}^{n} \frac{1}{(x+a_i)^2} \right\}, \tag{8}$$

where

$$T_m(x) = \sum_{\substack{i=1\\ i\neq k}}^{n} \frac{1}{(x+a_i)^m},\ (1 \le m \le \lambda).$$

Proof. In Theorem 2.1, take $r = 0,\ 1,\ 2.$ □

Corollary 3.4

$$\sum_{k=1}^{n} \frac{f(-a_k)}{(x+a_k) \prod_{\substack{j=1\\ j\neq k}}^{n} (a_j-a_k)} = \frac{f(x)}{(x+a_1)(x+a_2)\ldots(x+a_n)}, \tag{9}$$

$$\sum_{k=1}^{n} \frac{f(-a_k)}{(x+a_k)^2 \prod_{\substack{j=1\\ j\neq k}}^{n} (a_j-a_k)^2} \left\{ \frac{r+1}{x+a_k} + \frac{f'(-a_k)}{f(-a_k)} - 2 \sum_{\substack{i=1\\ i\neq k}}^{n} \frac{1}{x+a_i} \right\}$$

$$= -\frac{1}{(x+a_1)^2(x+a_2)^2\ldots(x+a_n)^2} \left\{ f'(x) - 2f(x) \sum_{i=1}^{n} \frac{1}{x+a_i} \right\} \tag{10}$$

and

$$\sum_{k=1}^{n}\frac{f(-a_k)}{(x+a_k)^3\prod_{\substack{j=1\\ j\neq k}}^{n}(a_j-a_k)^3}\left\{\frac{(r+2)(r+1)}{(x+a_k)^2}+2(r+1)\frac{\frac{f'(-a_k)}{f(-a_k)}-3\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{x+a_i}}{x+a_k}\right.$$

$$+\left.\frac{f''(-a_k)}{f(-a_k)}-6\frac{f'(-a_k)}{f(-a_k)}\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{x+a_i}+9\left(\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{x+a_i}\right)^2+3\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1}{(x+a_i)^2}\right\}$$

$$=\frac{1}{2(x+a_1)^3(x+a_2)^3\dots(x+a_n)^3}\left\{f''(x)-6f'(x)\sum_{i=1}^{n}\frac{1}{x+a_i}\right.$$

$$\left.+9f(x)\left(\sum_{i=1}^{n}\frac{1}{x+a_i}\right)^2+3f(x)\sum_{i=1}^{n}\frac{1}{(x+a_i)^2}\right\}. \qquad (11)$$

Proof. Take $r = 0,\ 1,\ 2$ in Corollary 3.2 or $\lambda = 1,\ 2,\ 3$ in Corollary 3.3. □

4. Some Applications

It is seen that the identities in Corollaries 3.2, 3.3 and 3.4 are independent of the choice of a_k and $f(x)$. So we can obtain some interesting identities by taking special values for a_k and $f(x)$.

4.1 The case: $a_k = -k$

In [5, 11], some interesting identities involving the harmonic numbers were studied. In this section, by taking $a_k = -k$, we show a number of identities involving the harmonic numbers.

Theorem 4.5

$$\sum_{k=1}^{n}(-1)^{(k+1)\lambda}\binom{n}{k}^{\lambda}\sum_{\ell=0}^{\lambda-1}\frac{(-1)^{\ell}}{\ell!}\sum_{s=0}^{\ell}\binom{\ell}{s}f^{(\ell-s)}(k)A\left(\lambda T_1,\ \lambda T_2,\ \dots,\ \lambda T_s\right)\frac{\binom{\lambda-\ell+r-1}{r}}{k^{\lambda-\ell+r-1}}$$

$$=\frac{1}{r!}\sum_{j=0}^{r}\binom{r}{j}f^{(r-j)}(0)A\left(\lambda S_1,\ \lambda S_2,\ \dots,\ \lambda S_j\right), \qquad (12)$$

where

$$T_m = \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{i^m}, \ (1 \le m \le \lambda),$$

and

$$S_m = \sum_{i=1}^{n} \frac{1}{i^m}, \quad (1 \le m \le r).$$

Proof. Take $a_k = -k$ and $x = 0$ in Theorem 2.1. □

When $\lambda = 1$, then we can obtain the main results of paper [9] by Mercier:

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}\binom{n}{k} f(k)}{k^r} = \frac{1}{r!} \sum_{j=0}^{r} \binom{r}{j} f^{(r-j)}(0) A\left(S_1,\ S_2,\ \ldots,\ S_j\right). \tag{13}$$

In [9], from this identity, Mercier obtained a lot of combinatorial identities. For example,

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} f(k) = f(0), \tag{14}$$

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}\binom{n}{k}}{k} f(k) = f'(0) + f(0) \sum_{i=1}^{n} \frac{1}{i}, \tag{15}$$

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}\binom{n}{k}}{k^2} f(k) = \frac{1}{2} \left\{ f''(0) + 2f'(0) \sum_{i=1}^{n} \frac{1}{i} + f(0) \left(\left(\sum_{i=1}^{n} \frac{1}{i} \right)^2 + \sum_{i=1}^{n} \frac{1}{i^2} \right) \right\}. \tag{16}$$

For $\lambda = 2,\ 3$, we have

$$\begin{aligned} &\sum_{k=1}^{n} \frac{\binom{n}{k}^2 f(k)}{k^r} \left\{ \frac{r+1}{k} - \frac{f'(k)}{f(k)} - 2 \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{i} \right\} \\ = &\ \frac{1}{r!} \sum_{j=0}^{r} \binom{r}{j} f^{(r-j)}(0) A\left(2S_1,\ 2S_2,\ \ldots,\ 2S_j\right) \end{aligned} \tag{17}$$

and

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}\binom{n}{k}^3 f(k)}{k^r} \left\{ \frac{(r+2)(r+1)}{k^2} - 2\frac{r+1}{k} \left(\frac{f'(k)}{f(k)} + 3 \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{i} \right) + \frac{f''(k)}{f(k)} \right.$$

$$+6\frac{f'(k)}{f(k)}\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i}+9\left(\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i}\right)^2+3\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i^2}\Bigg\}$$

$$= \frac{2}{r!}\sum_{j=0}^{r}\binom{r}{j}f^{(r-j)}(0)A\left(3S_1,\ 3S_2,\ \ldots,\ 3S_j\right). \tag{18}$$

In particular,

$$\sum_{k=1}^{n}\binom{n}{k}^2 f(k)\left\{\frac{1}{k}-\frac{f'(k)}{f(k)}-2\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i}\right\}=f(0), \tag{19}$$

$$\sum_{k=1}^{n}\frac{\binom{n}{k}^2}{k}f(k)\left\{\frac{2}{k}-\frac{f'(k)}{f(k)}-2\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i}\right\}=f'(0)+2f(0)\sum_{i=1}^{n}\frac{1}{i}, \tag{20}$$

$$\sum_{k=1}^{n}\frac{\binom{n}{k}^2}{k^2}f(k)\left\{\frac{3}{k}-\frac{f'(k)}{f(k)}-2\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i}\right\}$$

$$= \frac{1}{2}\left\{f''(0)+4f'(0)\sum_{i=1}^{n}\frac{1}{i}+f(0)\left(\left(4\sum_{i=1}^{n}\frac{1}{i}\right)^2+2\sum_{i=1}^{n}\frac{1}{i^2}\right)\right\}, \tag{21}$$

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}^3 f(k)\Bigg\{\frac{2}{k^2}-\frac{2}{k}\left(\frac{f'(k)}{f(k)}+3\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i}\right)+\frac{f''(k)}{f(k)}+6\frac{f'(k)}{f(k)}\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i}$$

$$+9\left(\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i}\right)^2+3\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i^2}\Bigg\}$$

$$= 2f(0), \tag{22}$$

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}\binom{n}{k}^3}{k}f(k)\left\{\frac{6}{k^2}-\frac{4}{k}\left(\frac{f'(k)}{f(k)}+3\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i}\right)+\frac{f''(k)}{f(k)}+6\frac{f'(k)}{f(k)}\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i}\right.$$
$$\left.+9\left(\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i}\right)^2+3\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i^2}\right\}$$
$$= 2f'(0)+6f(0)\sum_{i=1}^{n}\frac{1}{i}. \tag{23}$$

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}\binom{n}{k}^3}{k^2}f(k)\left\{\frac{12}{k^2}-\frac{6}{k}\left(\frac{f'(k)}{f(k)}+3\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i}\right)+\frac{f''(k)}{f(k)}+6\frac{f'(k)}{f(k)}\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i}\right.$$
$$\left.+9\left(\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i}\right)^2+3\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i^2}\right\}$$
$$= f''(0)+6f'(0)\sum_{i=1}^{n}\frac{1}{i}+f(0)\left(9\left(\sum_{i=1}^{n}\frac{1}{i}\right)^2+3\sum_{i=1}^{n}\frac{1}{i^2}\right). \tag{24}$$

In identities (15)-(16) and (19)-(24), according to different choices of $f(x)$, we can get a lot of identities. For example, if $f(x)=1$, then we have

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}\binom{n}{k}}{k}=\sum_{i=1}^{n}\frac{1}{i}, \tag{25}$$

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}\binom{n}{k}}{k^2}=\frac{1}{2}\left\{\left(\left(\sum_{i=1}^{n}\frac{1}{i}\right)^2+\sum_{i=1}^{n}\frac{1}{i^2}\right)\right\}, \tag{26}$$

$$\sum_{k=1}^{n}\binom{n}{k}^2\left\{\frac{1}{k}-2\sum_{\substack{i=1\\i\neq k}}^{n}\frac{1}{i}\right\}=1, \tag{27}$$

$$\sum_{k=1}^{n} \frac{\binom{n}{k}^2}{k} \left\{ \frac{2}{k} - 2 \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{i} \right\} = 2 \sum_{i=1}^{n} \frac{1}{i}, \tag{28}$$

$$\sum_{k=1}^{n} \frac{\binom{n}{k}^2}{k^2} \left\{ \frac{3}{k} - 2 \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{i} \right\} = 2 \left(\sum_{i=1}^{n} \frac{1}{i} \right)^2 + \sum_{i=1}^{n} \frac{1}{i^2}, \tag{29}$$

$$\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k}^3 \left\{ \frac{2}{k^2} - \frac{6}{k} \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{i} + 9 \left(\sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{i} \right)^2 + 3 \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{i^2} \right\} = 2, \tag{30}$$

$$\sum_{k=1}^{n} \frac{(-1)^{k+1} \binom{n}{k}^3}{k} \left\{ \frac{2}{k^2} - \frac{4}{k} \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{i} + 3 \left(\sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{i} \right)^2 + \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{i^2} \right\} = 2 \sum_{i=1}^{n} \frac{1}{i}. \tag{31}$$

$$\sum_{k=1}^{n} \frac{(-1)^{k+1} \binom{n}{k}^3}{k^2} \left\{ \frac{4}{k^2} - \frac{6}{k} \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{i} + 3 \left(\sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{i} \right)^2 + \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{i^2} \right\} = 3 \left(\sum_{i=1}^{n} \frac{1}{i} \right)^2 + \sum_{i=1}^{n} \frac{1}{i^2}. \tag{32}$$

4.2 The case: $a_k = -\frac{1-q^k}{1-q}$

In this section, taking $a_k = -\frac{1-q^k}{1-q}$ we obtain some identities involving the Gaussian binomial coefficient.

Theorem 4.6

$$\sum_{k=1}^{n} \frac{(-1)^{(k+1)\lambda} \begin{bmatrix} n \\ k \end{bmatrix}^{\lambda}}{q^{\lambda\left(nk - \frac{1}{2}k(k+1)\right)} \left(\frac{1-q^k}{1-q}\right)^{\lambda+r-1}} \sum_{\ell=0}^{\lambda-1} \frac{(-1)^{\ell}}{\ell!} \sum_{s=0}^{\ell} \binom{\ell}{s} f^{(\ell-s)} \left(\frac{1-q^k}{1-q} \right)$$

$$\times A\left(\lambda T_1,\ \lambda T_2,\ \ldots,\ \lambda T_s\right)\binom{\lambda-\ell+r-1}{r}\left(\frac{1-q^k}{1-q}\right)^{\ell}$$

$$= \frac{1}{r!}\sum_{j=0}^{r}\binom{r}{j}f^{(r-j)}(0)A\left(\lambda S_1,\ \lambda S_2,\ \ldots,\ \lambda S_j\right) \tag{33}$$

where

$$T_m = \sum_{\substack{i=1\\ i\neq k}}^{n}\left(\frac{1-q}{1-q^i}\right)^m,\ (1\le m\le\lambda),$$

and

$$S_m = \sum_{i=1}^{n}\left(\frac{1-q}{1-q^i}\right)^m, \quad (1\le m\le r).$$

Proof. Take $a_k = -\frac{1-q^k}{1-q}$ and $x=0$ in Theorem 2.1. □

When $\lambda=1$, this reduces to an identity due to Mercier [10]:

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}\begin{bmatrix} n\\ k\end{bmatrix} f\left(\frac{1-q^k}{1-q}\right)}{q^{nk-\frac{1}{2}k(k+1)}\left(\frac{1-q^k}{1-q}\right)^r} = \frac{1}{r!}\sum_{j=0}^{r}\binom{r}{j}f^{(r-j)}(0)A\left(S_1,\ S_2,\ \ldots,\ S_j\right). \tag{34}$$

In particular, for $r=0,\ 1$, we have

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}\begin{bmatrix} n\\ k\end{bmatrix} f\left(\frac{1-q^k}{1-q}\right)}{q^{nk-\frac{1}{2}k(k+1)}} = f(0), \tag{35}$$

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}\begin{bmatrix} n\\ k\end{bmatrix} f\left(\frac{1-q^k}{1-q}\right)}{q^{nk-\frac{1}{2}k(k+1)}\left(\frac{1-q^k}{1-q}\right)} = f'(0)+f(0)\sum_{i=1}^{n}\frac{1-q}{1-q^i}. \tag{36}$$

Take $f(x)=1$ and let $q\to 1/q$. Then (36) becomes the result of Van Hamme [13]:

$$\sum_{k=1}^{n}\begin{bmatrix} n\\ k\end{bmatrix}\frac{(-1)^{k-1}q^{\binom{k+1}{2}}}{1-q^k} = \sum_{i=1}^{n}\frac{q^i}{1-q^i}. \tag{37}$$

When $\lambda=2$, we obtain

$$\sum_{k=1}^{n}\frac{\begin{bmatrix} n\\ k\end{bmatrix}^2 f\left(\frac{1-q^k}{1-q}\right)}{q^{2nk-k(k+1)}\left(\frac{1-q^k}{1-q}\right)^r}\left\{(r+1)\frac{1-q}{1-q^k}-\frac{f'\left(\frac{1-q^k}{1-q}\right)}{f\left(\frac{1-q^k}{1-q}\right)}-2\sum_{\substack{i=1\\ i\neq k}}^{n}\frac{1-q}{1-q^i}\right\}$$

$$= \frac{1}{r!}\sum_{j=0}^{r}\binom{r}{j}f^{(r-j)}(0)A\left(2S_1,\ 2S_2,\ \ldots,\ 2S_j\right). \tag{38}$$

In particular, for $r = 0,\ 1$ in (38), the following identities hold.

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}^2 q^{k(k+1)-2nk} f\left(\frac{1-q^k}{1-q}\right) \left\{ \frac{1-q}{1-q^k} - \frac{f'\left(\frac{1-q^k}{1-q}\right)}{f\left(\frac{1-q^k}{1-q}\right)} - 2 \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1-q}{1-q^i} \right\} = f(0), \quad (39)$$

and

$$\sum_{k=1}^{n} \frac{\begin{bmatrix} n \\ k \end{bmatrix}^2 f\left(\frac{1-q^k}{1-q}\right)}{q^{2nk-k(k+1)}\left(\frac{1-q^k}{1-q}\right)} \left\{ 2\frac{1-q}{1-q^k} - \frac{f'\left(\frac{1-q^k}{1-q}\right)}{f\left(\frac{1-q^k}{1-q}\right)} - 2 \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1-q}{1-q^i} \right\}$$
$$= f'(0) + 2f(0) \sum_{i=1}^{n} \frac{1-q}{1-q^i}. \quad (40)$$

Specifically, for $f(x) = 1$, we have

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}^2 q^{k(k+1)-2nk} \left\{ \frac{1}{1-q^k} - 2 \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{1-q^i} \right\} = \frac{1}{1-q}, \quad (41)$$

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix}^2 \frac{q^{k(k+1)-2nk}}{1-q^k} \left\{ \frac{1}{1-q^k} - \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{1}{1-q^i} \right\} = \frac{1}{1-q} \sum_{i=1}^{n} \frac{1}{1-q^i}. \quad (42)$$

4.3 An identity of Fu and Lascoux

Theorem 4.7

$$\frac{a^{n\lambda}\left(\frac{bq}{a};q\right)_n^{\lambda}}{(az-bc)^{(n-1)\lambda}(q;q)_n} \sum_{k=1}^{n} (-1)^{(k-1)\lambda} \begin{bmatrix} n \\ k \end{bmatrix}^{\lambda} q^{\lambda\binom{k+1}{2}-\lambda nk}(1-q^k)(a-bq^k)^{\lambda(n-2)}$$
$$\times \sum_{\ell=0}^{\lambda-1} \frac{1}{\ell!} \sum_{s=0}^{\ell} (-1)^s \binom{\ell}{s} f^{(\ell-s)}\left(-\frac{c-zq^k}{a-bq^k}\right) A\left(\lambda T_1(x),\ \lambda T_2(x),\ \ldots,\ \lambda T_s(x)\right) \frac{\binom{\lambda-\ell+r-1}{r}}{\left(x+\frac{c-zq^k}{a-bq^k}\right)^{\lambda-\ell+r}}$$
$$= \frac{(-1)^r}{r!\left(x+\frac{c-zq}{a-bq}\right)^{\lambda}\left(x+\frac{c-zq^2}{a-bq^2}\right)^{\lambda}\cdots\left(x+\frac{c-zq^n}{a-bq^n}\right)^{\lambda}}$$
$$\times \sum_{j=0}^{r} (-1)^j \binom{r}{j} f^{(r-j)}(x) A\left(\lambda S_1(x),\ \lambda S_2(x),\ \ldots,\ \lambda S_j(x)\right), \quad (43)$$

where

$$T_m(x) = \sum_{\substack{i=1 \\ i \neq k}}^{n} \frac{(a-bq^i)^m}{(ax+c-q^i(bx+z))^m}, \ (1 \leq m \leq \lambda),$$

and

$$S_m(x) = \sum_{i=1}^{n} \frac{(a-bq^i)^m}{(ax+c-q^i(bx+z))^m}, \quad (1 \leq m \leq r).$$

Proof. In Theorem 2.1, take $a_k = \frac{c-zq^k}{a-bq^k}$. □

When $\lambda = 1$, $r = 0$, $x \to -x$ and $f(x) = 1$, this reduces to

$$\frac{c^n \left(\frac{zq}{c};q\right)_n}{(az-bc)^{n-1}(q;q)_n} \sum_{k=1}^{n} (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k+1}{2}-nk} (1-q^k) \frac{(a-bq^k)^{n-1}}{c-zq^k} \cdot \frac{1}{1-\frac{a-bq^k}{c-zq^k}x}$$
$$= \frac{1}{\left(1-\frac{a-bq}{c-zq}x\right)\left(1-\frac{a-bq^2}{c-zq^2}x\right)\dots\left(1-\frac{a-bq^n}{c-zq^n}x\right)}. \quad (44)$$

Comparing the coefficients of x^τ on both sides of the above equation, we have an identity of Fu and Lascoux [8]:

$$\frac{c^n \left(\frac{zq}{c};q\right)_n}{(az-bc)^{n-1}(q;q)_n} \sum_{k=1}^{n} (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k+1}{2}-nk} (1-q^k) \frac{(a-bq^k)^{n-1+\tau}}{(c-zq^k)^{\tau+1}}$$
$$= h_\tau \left(\frac{a-bq}{c-zq}, \frac{a-bq^2}{c-zq^2}, \dots, \frac{a-bq^n}{c-zq^n}\right), \quad (45)$$

where $h_\tau(a_1, a_2, \dots, a_n)$ is the τ-th complete symmetric function defined by

$$h_\tau(a_1, a_2, \dots, a_n) = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_\tau \leq n} a_{i_1} a_{i_2} \dots a_{i_\tau}.$$

Taking $a = 0$, $c = 1$ and $b = -1$ in (45) we have

$$\sum_{k=1}^{n} (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}+\tau k} \frac{1-q^k}{(1-zq^k)^{\tau+1}} = \frac{(q;q)_n}{(zq;q)_n} h_\tau \left(\frac{q}{1-zq}, \frac{q^2}{1-zq^2}, \dots, \frac{q^n}{1-zq^n}\right). \quad (46)$$

When $z = 1$, this identity reduces to Dilcher's identity [7]:

$$\sum_{k=1}^{n} (-1)^{k-1} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}+\tau k} \frac{1}{(1-q^k)^\tau} = h_\tau \left(\frac{q}{1-q}, \frac{q^2}{1-q^2}, \dots, \frac{q^n}{1-q^n}\right). \quad (47)$$

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References

[1] S. Ahlgren and K. Ono, A Gaussian hypergeometric series evaluation and Apéry number congruences, J. Reine Angew. Math. 518(2000): 187-212.

[2] S. Ahlgren, S. B. Ekhad, K. Ono and D. Zeilberger, A binomial coefficient identity associated to a conjecture of Beukers, The Electronic J. Combinatorics 5(1998), #R10.

[3] F. Beukers, Another congruence for Apéry numbers, J. Number Theory 25(1987): 201-210.

[4] W.-C. Chu, A binomial coefficient identity associated with Beukers' Conjecture on Apéry numbers, The Electronic J. Combinatorics 11(2004), #N15.

[5] W. Chu and L. De. Donno, Hypergeometric series and harmonic number identities, Adv. Appl. math., 34(2005): 123-137.

[6] W.-C. Chu, Partial-fraction decompositions and harmonic number identities, arXiv:math.CO/0502537 vl.

[7] K. Dilcher, Some q-series identities related to divisor functions, Discrete Math., 145(1995): 83-93.

[8] A. M. Fu and A. Lascoux, q-identities from Lagrange and Newton interpolation, Adv. Appl. math., 31(2003): 527-531.

[9] A. Mercier, Quelques identites de L'analyse combinatoire, Discrete math., 49(1984): 139-149.

[10] A. Mercier, Identities containing Gaussian binomial coefficients, Discrete Math., 76(1989): 67-73.

[11] P. Paule and C. Schneider, Computer proofs of a new family of harmonic number identities, Adv. Appl. math., 31(2003): 359-378.

[12] J. Riordan, An Introduction to Combinatorial Analysis, Princeton University Press, Princeton, NY, 1978.

[13] L. Van Hamme, Advanced problem 6407, American Math. Monthly, 40(1982): 703-704.

[14] Zhizheng Zhang, Several identities relatrd to Mercier's, Mathematica Montisnigri, 11(1999): 159-171.

[15] Zhizheng Zhang, Generalizations of several identities, Northeast. Math. J., 10.1(1994): 93-98.