PARITY THEOREMS FOR STATISTICS ON PERMUTATIONS AND CATALAN WORDS

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Abstract

We establish parity theorems for statistics on the symmetric group S_n, the derangements D_n, and the Catalan words C_n, giving both algebraic and bijective proofs. For the former, we evaluate q-generating functions at $q = -1$; for the latter, we define appropriate sign-reversing involutions. Most of the statistics involve counting inversions or finding the major index of various words.

Keywords: Permutation statistic, inversion, major index, derangement, Catalan numbers.

1. Introduction

We'll use the following notational conventions: $\mathbb{N} := \{0, 1, 2, \dots\}$, $\mathbb{P} := \{1, 2, \dots\}$, $[0] := \varnothing$, and $[n] := \{1, \dots, n\}$ for $n \in \mathbb{P}$. Empty sums take the value 0 and empty products the value 1, with $0^0 := 1$. The letter q denotes an indeterminate, with $0_q := 0$, $n_q := 1 + q + \cdots + q^{n-1}$ for $n \in \mathbb{P}$, $0_q^! := 1$, $n_q^! := 1_q 2_q \cdots n_q$ for $n \in \mathbb{P}$, and $\binom{n}{k}_q := n_q^!/k_q^!(n-k)_q^!$ for $n \in \mathbb{N}$ and $0 \leqslant k \leqslant n$. The binomial coefficient $\binom{n}{k}$ is equal to zero if k is a negative integer or if $0 \leqslant n < k$.

Let Δ be a finite set of discrete structures and $I : \Delta \to N$, with generating function

$$G(I, \Delta; q) := \sum_{\delta \in \Delta} q^{I(\delta)} = \sum_k |\{\delta \in \Delta : I(\delta) = k\}|\, q^k. \tag{1.1}$$

Of course, $G(I, \Delta; 1) = |\Delta|$. If $\Delta^+ := \{\delta \in \Delta : I(\delta) \text{ is even}\}$ and $\Delta^- := \{\delta \in \Delta : I(\delta) \text{ is odd}\}$, then $G(I, \Delta; -1) = |\Delta^+| - |\Delta^-|$. Hence if $G(I, \Delta; -1) = 0$, the set Δ is "balanced" with respect to the parity of I. For example, setting $q = -1$ in the binomial theorem,

$$(1+q)^n = \sum_{S \subseteq [n]} q^{|S|} = \sum_{k=0}^{n} \binom{n}{k} q^k, \tag{1.2}$$

yields the familiar result that a finite nonempty set has as many subsets of odd cardinality as it has subsets of even cardinality.

When $G(I, \Delta; -1) = 0$ and hence $|\Delta^+| = |\Delta^-|$, it is instructive to identify an $I-$parity changing involution of Δ. For the statistic $|S|$ in (1.2), the map

$$S \mapsto \begin{cases} S \cup \{1\}, & \text{if } 1 \notin S; \\ S - \{1\}, & \text{if } 1 \in S, \end{cases}$$

furnishes such an involution. More generally, if $G(I, \Delta; -1) = |\Delta^+| - |\Delta^-| = c$, it suffices to identify a subset Δ^* of Δ of cardinality $|c|$ contained completely within Δ^+ or Δ^- and then to define an I-parity changing involution on $\Delta - \Delta^*$. The subset Δ^* thus captures both the sign and magnitude of $G(I, \Delta; -1)$. Evaluation of q-generating functions as in (1.1) at $q = -1$ has yielded parity theorems for statistics on set partitions [9, 13], lattice paths [10], domino arrangements [11], and Laguerre configurations [10].

Since each member of $\Delta - \Delta^*$ is paired with another of opposite I-parity, we have $|\Delta| \equiv |\Delta^*| \pmod 2$. Thus, the I-parity changing involutions described above also yield combinatorial proofs of congruences of the form $a_n \equiv b_n \pmod 2$. Shattuck [9] has, for example, given such a combinatorial proof of the congruence

$$S(n,k) \equiv \binom{n - \lfloor k/2 \rfloor - 1}{n-k} \pmod 2$$

for Stirling numbers of the second kind, answering a question posed by Stanley [12, p. 46, Exercise 17b].

In §2 below, we establish parity theorems for several permutation statistics defined on all of S_n, algebraically by evaluating q-generating functions at $q = -1$ and combinatorially by identifying appropriate parity changing involutions. In §3, we analyze the parity of some statistics on D_n, the set of derangements of $[n]$ (i.e., permutations of $[n]$ having no fixed points).

Shattuck and Wagner [10] derive a parity theorem for the number of inversions in binary words of length n with k 1's. In §4, we obtain comparable results for C_n, the set of binary words of length $2n$ with n 1's and with no initial segment containing more 1's than 0's (termed *Catalan words*).

Recall that the inversion and major index statistics for a word $w = w_1 w_2 \cdots w_m$ in some alphabet are given by

$$maj(w) := \sum_{i \in D(w)} i, \quad \text{where } D(w) := \{1 \le i \le m-1 : \ w_i > w_{i+1}\},$$

and

$$inv(w) := |\{(i,j) : \; i < j \text{ and } w_i > w_j\}| \, .$$

2. Permutation Statistics

2.1 Some Balanced Permutation Statistics

Let S_n be the set of permutations of $[n]$. A function $f : \; S_n \to \mathbb{N}$ is called a permutation statistic. Two important permutation statistics are *inv* and *maj*, which record the number of inversions and the major index, respectively, of a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, expressed as a word. The statistics *inv* and *maj* have the same q-generating function over S_n:

$$\sum_{\sigma\in S_n} q^{inv(\sigma)} = n_q^! = \sum_{\sigma\in S_n} q^{maj(\sigma)}, \tag{2.1}$$

[12, Corollary 1.3.10] and [1, Corollary 3.8].

Substituting $q = -1$ into (2.1) reveals that $n_{-1}^! = 0$ if $n \geq 2$, and hence *inv* and *maj* are both balanced if $n \geq 2$. Interchanging σ_1 and σ_2 in $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$ changes the parity of both *inv* and *maj* and thus furnishes an appropriate involution. Note that switching the elements 1 and 2 in σ changes the *inv*-parity, but not necessarily the *maj*-parity.

Now express $\sigma \in S_n$ in the *standard cycle form*

$$\sigma = (\alpha_1)(\alpha_2)\cdots ,$$

where $\alpha_1, \alpha_2, \ldots$ are the cycles of σ, ordered by increasing smallest elements with each cycle (α_i) written with its smallest element in the first position. Let $S_{n,k}$ denote the set of permutations of $[n]$ with k cycles and $c(n,k) := |S_{n,k}|$, the signless Stirling number of the first kind. The $c(n,k)$ are connection constants in the polynomial identities

$$q(q+1)\cdots(q+n-1) = \sum_{k=0}^{n} c(n,k)q^k. \tag{2.2}$$

Setting $q = -1$ in (2.2) reveals that there are as many permutations of $[n]$ with an even number of cycles as there are with an odd number of cycles if $n \geq 2$. Alternatively, breaking apart or merging α_1 and α_2 as shown below, leaving the other cycles undisturbed, changes the parity of the number of cycles:

$$\alpha_1 = (1\cdots 2\cdots), \ldots \;\leftrightarrow\; \alpha_1 = (1\cdots),\; \alpha_2 = (2\cdots), \ldots .$$

This involution also shows that the statistic recording the number of cycles of σ with even cardinality is balanced if $n \geq 2$.

Given $\sigma = (\alpha_1)(\alpha_2)\cdots$, expressed in standard cycle form, let

$$w(\sigma) := \sum_i (i-1)|\alpha_i|.$$

Edelman, Simion, and White [4] show that

$$\sum_{\sigma \in S_n} x^{|\sigma|} q^{w(\sigma)} = \prod_{i=0}^{n-1} (xq^i + i), \tag{2.3}$$

where $|\sigma|$ denotes the number of cycles. Setting $x = 1$ in (2.3) yields

$$\sum_{\sigma \in S_n} q^{w(\sigma)} = \prod_{i=0}^{n-1} (q^i + i), \tag{2.4}$$

another q-generalization of $n!$.

Setting $q = -1$ in (2.4) shows that the w statistic is balanced if $n \geq 2$. Alternatively, if the last cycle has cardinality greater than one, break off the last member and form a 1-cycle with it; if the last cycle contains a single member, place it at the end of the penultimate cycle.

2.2. An Unbalanced Permutation Statistic

Carlitz [2] defines the statistic inv_c on S_n as follows: express $\sigma \in S_n$ in standard cycle form; then remove parentheses and count inversions in the resulting word to obtain $inv_c(\sigma)$. As an illustration, for the permutation $\sigma \in S_7$ given by 3241756, we have $inv_c(\sigma) = 3$, the number of inversions in the word 1342576.

Let

$$c_q(n,k) := \sum_{\sigma \in S_{n,k}} q^{inv_c(\sigma)}, \tag{2.5}$$

where $S_{n,k}$ is the set of permutations of $[n]$ with k cycles. Then $c_q(n,0) = \delta_{n,0},\ c_q(0,k) = \delta_{0,k}$, and

$$c_q(n,k) = c_q(n-1,k-1) + (n-1)_q c_q(n-1,k),\ \forall n,k \in \mathbb{P}, \tag{2.6}$$

since n may go in a cycle by itself or come directly after any member of $[n-1]$ within a cycle.

Using (2.6), it is easy to show that

$$x(x+1_q)\cdots(x+(n-1)_q)=\sum_{k=0}^{n}c_q(n,k)x^k. \tag{2.7}$$

Setting $x=1$ in (2.7) gives

$$c_q(n):=\sum_{k=0}^{n}c_q(n,k)=\sum_{\sigma\in S_n}q^{inv_c(\sigma)}=\prod_{j=0}^{n-1}(1+j_q). \tag{2.8}$$

Theorem 2.1. *For all $n\in N$,*

$$c_{-1}(n):=\sum_{\sigma\in S_n}(-1)^{inv_c(\sigma)}=2^{\lfloor n/2\rfloor}. \tag{2.9}$$

Proof. Put $q=-1$ in (2.8) and note that

$$j_q|_{q=-1}=\begin{cases}0, & \text{if j is even;}\\ 1, & \text{if j is odd.}\end{cases}$$

Alternatively, with S_n^+, S_n^- denoting the members of S_n with even or odd inv_c values, respectively, we have $c_{-1}(n)=|S_n^+|-|S_n^-|$. To prove (2.9), it thus suffices to identify a subset S_n^* of S_n^+ such that $|S_n^*|=2^{\lfloor n/2\rfloor}$ along with an inv_c-parity changing involution of $S_n-S_n^*$.

First assume n is even. In this case, the set S_n^* consists of those permutations expressible in standard cycle form as a product of 1-cycles and the transpositions $(2i-1,\ 2i)$, $1\le i\le n/2$. Note that $S_n^*\subseteq S_n^+$ with zero inv_c value for each of its $2^{n/2}$ members.

Before giving the involution on $S_n-S_n^*$, we make a definition: given $\sigma=(\alpha_1)(\alpha_2)\cdots\in S_m$ in standard cycle form and $j,\ 1\le j\le m$, let $\sigma_{[j]}$ be the permutation of $[j]$ (in standard cycle form) obtained by writing the members of $[j]$ in the order as they appear within the cycles of σ (e.g., if $\sigma=(163)(25)(4)(7)\in S_7$ and $j=4$, then $\sigma_{[4]}=(13)(2)(4)$ and $\sigma_{[7]}=\sigma$).

Suppose now $\sigma\in S_n-S_n^*$ is expressed in standard cycle form and that i_0 is the smallest integer $i,\ 1\le i\le n/2$, for which $\sigma_{[2i]}\in S_{2i}-S_{2i}^*$. Then it must be the case for σ that

(i) neither $2i_0-1$ nor $2i_0$ starts a cycle, or

(ii) exactly one of $2i_0-1,\ 2i_0$ starts a cycle with $2i_0-1$ and $2i_0$ not belonging to the same cycle.

Switching $2i_0 - 1$ and $2i_0$ within σ, written in standard cycle form, changes the inv_c value by one, and the resulting map is thus a parity changing involution of $S_n - S_n^*$.

If n is odd, let $S_n^* \subseteq S_n^+$ consist of those permutations expressible as a product of 1-cycles and the transpositions $(2i, 2i+1)$, $1 \le i \le \frac{n-1}{2}$. Switch $2i_0$ and $2i_0+1$ within $\sigma \in S_n - S_n^*$, where i_0 is the smallest i, $1 \le i \le \frac{n-1}{2}$, for which $\sigma_{[2i+1]} \in S_{2i+1} - S_{2i+1}^*$. $\square$

The preceding parity theorem has the refinement

Theorem 2.2. *For all $n \in N$,*

$$c_{-1}(n,k) := \sum_{\sigma \in S_{n,k}} (-1)^{inv_c(\sigma)} = \binom{\lfloor n/2 \rfloor}{n-k}, \qquad 0 \le k \le n. \tag{2.10}$$

Proof. Set $q = -1$ in (2.7) to get

$$\sum_{k=0}^{n} c_{-1}(n,k) x^k = x^{\lceil n/2 \rceil}(x+1)^{\lfloor n/2 \rfloor} = \sum_{k=\lceil n/2 \rceil}^{n} \binom{\lfloor n/2 \rfloor}{n-k} x^k.$$

Or let $S_{n,k}^{\pm} := S_{n,k} \cap S_n^{\pm}$ and $S_{n,k}^* := S_{n,k} \cap S_n^*$. Then $S_{n,k}^* \subseteq S_{n,k}^+$ and its cardinality agrees with the right-hand side of (2.10). The restriction of the map used for Theorem 2.1 to $S_{n,k} - S_{n,k}^*$ is again an involution and inherits the parity changing property. $\square$

Remark. The bijection of Theorem 2.2 also proves combinatorially that

$$c(n,k) \equiv \binom{\lfloor n/2 \rfloor}{n-k} \pmod 2, \qquad 0 \leqslant k \leqslant n, \tag{2.11}$$

since off of a set of cardinality $\binom{\lfloor n/2 \rfloor}{n-k}$, each permutation $\sigma \in S_{n,k}$ is paired with another of opposite inv_c-parity. The congruences in (2.11) can also be obtained by taking mod 2 the polynomial identities in (2.2) (cf. [12, p. 46, Exercise 17c]).

3. Some Statistics for Derangements

A permutation σ of $[n]$ having no fixed points (i.e., $i \in [n]$ such that $\sigma(i) = i$) is called a derangement. Let D_n denote the set of derangements of $[n]$ and $d_n := |D_n|$. A typical inclusion-exclusion argument gives the well known formula

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}, \qquad \forall n \in \mathbb{N}. \tag{3.1}$$

Given $\sigma \in D_n$, express it in the form

$$\sigma = (\alpha_1)(\alpha_2)\cdots,$$

where $\alpha_1, \alpha_2, \ldots$ are the cycles of σ arranged as follows:

(i) the cycles $\alpha_1, \alpha_2, \ldots$ are ordered by increasing second smallest elements;

(ii) each cycle (α_i) is written with the second smallest element in the last position.

Garsia and Remmel [6] term this the *ordered cycle factorization* (OCF for brief) of σ.

Define the statistic inv_o on D_n as follows: write out the cycles of $\sigma \in D_n$ in OCF form; then remove parentheses and count inversions in the resulting word to obtain $inv_o(\sigma)$. As an illustration, for the derangement $\sigma \in D_7$ given by 4321756, we have $inv_o(\sigma) = 3$, the number of inversions in the word 2314576.

The statistic inv_o is due to Garsia and Remmel [6], who show that the generating function

$$D_q(n) := \sum_{\sigma \in D_n} q^{inv_o(\sigma)} = n_q^! \sum_{k=0}^{n} \frac{(-1)^k}{k_q^!}, \qquad \forall n \in \mathbb{N}, \tag{3.2}$$

which generalizes (3.1).

Theorem 3.1. *For all $n \in \mathbb{N}$,*

$$D_{-1}(n) = \begin{cases} 1, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{3.3}$$

Proof. Formula (3.3) is an immediate consequence of (3.2), for

$$\sum_{k=0}^{n} \frac{(-1)^k n_q^!}{k_q^!}\bigg|_{q=-1} = \sum_{k=0}^{n} (-1)^k \prod_{i=k+1}^{n} i_q \bigg|_{q=-1} = (-1)^{n-1} n_{-1} + (-1)^n,$$

as

$$j_{-1} = \begin{cases} 0, & \text{if } j \text{ is even;} \\ 1, & \text{if } j \text{ is odd.} \end{cases}$$

Alternatively, let $\sigma = (\alpha_1)(\alpha_2) \cdots \in D_n$ be expressed in OCF form, first assuming n is odd. Locate the leftmost cycle of σ containing at least three members and interchange the first two members of this cycle. Now assume n is even. If σ has a cycle of length greater than two, proceed as in the odd case. If all cycles of σ are transpositions and $\sigma \neq (1,2)(3,4) \cdots (n-1,n)$, let i_0 be the smallest integer i for which the transposition $(2i-1, 2i)$ fails to occur in σ. Switch $2i_0 - 1$ and $2i_0$ in σ, noting that $2i_0 - 1$ and $2i_0$ must both start cycles. Thus whenever n is even, every $\sigma \in D_n$ is paired with another of opposite inv_o-parity except for $(1,2)(3,4) \cdots (n-1,n)$, which has inv_o value zero. $\square$

Now consider the generating function $d_q(n)$ resulting when one restricts inv to D_n, i.e.,

$$d_q(n) := \sum_{\sigma \in D_n} q^{inv(\sigma)}. \tag{3.4}$$

We have been unable to find a simple formula for $d_q(n)$ which generalizes (3.1) or a recurrence satisfied by $d_q(n)$ that generalizes one for d_n. However, we do have the following parity result.

Theorem 3.2. *For all $n \in \mathbb{N}$,*

$$d_{-1}(n) = (-1)^{n-1}(n-1). \tag{3.5}$$

Proof. Equivalently, we show that the numbers $d_{-1}(n)$ satisfy

$$d_{-1}(n) = -d_{-1}(n-1) + (-1)^{n-1}, \qquad \forall n \in \mathbb{P}, \tag{3.6}$$

with $d_{-1}(0) = 1$. Let $n \geqslant 2$, $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in D_n$, and $D_n^* \subseteq D_n$ consist of those derangements σ for which $\sigma_1 = 2$ and $\sigma_2 \geqslant 3$. Define an *inv*-parity changing involution f on $D_n - D_n^* - \{n12\cdots n-1\}$ as follows:

(i) if $\sigma_2 \geqslant 3$, whence $\sigma_1 \geqslant 3$, switch 1 and 2 in σ to obtain $f(\sigma)$;

(ii) if $\sigma_2 = 1$, let k_0 be the smallest integer k, $1 \leqslant k \leqslant \lfloor \frac{n-1}{2} \rfloor$, such that $\sigma_{2k}\sigma_{2k+1} \neq (2k-1)(2k)$; switch $2k_0$ and $2k_0+1$ if $\sigma_{2k_0} = 2k_0 - 1$ or switch $2k_0 - 1$ and $2k_0$ if $\sigma_{2k_0} \geqslant 2k_0 + 1$ to obtain $f(\sigma)$.

Thus,

$$d_{-1}(n) := \sum_{\sigma \in D_n} (-1)^{inv(\sigma)} = \sum_{\sigma \in D_n^* \cup \{n12\cdots n-1\}} (-1)^{inv(\sigma)}. \tag{3.7}$$

One can regard members σ of D_n^* as 2 followed by a derangement of $[n-1]$ since within the terminal segment $\sigma' := \sigma_2\sigma_3\cdots\sigma_n$, we must have $\sigma_2 \neq 1$ and $\sigma_k \neq k$ for all $k \geqslant 3$. Thus,

$$\sum_{\sigma' : \sigma \in D_n^*} (-1)^{inv(\sigma')} = d_{-1}(n-1),$$

from which

$$\sum_{\sigma \in D_n^*} (-1)^{inv(\sigma)} = -d_{-1}(n-1), \tag{3.8}$$

since the initial 2 adds an inversion. The recurrence (3.6) follows immediately from (3.7) and (3.8) upon adding the contribution of $(-1)^{n-1}$ from the singleton $\{n12\cdots n-1\}$. □

Now consider the generating function $r_q(n)$ resulting when one restricts *maj* to D_n, i.e.,

$$r_q(n) := \sum_{\sigma \in D_n} q^{maj(\sigma)}. \tag{3.9}$$

We were unable to find a simple formula for $r_q(n)$ which generalizes (3.1). Yet when $q = -1$ we have

Theorem 3.3. *For all* $n \in \mathbb{N}$,

$$r_{-1}(n) = \begin{cases} (-1)^{n/2}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{3.10}$$

Proof. First verify (3.10) for $0 \leqslant n \leqslant 3$. Let $n \geqslant 4$ and $D_n^* \subseteq D_n$ consist of those derangements starting with 2143 when expressed as a word. We define a *maj*-parity changing involution of $D_n - D_n^*$ below. Note that for derangements of the form $\sigma = 2143\sigma_5 \cdots \sigma_n$, the subword $\sigma_5 \cdots \sigma_n$ is itself a derangement on $n-4$ elements. Thus for $n \geqslant 4$,

$$r_{-1}(n) := \sum_{\sigma \in D_n} (-1)^{maj(\sigma)} = \sum_{\sigma \in D_n^*} (-1)^{maj(\sigma)} = r_{-1}(n-4),$$

which proves (3.10).

We now define a *maj*-parity changing involution f of $D_n - D_n^*$ when $n \geqslant 4$. Let $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in D_n - D_n^*$ be expressed as a word. If possible, pair σ with $\sigma' = f(\sigma)$ according to (I) and (II) below:

(I) first, if both $\sigma_1 \neq 2$ and $\sigma_2 \neq 1$, then switch σ_1 and σ_2 within σ to obtain σ';

(II) if (I) cannot be implemented (i.e., $\sigma_1 = 2$ or $\sigma_2 = 1$) but $\sigma_3 \neq 4$ and $\sigma_4 \neq 3$, then switch σ_3 and σ_4 within σ to obtain σ'.

We now define f for the cases that remain. To do so, consider $S_\sigma := \sigma_1\sigma_2\sigma_3\sigma_4 \cap [4]$, where $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in D_n - D_n^*$ is of a form not covered by rules (I) and (II) above. We consider cases depending upon $|S_\sigma|$. If $|S_\sigma| = 2$ or if $|S_\sigma| = 4$, first multiply σ by the transposition (34) and then exchange the letters in the third and fourth positions to obtain σ'. This corresponds to the pairings

i) $\sigma = a1b3\ldots4\ldots \;\leftrightarrow\; \sigma' = a14b\ldots3\ldots$;

ii) $\sigma = 2ab3\ldots4\ldots \;\leftrightarrow\; \sigma' = 2a4b\ldots3\ldots$;

iii) $\sigma = 2341\ldots \;\leftrightarrow\; \sigma' = 2413\ldots$;

iv) $\sigma = 4123\ldots \;\leftrightarrow\; \sigma' = 3142\ldots$,

where $a, b \geqslant 5$.

If $|S_\sigma| = 3$, then pair according to one of six cases shown below where $a \geqslant 5$, leaving the other letters undisturbed:

i) $\sigma = 314a\ldots \;\leftrightarrow\; \sigma' = 41a3\ldots$;

ii) $\sigma = 234a\ldots \;\leftrightarrow\; \sigma' = 24a3\ldots;$

iii) $\sigma = a123\ldots \;\leftrightarrow\; \sigma' = 2a13\ldots;$

iv) $\sigma = a142\ldots \;\leftrightarrow\; \sigma' = 2a41\ldots;$

v) $\sigma = 21a3\ldots4\ldots \;\leftrightarrow\; \sigma' = 214a\ldots3\ldots;$

vi) $\sigma = a143\ldots2\ldots \;\leftrightarrow\; \sigma' = 2a43\ldots1\ldots.$

It is easy to verify that σ and σ' have opposite *maj*-parity in all cases. □

4. Statistics for Catalan Words

The Catalan numbers c_n are defined by the closed form

$$c_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n \in \mathbb{N}, \tag{4.1}$$

as well as by the recurrence

$$c_{n+1} = \sum_{j=0}^{n} c_j c_{n-j}, \qquad c_0 = 1. \tag{4.2}$$

If one defines the generating function

$$f(x) = \sum_{n \geqslant 0} c_n x^n, \tag{4.3}$$

then (4.2) is equivalent to

$$f(x) = 1 + x f(x)^2. \tag{4.4}$$

Due to (4.2), the Catalan numbers enumerate many combinatorial structures, among them the set C_n consisting of words $w = w_1 w_2 \cdots w_{2n}$ of n 1's and n 0's for which no initial segment contains more 1's than 0's (termed *Catalan words*). In this section, we'll look at two q-analogues of the Catalan numbers, one of Carlitz which generalizes (4.4) and another of MacMahon which generalizes (4.1), when $q = -1$. These q-analogues arise as generating functions for statistics on C_n.

If

$$\tilde{C}_q(n) := \sum_{w \in C_n} q^{inv(w)}, \tag{4.5}$$

then

$$\tilde{C}_q(n+1) = \sum_{k=0}^{n} q^{(k+1)(n-k)} \tilde{C}_q(k)\tilde{C}_q(n-k), \qquad \tilde{C}_q(0) = 1, \tag{4.6}$$

upon decomposing a Catalan word $w \in C_{n+1}$ into $w = 0w_1 1w_2$ with $w_1 \in C_k$, $w_2 \in C_{n-k}$ for some k, $0 \leqslant k \leqslant n$, and noting that the number of inversions of w is given by

$$inv(w) = inv(w_1) + inv(w_2) + (k+1)(n-k).$$

Taking reciprocal polynomials of both sides of (4.6) and writing

$$C_q(n) = q^{\binom{n}{2}} \tilde{C}_{q^{-1}}(n) \tag{4.7}$$

yields the recurrence [5]

$$C_q(n+1) = \sum_{k=0}^{n} q^k C_q(k) C_q(n-k), \qquad C_q(0) = 1. \tag{4.8}$$

If one defines the generating function

$$f(x) = \sum_{n \geqslant 0} C_q(n) x^n, \tag{4.9}$$

then (4.8) is equivalent to the functional equation [3, 5]

$$f(x) = 1 + x f(x) f(qx), \tag{4.10}$$

which generalizes (4.4).

Theorem 4.1. *For all $n \in \mathbb{N}$,*

$$C_{-1}(n) = \begin{cases} \delta_{n,0}, & \text{if } n \text{ is even;} \\ (-1)^{\frac{n-1}{2}} c_{\frac{n-1}{2}}, & \text{if } n \text{ is odd.} \end{cases} \tag{4.11}$$

Proof. Setting $q = -1$ in (4.10) gives

$$f(x) = 1 + x f(x) f(-x). \tag{4.12}$$

Putting $-x$ for x in (4.12), solving the resulting system in $f(x)$ and $f(-x)$, and noting $f(0) = 1$ yields

$$\begin{aligned} f(x) &= \sum_{n \geqslant 0} C_{-1}(n) x^n \\ &= \frac{(2x-1) + \sqrt{4x^2+1}}{2x} = 1 + \sum_{n \geqslant 1} (-1)^{n-1} \frac{1}{n} \binom{2n-2}{n-1} x^{2n-1}, \end{aligned}$$

which implies (4.11).

Alternatively, note that

$$C_{-1}(n) = (-1)^{\binom{n}{2}} \sum_{w \in C_n} (-1)^{inv(w)},$$

by (4.5) and (4.7). So (4.11) is equivalent to

$$\sum_{w \in C_n} (-1)^{inv(w)} = \begin{cases} \delta_{n,0}, & \text{if } n \text{ is even;} \\ c_{\frac{n-1}{2}}, & \text{if } n \text{ is odd.} \end{cases} \tag{4.13}$$

To prove (4.13), let C_n^+, $C_n^- \subseteq C_n$ consist of the Catalan words with even or odd *inv* values, respectively, and $C_n^* \subseteq C_n$ consist of those words $w = w_1 w_2 \cdots w_{2n}$ for which

$$w_{2i} w_{2i+1} = 00 \text{ or } 11, \qquad 1 \leqslant i \leqslant n-1. \tag{4.14}$$

Clearly, $C_n^* \subseteq C_n^+$ with cardinality matching the right-hand side of (4.13). Suppose $w \in C_n - C_n^*$ and that i_0 is the smallest index for which (4.14) fails to hold. Switch w_{2i_0} and w_{2i_0+1} in w. The resulting map is a parity changing involution of $C_n - C_n^*$, which proves (4.13) and hence (4.11). □

Another q-Catalan number arises as the generating function for the major index statistic on C_n [8]. If

$$\tilde{c}_q(n) := \sum_{w \in C_n} q^{maj(w)}, \tag{4.15}$$

then there is the closed form (see [5], [8, p. 215])

$$\tilde{c}_q(n) = \frac{1}{(n+1)_q} \binom{2n}{n}_q, \qquad \forall n \in \mathbb{N}, \tag{4.16}$$

which generalizes (4.1).

Theorem 4.2. *For all $n \in \mathbb{N}$,*

$$\tilde{c}_{-1}(n) = \binom{n}{\lfloor n/2 \rfloor}. \tag{4.17}$$

Proof. If n is even, then by (4.16),

$$\tilde{c}_{-1}(n) = \lim_{q \to -1} \tilde{c}_q(n) = \lim_{q \to -1} \frac{1}{(n+1)_q} \prod_{i=0}^{n-1} \frac{(2n-i)_q}{(n-i)_q}$$
$$= \prod_{\substack{i=0 \\ i \text{ even}}}^{n-2} \lim_{q \to -1} \left(\frac{q^{2n-i}-1}{q^{n-i}-1} \right) = \prod_{\substack{i=0 \\ i \text{ even}}}^{n-2} \frac{2n-i}{n-i} = \prod_{\substack{i=0 \\ i \text{ even}}}^{n-2} \frac{n-i/2}{n/2-i/2} = \binom{n}{n/2},$$

with the odd case handled similarly.

Alternatively, let C_n^+, $C_n^- \subseteq C_n$ consist of the Catalan words with even or odd major index value, respectively, and $C_n^* \subseteq C_n$ consist of those words $w = w_1 w_2 \cdots w_{2n}$ which satisfy the following two requirements:

(i) one can express w as $w = x_1 x_2 \cdots x_n$, where $x_i = 00$, 11, or 01, $1 \leqslant i \leqslant n$;

(ii) for each i, $x_i = 01$ only if the number of 00's in the initial segment $x_1 x_2 \cdots x_{i-1}$ equals the number of 11's. (A word in C_n^* may start with either 01 or 00.)

Clearly, $C_n^* \subseteq C_n^+$ and below it is shown that $|C_n^*| = \binom{n}{\lfloor n/2 \rfloor}$. Suppose $w = w_1 w_2 \cdots w_{2n} \in C_n - C_n^*$ and that i_0 is the smallest integer i, $1 \leqslant i \leqslant n$, such that one of the following two conditions holds:

(i) $w_{2i-1} w_{2i} = 10$, or

(ii) $w_{2i-1} w_{2i} = 01$ and the number of 0's in the initial segment $w_1 w_2 \cdots w_{2i-2}$ is strictly greater than the number of 1's.

Switching w_{2i_0-1} and w_{2i_0} in w changes the major index by an odd amount and the resulting map is a parity changing involution of $C_n - C_n^*$.

We now show $|C_n^*| = \binom{n}{\lfloor n/2 \rfloor}$ by defining a bijection between C_n^* and the set $\Lambda(n)$ of (minimal) lattice paths from $(0,0)$ to $(\lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor)$. Given $w = x_1 x_2 \cdots x_n \in C_n^*$ as described in (i) and (ii) above, we construct a lattice path $\lambda_w \in \Lambda(n)$ as follows. Let $j_1 < j_2 < \ldots$ be the set of indices j, possibly empty and denoted $S(w)$, for which $x_j = 01$, with $j_0 := 0$. For $s \geqslant 1$, let step j_s in λ_w be a V (vertical step) if s is odd and an H (horizontal step) if s is even.

Suppose now $i \in [n] - S(w)$ and that t, $t \geqslant 0$, is the greatest integer such that $j_t < i$. If t is even, put a V (resp., H) for the i^{th} step of λ_w if $x_i = 11$ (resp., 00). If t is odd, put a V (resp., H) for the i^{th} step of λ_w if $x_i = 00$ (resp., 11), which now specifies λ_w completely. The map $w \mapsto \lambda_w$ is seen to be a bijection between C_n^* and $\Lambda(n)$; note that $S(w)$ corresponds to the steps of λ_w in which it either rises above the line $y = x$ or returns to $y = x$ from above. □

Note that the preceding supplies a combinatorial proof of the congruence $\frac{1}{n+1}\binom{2n}{n} \equiv \binom{n}{\lfloor n/2 \rfloor} \pmod 2$ for $n \in \mathbb{N}$ since off of a set of cardinality $\binom{n}{\lfloor n/2 \rfloor}$, each Catalan word $w \in C_n$ is paired with another of opposite *maj*-parity.

Let $P_n \subseteq S_n$ consist of those permutations $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ avoiding the pattern 312, i.e., there are no indices $i < j < k$ such that $\sigma_j < \sigma_k < \sigma_i$ (termed *Catalan permutations*).

Knuth [7, p. 238] describes a bijection g between P_n and C_n in which

$$inv(\sigma) = \binom{n}{2} - inv(g(\sigma)), \qquad \forall \sigma \in P_n,$$

and hence

$$C_q(n) := \sum_{w \in C_n} q^{\binom{n}{2} - inv(w)} = \sum_{\sigma \in P_n} q^{inv(\sigma)}. \tag{4.18}$$

By (4.11) and (4.18), we then have the parity result

$$\sum_{\sigma \in P_n} (-1)^{inv(\sigma)} = \begin{cases} \delta_{n,0}, & \text{if } n \text{ is even;} \\ (-1)^{\frac{n-1}{2}} c_{\frac{n-1}{2}}, & \text{if } n \text{ is odd.} \end{cases} \tag{4.19}$$

The composite map $g^{-1} \circ h \circ g$, where h is the involution establishing (4.13), furnishes an appropriate involution for (4.19).

References

[1] G. Andrews, The theory of partitions, *Encyclopedia of Mathematics and Applications, Vol.* II, Addison-Wesley (1976).

[2] L. Carlitz, Generalized Stirling numbers, *Combinatorial Analysis Notes*, Duke University (1968), 1–7.

[3] L. Carlitz and R. Scoville, A note on weighted sequences, *Fib. Quart.* 13 (1975), 303–306.

[4] P. Edelman, R. Simion, and D. White, Partition statistics on permutations, *Discrete Math.* 99 (1992), 63–68.

[5] J. Fürlinger and J. Hofbauer, q-Catalan numbers, *J. Combin. Theory, Ser. A*, 40 (1985), 248–264.

[6] A. Garsia and J. Remmel, A combinatorial interpretation of q-derangement and q-Laguerre numbers, *Europ. J. Combinatorics* 1 (1980), 47–59.

[7] D. Knuth, *The Art of Computer Programming: Fundamental Algorithms, Vol.* I, Addison-Wesley (1968).

[8] P. MacMahon, *Combinatorial Analysis, Vol.* II, Cambridge Univ. Press, 1915–1916. Reprinted, Chelsea, New York, 1960.

[9] M. Shattuck, Bijective proofs of parity theorems for partition statistics, *J. Integer Seq.* 8 (2005), Art. 5.1.5.

[10] M. Shattuck and C. Wagner, Parity theorems for statistics on lattice paths and Laguerre configurations, *Research Report*, Department of Mathematics, University of Tennessee (2004).

[11] M. Shattuck and C. Wagner, Parity theorems for statistics on domino arrangements, *Elect. J. Combin.*, to appear.

[12] R. Stanley, *Enumerative Combinatorics, Vol.* I, Wadsworth and Brooks/Cole, 1986.

[13] C. Wagner, Partition statistics and q-Bell numbers ($q = -1$), *J. Integer Seq.* 7 (2004), Art. 4.1.1.