ON SUMS OF SQUARES OF PELL-LUCAS NUMBERS

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Abstract

In this paper we prove several formulas for sums of squares of even Pell-Lucas numbers, sums of squares of odd Pell-Lucas numbers, and sums of products of even and odd Pell-Lucas numbers. These sums have nice representations as products of appropriate Pell and Pell-Lucas numbers with terms from certain integer sequences.

1. Introduction

The Pell and Pell-Lucas sequences P_n and Q_n are defined by the recurrence relations

$$P_0 = 0, \qquad P_1 = 2, \qquad P_n = 2\,P_{n-1} + P_{n-2} \text{ for } n \geqslant 2,$$

and

$$Q_0 = 2, \qquad Q_1 = 2, \qquad Q_n = 2\,Q_{n-1} + Q_{n-2} \text{ for } n \geqslant 2.$$

In the Sections 2–4 we consider sums of squares of odd and even terms of the Pell-Lucas sequence and sums of their products. These sums have nice representations as products of appropriate Pell and Pell-Lucas numbers.

The numbers Q_k make the integer sequence $A002203$ from [8] while the numbers $\frac{1}{2}\,P_k$ make $A000129$. In this paper we shall also need the sequences $A001109$, $A029546$, $A029547$ and $A077420$ that we shorten to a_k, b_k, c_k and d_k with $k \geqslant 0$. For the convenience of the reader we shall now explicitly define these sequences.

The first ten terms of the sequence a_k are 0, 1, 6, 35, 204, 1189, 6930, 40391, 235416 and 1372105, it satisfies the recurrence relations $a_0 = 0$, $a_1 = 1$ and $a_n = 6\,a_{n-1} - a_{n-2}$ for $n \geq 2$, and it is given by the formula $a_n = \frac{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n}{4\sqrt{2}}$.

The first seven terms of the sequence b_k are 1, 35, 1190, 40426, 1373295, 46651605 and 1584781276, it satisfies the recurrence relations $b_0 = 1$, $b_1 = 35$, $b_2 = 1190$ and

$$b_n = 35\,(b_{n-1} - b_{n-2}) - b_{n-3}$$

for $n \geq 3$, and it is given by the formula $b_n = \frac{1}{192}\,(f_+\,F_+^n + f_-\,F_-^n - 6)$, where $f_\pm = 99 \pm 70\sqrt{2}$ and $F_\pm = 17 \pm 12\sqrt{2}$.

The first seven terms of the sequence c_k are 1, 34, 1155, 39236, 1332869, 45278310 and 1538129671, it satisfies the recurrence relations $c_0 = 1$, $c_1 = 34$ and $c_n = 34\,c_{n-1} - c_{n-2}$ for $n \geq 2$, and it is given by the formula $c_n = \frac{F_+^{n+1} - F_-^{n+1}}{F_+ - F_-}$.

Finally, the first seven terms of the sequence d_k are 1, 33, 1121, 38081, 1293633, 43945441 and 1492851361, it satisfies the recurrence relations $d_0 = 1$, $d_1 = 33$ and $d_n = 34\,d_{n-1} - d_{n-2}$ for $n \geq 2$, and it is given by the formula $d_n = \frac{3+2\sqrt{2}}{6}\,F_+^n + \frac{3-2\sqrt{2}}{6}\,F_-^n$.

In the last three sections we look into the alternating sums of squares of odd and even terms of the Pell-Lucas sequence and the alternating sums of products of two consecutive Pell-Lucas numbers. These sums also have nice representations as products of appropriate Pell and Pell-Lucas numbers with terms from the above four integer sequences.

These formulas for ordinary sums and for alternating sums have been discovered with the help of a PC computer and all algebraic identities needed for the verification of our theorems can be easily checked in either Derive, Mathematica or Maple V. Running times of all these calculations are in the range of a few seconds.

Similar results for Fibonacci and Lucas numbers have recently been discovered by the first author in papers [1], [2], [3], and [4]. They improved some results in [7].

2. Pell-Lucas even squares

The following lemma is needed to accomplish the inductive step in the proof of our first theorem.

Lemma 1. *For every $m \geqslant 0$ and $k \geqslant 0$ the following equality holds:*

$$32\,a_{m+1}^2 + Q_{2k+2m+2}^2 + a_{m+1} \cdot Q_{2k} \cdot Q_{2k+2m} = a_{m+2} \cdot Q_{2k} \cdot Q_{2k+2m+2}. \qquad (2.1)$$

Proof. Let $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. Notice that $\alpha + \beta = 2$ and $\alpha \cdot \beta = -1$ so that the real numbers α and β are solutions of the equation $x^2 - 2\,x - 1 = 0$. Since $Q_j = \alpha^j + \beta^j$

and

$$a_j = \frac{7\alpha+\beta}{16}\,\alpha^{2j-2} + \frac{\alpha+7\beta}{16}\,\beta^{2j-2}$$

for every $j \geqslant 0$, the difference of the right hand side and the left hand side of the relation (2.1) (after the substitutions $\beta = -\frac{1}{\alpha}$ and $\alpha = 1+\sqrt{2}$ and the replacement of $99+70\sqrt{2}$, $17+12\sqrt{2}$, $7+5\sqrt{2}$, $3+2\sqrt{2}$, $1+\sqrt{2}$, and $-1+\sqrt{2}$ with α^6, α^4, α^3, α^2, α and $\frac{1}{\alpha}$) is equal to $\frac{M}{16\,\alpha^4}$, where $M = p_0\,\alpha^{4k+4m} + p_-\,\alpha^{4k}$ with $p_0 = p_+ - 9232 - 6528\sqrt{2}$ and

$$p_\pm = 3\,\alpha^8 + 17\,\alpha^7 + 30\,\alpha^6 \pm 14\,\alpha^5 - 5\,\alpha^4 \mp 3\,\alpha^3.$$

Since both polynomials p_0 and p_- contain $\alpha - 1 - \sqrt{2}$ as a factor we conclude that $M = 0$ and the proof is complete. □

Theorem 1. *For every $m \geqslant 0$ and $k \geqslant 0$ the following equality holds:*

$$\alpha_m + \sum_{i=0}^{m} Q_{2k+2i}^2 = \begin{cases} Q_{4\,k}, & \text{if } m = 0; \\ a_{m+1}\cdot Q_{2k}\cdot Q_{2k+2m}, & \text{if } m \geqslant 1, \end{cases} \tag{2.2}$$

where the sequence α_m is defined as follows: $\alpha_0 = -2$, $\alpha_1 = 32$, and $\alpha_m - \alpha_{m-1} = 32\cdot a_m^2$ for $m \geqslant 2$ (i. e., $\alpha_m = 32\cdot\left(\sum_{j=1}^m a_j^2\right)$).

Proof. When $m = 0$ we obtain

$$Q_{2k}^2 - Q_{4k} - 2 = \left(\alpha^{2\,k} + \beta^{2\,k}\right)^2 - \alpha^{4\,k} - \beta^{4\,k} - 2 = 2\,\left(\alpha^k\right)^2\left(\beta^k\right)^2 - 2 = 0.$$

For $m \geqslant 1$ the proof is by induction on m. When $m = 1$ the relation (2.2) is

$$32 + Q_{2k}^2 + Q_{2k+2}^2 - 6\cdot Q_{2k}\cdot Q_{2k+2} = 0$$

which is true since its left hand side is $(\alpha^2 - 2\,\alpha - 1)(\alpha^2 + 2\,\alpha - 1)(\alpha^{4k} + \alpha^{-4k-4} - 6\,\alpha^{-2})$.

Assume that the relation (2.2) is true for $m = r$. Then

$$\alpha_{r+1} + \sum_{i=0}^{r+1} Q_{2k+2i}^2 = 32\,a_{r+1}^2 + \alpha_r + Q_{2k+2r+2}^2 + \sum_{i=0}^{r} Q_{2k+2i}^2 =$$

$$32\,a_{r+1}^2 + Q_{2k+2r+2}^2 + a_{r+1}\cdot Q_{2k}\cdot Q_{2k+2r} = a_{r+2}\cdot Q_{2k}\cdot Q_{2k+2r+2},$$

where the last step uses Lemma 1 for $m = r+1$. Hence, (2.2) is true for $m = r+1$ and the proof is completed. □

Remark 1. The following are three additional versions of Theorem 1:

For every $j \geqslant 0$ and $k \geqslant 1$ the following equalities holds:

$$\alpha_{j+1} + \sum_{i=0}^{j} Q_{2k+2i}^2 = a_{j+1}\cdot Q_{2k-2}\cdot Q_{2k+2j+2}, \tag{2.3}$$

$$\sum_{i=0}^{j} Q_{2k+2i}^2 = \gamma_j + a_{j+1}\cdot Q_{2k-1}\cdot Q_{2k+2j+1}, \tag{2.4}$$

$$\sum_{i=0}^{j} Q_{2k+2i}^2 = \delta_j + a_{j+1} \cdot Q_{2k+1} \cdot Q_{2k+2j-1}, \tag{2.5}$$

where $\gamma_0 = \delta_0 = 8$ and $\gamma_j - \gamma_{j-1} = 2 \cdot P_{2\,j+1}^2$ and $\delta_j - \delta_{j-1} = 2 \cdot P_{2\,j-1}^2$ for every $j \geqslant 1$.

3. Pell-Lucas odd squares

The initial step in an inductive proof of our second theorem uses the following lemma.

Lemma 2. *For every $k \geqslant 1$ the following identity holds:*

$$Q_{2k+1}^2 - Q_{2k-1}\, Q_{2k+3} = 32. \tag{3.1}$$

Proof. By the Binet formula $Q_n = \alpha^n + \beta^n$ so that we have

$$\begin{aligned} Q_{2k+1}^2 - Q_{2k-1}\, Q_{2k+3} = &\\ &(\alpha^{2k+1} + \beta^{2k+1})^2 - (\alpha^{2k-1} + \beta^{2k-1})(\alpha^{2k+3} + \beta^{2k+3}) = \\ &\quad 2\,(\alpha \cdot \beta)^{2\,k+1} - (\alpha \cdot \beta)^{2\,k-1}(\alpha^4 + \beta^4) = -2 + Q_4 = -2 + 34 = 32. \end{aligned}$$

□

The following lemma is needed to accomplish the inductive step in the proof of our second theorem.

Lemma 3. *For every $j \geqslant 0$ and $k \geqslant 1$ the following equality holds:*

$$Q_{2k+2j+3}^2 + Q_{2k-1}(a_{j+1}\, Q_{2k+2j+3} - a_{j+2}\, Q_{2k+2j+5}) = 32\, a_{j+2}^2. \tag{3.2}$$

Proof. The difference of the left hand side and the right hand side of (3.2) (after the substitutions $\beta = -\frac{1}{\alpha}$ and $\alpha = 1 + \sqrt{2}$ and the replacement of $99 + 70\sqrt{2}$, $17 + 12\sqrt{2}$, $7 + 5\sqrt{2}$, $3 + 2\sqrt{2}$, $1 + \sqrt{2}$, and $-1 + \sqrt{2}$ with α^6, α^4, α^3, α^2, α and $\frac{1}{\alpha}$) is equal to $\frac{M}{16\,\alpha^8}$, where

$$M = p_0\, \alpha^{4k+4j} + p_-\, \alpha^{4k}$$

with $p_0 = p_+ - 1827888 - 1292512\,\sqrt{2}$ and

$$p_\pm = 3\,\alpha^{14} \pm 17\,\alpha^{13} + 30\,\alpha^{12} \pm 14\,\alpha^{11} - 5\,\alpha^{10} \mp 3\,\alpha^9.$$

Since both polynomials p_0 and p_- contain $\alpha - 1 - \sqrt{2}$ as a factor we conclude that $M = 0$ and the proof is complete. □

Theorem 2. *For every $j \geqslant 0$ and $k \geqslant 1$ the following equality holds:*

$$\sum_{i=0}^{j} Q_{2k+2i+1}^2 = \alpha_{j+1} + a_{j+1} \cdot Q_{2k-1} \cdot Q_{2k+2j+3}. \tag{3.3}$$

Proof. The proof is by induction on j. For $j = 0$ the relation (3.3) is

$$Q_{2k+1}^2 = 32 + Q_{2k-1}\, Q_{2k+3}$$

which is true by the relation (3.1) in Lemma 2. Assume that the relation (3.3) is true for $j = r$. Then

$$\sum_{i=0}^{r+1} Q_{2k+2i+1}^2 = Q_{2k+2r+3}^2 + \sum_{i=0}^{r} Q_{2k+2i+1} =$$

$$Q_{2k+2r+3}^2 + \alpha_{r+1} + a_{r+1} \cdot Q_{2k-1} \cdot Q_{2k+2r+3} =$$

$$\alpha_{r+1} + 32\, a_{r+2}^2 + a_{r+2} \cdot Q_{2k-1} \cdot Q_{2k+2r+5} =$$

$$\alpha_{(r+1)+1} + a_{r+2} \cdot Q_{2k-1} \cdot Q_{2k+2(r+1)+3},$$

where the third step uses Lemma 3. Hence, (3.3) is true also for $j = r + 1$ and the proof is complete. □

Another version of Theorem 2 is the following statement:

Theorem 3. *For every $j \geqslant 0$ and $k \geqslant 1$ the following equality holds:*

$$\beta_j + \sum_{i=0}^{j} Q_{2k+2i+1}^2 = a_{j+1} \cdot Q_{2k-2} \cdot Q_{2k+2j+4}, \tag{3.4}$$

with $\beta_0 = 200$ and $\beta_j - \beta_{j-1} = 2 \cdot P_{2j+3}^2$ for $j \geqslant 1$.

Proof. We shall only outline the key steps in an inductive proof leaving the details to the reader because they are analogous to the proof of Theorem 2.

The initial step is the equality $200 + Q_{2k+1}^2 = Q_{2k-2}\, Q_{2k+4}$ which holds for every $k \geqslant 1$. On the other hand, the inductive step is realized with the following equality:

$$2\, P_{2r+5}^2 + Q_{2k+2r+3}^2 + a_{r+1}\, Q_{2k-2}\, Q_{2k+2r+4} = a_{r+2}\, Q_{2k-2}\, Q_{2k+2r+6},$$

which holds for every $k \geqslant 1$ and every $r \geqslant 1$. □

4. Pell-Lucas products

For the first two steps in a proof by induction of our next theorem we require the following lemma.

Lemma 4. *For every $k \geqslant 1$ the following equalities hold:*

$$Q_{2k}\, Q_{2k+1} = Q_{4k+1} + 2. \tag{4.1}$$

$$Q_{2k}\, Q_{2k+1} + Q_{2k+2}\, Q_{2k+3} = 6\, Q_{2k}\, Q_{2k+3} - 80. \tag{4.2}$$

Proof of (4.1). By the Binet formula we have

$$Q_{2k}\,Q_{2k+1} = \left(\alpha^{2k}+\beta^{2k}\right)\left(\alpha^{2k+1}+\beta^{2k+1}\right) = \alpha^{4k+1}+(\alpha\cdot\beta)^{2k}(\alpha+\beta)+\beta^{4k+1} = Q_{4k+1}+2.$$

□

Proof of (4.2). When we apply the Binet formula to the terms in the difference of the left hand side and the right hand side of (4.2) we get

$$80+\alpha^{4\,k+1}+\alpha\,(\alpha\,\beta)^{2\,k}+\beta\,(\alpha\,\beta)^{2\,k}+\beta^{4\,k+1}+\alpha^{4\,k+5}+\alpha^3\beta^2\,(\alpha\,\beta)^{2\,k}+ \beta^3\alpha^2\,(\alpha\,\beta)^{2\,k}+\beta^{4\,k+5}-6\alpha^{4\,k+3}-6\alpha^3\,(\alpha\,\beta)^{2\,k}-6\beta^3\,(\alpha\,\beta)^{2\,k}-6\beta^{4\,k+3}.$$

Since $\alpha\,\beta=-1$, this simplifies to $p(\alpha)\,\alpha^{4k}+p(\beta)\,\beta^{4k}+q$, where $p(x) = x^5-6\,x^3+x$ and $q=\alpha^3\,\beta^2+\alpha^2\,\beta^3-6\,\alpha^3-6\,\beta^3+\alpha+\beta+80$. Now it is easy to check (by the substitutions $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$) that $p(\alpha)=0$, $p(\beta)=0$ and $q=0$. This implies that the relation (4.2) holds. □

With the following lemma we shall make the inductive step in the proof of the third theorem.

Lemma 5. *For every $r \geqslant 1$ and $k \geqslant 1$ the following equality holds:*

$$Q_{2k}(a_{r+2}\,Q_{2k+2r+3}-a_{r+1}\,Q_{2k+2r+1})-Q_{2k+2r+2}\,Q_{2k+2r+3} = 2\,P_{2r+2}\,P_{2r+3}.$$

Proof. Let R denote the difference of the left hand side and the right hand side of the above relation. We need to show that $R=0$.

Let $A=\alpha^{2k}$, $B=\beta^{2k}$, $U=\beta^{2r}$, $V=\beta^{2r}$, $u=\frac{7\alpha+\beta}{16}$ and $v=\frac{\alpha+7\beta}{16}$. Note that a_{r+1} is equal to $u\,U+v\,V$. By the Binet formula we get

$$\begin{aligned}R = {} & (A+B)\left[(u\,\alpha^2\,U+v\,\beta^2\,V)(\alpha^3\,A\,U+\beta^3\,B\,V)-(u\,U+v\,V)\right.\\ & \left.(\alpha\,A\,U+\beta\,B\,V)\right]-(\alpha^2\,A\,U+\beta^2\,B\,V)(\alpha^3\,A\,U+\beta^3\,B\,V)-\\ & (\alpha^2\,U+\beta^2\,V)(\alpha^3\,U+\beta^3\,V) = \ell_+\left[u\,A\,B+\frac{(8\,u-m)\,A^2-m}{8}\right]U^2\\ & -2\,(A\,B-1)\,U\,V+(\ell_-\,v\,A\,B-n)\,V^2,\end{aligned}$$

where $\ell_\pm = 4(10\pm 7\sqrt{2})$, $m=4+3\sqrt{2}$ and $n=41-29\sqrt{2}$. Since $A\,B=1$, $u=\frac{m}{8}$ and $\ell_-\,v=n$ it follows that $R=0$. □

Theorem 4. *For every $j \geqslant 0$ and $k \geqslant 1$ the following equality holds:*

$$\alpha_j+\sum_{i=0}^{j} Q_{2k+2i}\cdot Q_{2k+2i+1} = \begin{cases} Q_{4\,k}, & \text{if } j=0;\\ a_{j+1}\cdot Q_{2k}\cdot Q_{2k+2j+1}, & \text{if } j \geqslant 1,\end{cases} \tag{4.3}$$

where $\alpha_0=-2$, $\alpha_1=80$, and $\alpha_{j+1}-\alpha_j = 2\cdot P_{2j+2}\cdot P_{2j+3}$ for $j \geqslant 1$.

Proof. The proof is by induction on j. For $j = 0$ the relation (4.3) is

$$Q_{2k}\,Q_{2k+1} - 2 = Q_{4k+1}$$

which is true by (4.1). For $j = 1$ the relation (4.3) is

$$80 + Q_{2k}\,Q_{2k+1} + Q_{2k+2}\,Q_{2k+3} = 6\,Q_{2k}\,Q_{2k+3}$$

which is true by (4.2).

Assume that the relation (4.3) is true for $j = r$. Then

$$\alpha_{r+1} + \sum_{i=0}^{r+1} Q_{2k+2i}\,Q_{2k+2i+1} = \alpha_r + 2\,P_{2r+2}\,P_{2r+3}$$
$$+ \sum_{i=0}^{r} Q_{2k+2i}\,Q_{2k+2i+1} + Q_{2k+2r+2}\,Q_{2k+2r+3} = a_{r+1}\,Q_{2k}\,Q_{2k+2r+1}$$
$$+ 2\,P_{2r+2}\,P_{2r+3} + Q_{2k+2r+2}\,Q_{2k+2r+3} = a_{r+2}\,Q_{2k}\,Q_{2k+2(r+1)+1},$$

where the last step uses Lemma 5. Hence, (4.3) is true also for $j = r + 1$. □

5. Alternating Pell-Lucas even squares

In this section we look for formulas that give closed forms for alternating sums of squares of Pell-Lucas numbers with even indices.

Lemma 6. *For every $k \geqslant 0$ we have*

$$Q_{2k}^2 = 8 + Q_{2k+1} \cdot Q_{2k-1}. \tag{5.1}$$

Proof. By the Binet formula we get

$$8 + Q_{2k-1}\,Q_{2k+1} = 8 + (\alpha^{2k-1} + \beta^{2k-1})(\alpha^{2k+1} + \beta^{2k+1}) = 8 + \alpha^{4k} +$$
$$\beta^2\,(\alpha\,\beta)^{2k-1} + \alpha^2\,(\alpha\,\beta)^{2k-1} + \beta^{4k} = \alpha^{4k} + 2 + \beta^{4k} = (\alpha^{2k} + \beta^{2k})^2 = Q_{2k}^2.$$

□

Lemma 7. *For every $k \geqslant 0$ we have*

$$Q_{2k}^2 - Q_{2k+2}^2 = 32 \cdot (1 - a_{k+1}\,Q_{2k}). \tag{5.2}$$

Proof. For real numbers a, b and c let $\mathfrak{p}_b^a$, $\mathfrak{q}_a$ and $\mathfrak{r}_c^{a;b}$ denote $\frac{\alpha^a - \beta^a}{\alpha^b - \beta^b}$, $\alpha^a + \beta^a$ and $\frac{a\,\alpha + b\,\beta}{c}$. The difference of the left hand side and the right hand side in (5.2) is

$$(\mathfrak{q}_{2k})^2 - (\mathfrak{q}_{2k+2})^2 - 32 + 32\left(\mathfrak{r}_{16}^{7;1}\,\alpha^{2k} + \mathfrak{r}_{16}^{1;7}\,\beta^{2k}\right)\,\mathfrak{q}_{2k}.$$

Let $\alpha^{2k} = A$ and $\beta^{2k} = B$. The above expression reduces to

$$p(\alpha)\,A^2 + q\,A\,B + p(\beta)\,B^2 - 32,$$

where $p(\alpha) = 1 + 14\,\alpha + 2\,\beta - \alpha^4$ and $q = 2 + 16\,\alpha + 16\,\beta - 2\,\alpha^2\,\beta^2$. If we replace α and β with $1+\sqrt{2}$ and $1-\sqrt{2}$ we get $p(\alpha) = p(\beta) = 0$ and $q = 32$. Hence, the difference is $32\,A\,B - 32 = 0$ because $A\,B = 1$. □

Lemma 8. *For every $k \geqslant 0$ and every $r \geqslant 0$ we have*

$$32 \cdot c_r \cdot (1 - a_{k+r+1}\,Q_{2k+2r}) + Q^2_{2k+4r+4} - Q^2_{2k+4r+6} \\ = 32 \cdot c_{r+1} \cdot (1 - a_{k+r+2}\,Q_{2k+2r+2}). \tag{5.3}$$

Proof. Notice that $c_r = \mathfrak{p}_4^{4r+4}$ for every $r \geqslant 0$. The difference of the left hand side and the right hand side in (5.3) is

$$32 \cdot \mathfrak{p}_4^{4r+4}\left[1 - \left(\mathfrak{r}_{16}^{7;1}\,\alpha^{2k+2r} + \mathfrak{r}_{16}^{1;7}\,\beta^{2k+2r}\right)\mathfrak{q}_{2k+2r}\right] + (\mathfrak{q}_{2k+4r+4})^2 - \\ (\mathfrak{q}_{2k+4r+6})^2 - 32 \cdot \mathfrak{p}_4^{4r+8}\left[1 - \left(\mathfrak{r}_{16}^{7;1}\,\alpha^{2k+2r+2} + \mathfrak{r}_{16}^{1;7}\,\beta^{2k+2r+2}\right)\mathfrak{q}_{2k+2r+2}\right].$$

Let $\alpha^{2k} = A$, $\beta^{2k} = B$, $\alpha^{2r} = U$ and $\beta^{2r} = V$. The above expression reduces to $\frac{M}{\alpha^4 - \beta^4}$, where M is

$$p(\alpha,\beta)\,A^2\,U^4 - q(\alpha,\beta)\,A^2\,U^2\,V^2 + r(\alpha,\beta)\,A\,B\,U^3\,V + 2\,s\,A\,B\,U^2\,V^2 - \\ p(\beta,\alpha)B^2\,V^4 + q(\beta,\alpha)B^2\,U^2\,V^2 - r(\beta,\alpha)A\,B\,U\,V^3 - 32(t(\alpha)U^2 - t(\beta)V^2),$$

where

$$p(\alpha,\beta) = t(\alpha)(\alpha^4\,\beta^4 - \alpha^8 + 14\,\alpha^5 + 2\,\alpha^4\,\beta + 14\,\alpha + 2\,\beta),$$

$$q(\alpha,\beta) = 2\,\beta^4(7\,\alpha + \beta)(\alpha^4\,\beta^4 - 1),$$

$$r(\alpha,\beta) = 16\,\alpha^4(\alpha + \beta)(\alpha^6\,\beta^2 - 1),$$

$s = \alpha^4\,\beta^4(\alpha^4 - \beta^4)(\alpha^2\,\beta^2 - 1)$ and $t(\alpha) = \alpha^4\,(\alpha^4 - 1)$. Replacing α and β with $1+\sqrt{2}$ and $1-\sqrt{2}$ we get

$$M = u_+\,A\,B\,U^3\,V - u_-\,A\,B\,U\,V^3 - u_+\,U^2 + u_-\,V^2,$$

where $u_\pm = 128(140 \pm 99\sqrt{2})$. Hence, $M = 0$ so that the difference is zero because $A\,B = 1$ and $U\,V = 1$. □

Lemma 9. *For every $k \geqslant 0$ and every $r \geqslant 0$ we have*

$$d_r \cdot Q^2_{2k+2r} - 64\,b_{r-1} - Q^2_{2k+4r+2} + Q^2_{2k+4r+4} = d_{r+1} \cdot Q^2_{2k+2r+2} - 64\,b_r.$$

Proof. Notice that

$$b_r = \mathfrak{r}_{384}^{169;29}\,\alpha^{4r} + \mathfrak{r}_{384}^{29;169}\,\beta^{4r} - \frac{1}{32}$$

and

$$d_r = \mathfrak{r}_{12}^{5;1}\,\alpha^{4r} + \mathfrak{r}_{12}^{1;5}\,\beta^{4r},$$

for every $r \geqslant 0$. The difference of the left hand side and the right hand side is

$$\left(\mathfrak{r}_{12}^{5;1}\,\alpha^{4r}+\mathfrak{r}_{12}^{1;5}\,\beta^{4r}\right)(\mathfrak{q}_{2k+2r})^2-\mathfrak{r}_6^{169;29}\,\alpha^{4r-4}-\mathfrak{r}_6^{29;169}\,\beta^{4r-4}-(\mathfrak{q}_{2k+4r+2})^2+ \\ (\mathfrak{q}_{2k+4r+4})^2-\left(\mathfrak{r}_{12}^{5;1}\,\alpha^{4r+4}+\mathfrak{r}_{12}^{1;5}\,\beta^{4r+4}\right)\cdot(\mathfrak{q}_{2k+2r+2})^2+\mathfrak{r}_6^{169;29}\,\alpha^{4r}-\mathfrak{r}_6^{29;169}\,\beta^{4r}.$$

Let $\alpha^{2k}=A$, $\beta^{2k}=B$, $\alpha^{2r}=U$ and $\beta^{2r}=V$. The above expression reduces to $\frac{-M}{12\,\alpha^4\,\beta^4}$, where M is

$$p(\alpha,\beta)A^2U^4+q(\alpha,\beta)(AUV)^2+r(\alpha,\beta)ABU^3V-24\,s\,AB(UV)^2+ \\ p(\beta,\alpha)B^2V^4+q(\beta,\alpha)(BUV)^2+r(\beta,\alpha)ABUV^3-t(\alpha,\beta)U^2-t(\beta,\alpha)V^2,$$

with $P=(\alpha\,\beta)^4$,

$$p(\alpha,\beta)=P\,(\alpha^4-1)(5\,\alpha^5+\alpha^4\,\beta-12\,\alpha^4+5\,\alpha+\beta),$$

$$q(\alpha,\beta)=P\,(P-1)(\alpha+5\,\beta),$$

$$r(\alpha,\beta)=2\,P\,(\alpha^6\,\beta^2-1)(5\,\alpha+\beta),$$

$s=P(P-\alpha^2\,\beta^2)$ and $t(\alpha,\beta)=2\,(P-\beta^4)(169\,\alpha+29\,\beta)$. Replacing α and β with $1+\sqrt{2}$ and $1-\sqrt{2}$ we get

$$M=u_+\,A\,B\,U^3\,V+u_-\,A\,B\,U\,V^3-u_+\,U^2-u_-\,V^2,$$

where $u_\pm=16(24\pm 17\sqrt{2})$. Hence, $M=0$ so that the difference is zero because $A\,B=1$ and $U\,V=1$. □

Theorem 5. *a) For every $m \geqslant 0$ and $k \geqslant 0$ the following equality holds:*

$$\sum_{i=0}^{2\,m+1}(-1)^i\cdot Q_{2k+2i}^2=32\cdot c_m\cdot(1-a_{k+m+1}\cdot Q_{2k+2m}), \tag{5.4}$$

b) For every $m \geqslant 1$ and $k \geqslant 0$ the following equality holds:

$$\sum_{i=0}^{2\,m}(-1)^i\cdot Q_{2k+2i}^2=d_m\cdot Q_{2k+2m}^2-64\cdot b_{m-1}. \tag{5.5}$$

Proof of a). The proof is by induction on m. For $m=0$ the relation (5.4) is

$$Q_{2k}^2-Q_{2k+2}^2=32\cdot(1-a_{k+1}\,Q_{2k})$$

(i. e., the relation (5.2)) which is true by Lemma 7.

Assume that the relation (5.4) is true for $m=r$. Then

$$\sum_{i=0}^{2(r+1)+1}(-1)^i\cdot Q_{2k+2i}^2=\sum_{i=0}^{2r+1}(-1)^i\cdot Q_{2k+2i}^2+Q_{2k+4r+4}^2-Q_{2k+4r+6}^2 \\ =32\cdot c_r\cdot(1-a_{k+r+1}\cdot Q_{2k+2r})+Q_{2k+4r+4}^2-Q_{2k+4r+6}^2 \\ =32\cdot c_{r+1}\cdot(1-a_{k+(r+1)+1}\cdot Q_{2k+2(r+1)}),$$

where the last step uses Lemma 8. Hence, (5.4) is true for $m = r + 1$ and the proof is completed. □

Proof of b). The proof is by induction on m. For $m = 0$ the relation (5.5) is

$$Q_{2k}^2 = d_0\, Q_{2k}^2 - 64\, b_{-1}$$

which is true since $d_0 = 1$ and $b_{-1} = 0$.

Assume that the relation (5.5) is true for $m = r$. Then

$$\sum_{i=0}^{2(r+1)} (-1)^i \cdot Q_{2k+2i}^2 = \sum_{i=0}^{2r} (-1)^i \cdot Q_{2k+2i}^2 - Q_{2k+4r+2}^2 + Q_{2k+4r+4}^2 =$$
$$d_r \cdot Q_{2k+2r}^2 - 64\, b_{r-1} - Q_{2k+4r+2}^2 + Q_{2k+4r+6}^2 = d_{r+1} \cdot Q_{2k+2r+2}^2 - 64\, b_r,$$

where the last step uses Lemma 9. Hence, (5.5) is true for $m = r + 1$ and the proof is completed. □

6. Alternating Pell-Lucas odd squares

Lemma 10. *For every $k \geqslant 0$ we have*

$$Q_{2k+3}^2 - Q_{2k+1}^2 = 32 \cdot a_{k+1} \cdot Q_{2k+2}. \tag{6.1}$$

Proof. By the Binet formulas the difference of the left hand side and the right hand side in (6.1) is

$$Q_{2k+3}^2 - Q_{2k+1}^2 - 32\, a_{k+1}\, Q_{2k+2} = (\alpha^{2k+1} + \beta^{2k+1})^2 - (\alpha^{2k+3} + \beta^{2k+3})^2$$
$$+ 32 \left(\frac{7\alpha + \beta}{16} \alpha^{2k} + \frac{\alpha + 7\beta}{16} \beta^{2k} \right) (\alpha^{2k+2} + \beta^{2k+2}),$$

Let $\alpha^{2k} = A$ and $\beta^{2k} = B$. The above expression reduces to

$$p(\alpha) A^2 - q\, A\, B + p(\beta) B^2,$$

where $q = 2\,\alpha^3\beta^3 - 2\,\alpha\,\beta - 2\,\alpha^3 - 14\,\beta^2\alpha - 14\,\alpha^2\beta - 2\,\beta^3$ and $p(\alpha) = \alpha^2(14\alpha - \alpha^4 + 2\beta + 1)$. It is easy to check (by substitutions $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$) that $p(\alpha) = 0$, $p(\beta) = 0$ and $q = 0$. This implies that the relation (6.1) is true. □

Lemma 11. *For every $k \geqslant 0$ we have*

$$8 + Q_{2k+1}^2 = Q_{2k} \cdot Q_{2k+2}. \tag{6.2}$$

Proof. Using the Binet representation of Pell-Lucas numbers we get

$$8 + Q_{2k+1}^2 - Q_{2k} Q_{2k+2} = 8 + (\alpha^{2k+1} + \beta^{2k+1})^2 - (\alpha^{2k} + \beta^{2k})(\alpha^{2k+2} + \beta^{2k+2})$$
$$= 8 + 2\,(\alpha\,\beta)^{2k+1} - (\alpha\,\beta)^{2k}(\alpha^2 + \beta^2) = 8 - 2 - 6 = 0,$$

because $\alpha\,\beta = -1$ and $\alpha^2+\beta^2 = 6$. Hence, $8 + Q_{2k+1}^2 = Q_{2k}Q_{2k+2}$. □

Lemma 12. *For all $k \geqslant 0$ and $r \geqslant 0$ we have $L = 0$, where L is*

$$Q_{2k+4r+5}^2 - Q_{2k+4r+7}^2 - 32\,c_r \cdot a_{k+r+1} \cdot Q_{2k+2r+2} + 32\,c_{r+1} \cdot a_{k+r+2} \cdot Q_{2k+2r+4}.$$

Proof. The expression L is in fact

$$(\mathfrak{q}_{2k+4r+5})^2 - (\mathfrak{q}_{2k+4r+7})^2 - 32\,\mathfrak{p}_4^{4r+4}\left(\mathfrak{r}_{16}^{7;1}\alpha^{2k+2r} + \mathfrak{r}_{16}^{1;7}\beta^{2k+2r}\right)\mathfrak{q}_{2k+2r+2} \\ + 32\,\mathfrak{p}_4^{4r+8}\left(\mathfrak{r}_{16}^{7;1}\alpha^{2k+2r+2} + \mathfrak{r}_{16}^{1;7}\beta^{2k+2r+2}\right)\mathfrak{q}_{2k+2r+4}.$$

Let $\alpha^{2k} = A$, $\beta^{2k} = B$, $\alpha^{2r} = U$ and $\beta^{2r} = V$. The above expression reduces to $\frac{M}{\alpha^4-\beta^4}$, where M is

$$p(\alpha,\beta)A^2U^4 + 2\,q(\alpha,\beta)(AUV)^2 + 2\,r(\alpha,\beta)ABU^3V - \\ 2\,s\,AB(UV)^2 - p(\beta,\alpha)B^2V^4 + 2\,q(\beta,\alpha)(BUV)^2 - 2\,r(\beta,\alpha)ABUV^3,$$

with

$$p(\alpha,\beta) = \alpha^6\,(\alpha^4-1)(\alpha^4\,\beta^4 - \alpha^8 + 14\,\alpha^5 + 2\,\alpha^4\,\beta + 14\,\alpha + 2\,\beta),$$

$$q(\alpha,\beta) = \alpha^2\beta^4((\alpha\beta)^4 - 1)(7\alpha+\beta),$$

$$r(\alpha,\beta) = \alpha^4(\alpha+\beta)(\alpha^6\,\beta^2 - 1)(\alpha^2 + 6\alpha\beta + \beta^2)$$

and $s = \alpha^5\beta^5((\alpha\beta)^2-1)(\alpha^4-\beta^4)$. Replacing α and β with $1+\sqrt{2}$ and $1-\sqrt{2}$ we see easily that all coefficients of the above polynomial vanish so that $L = 0$. □

Lemma 13. *For every $k \geqslant 0$ and every $r \geqslant 0$ we have*

$$2(P_{2r+3}^2 - P_{2r+1}^2) + d_r \cdot Q_{2k+2r} \cdot Q_{2k+2r+2} \\ - Q_{2k+4r+3}^2 + Q_{2k+4r+5}^2 = d_{r+1} \cdot Q_{2k+2r+2} \cdot Q_{2k+2r+4}.$$

Proof. The difference of the left hand side and the right hand side is

$$8\left[\left(\mathfrak{p}_1^{2r+3}\right)^2 - \left(\mathfrak{p}_1^{2r+1}\right)^2\right] + \left(\mathfrak{r}_{12}^{5;1}\,\alpha^{4r} + \mathfrak{r}_{12}^{1;5}\,\beta^{4r}\right)\mathfrak{q}_{2k+2r}\mathfrak{q}_{2k+2r+2} - \\ (\mathfrak{q}_{2k+4r+3})^2 + (\mathfrak{q}_{2k+4r+5})^2 - \left(\mathfrak{r}_{12}^{5;1}\,\alpha^{4r+4} + \mathfrak{r}_{12}^{1;5}\,\beta^{4r+4}\right)\mathfrak{q}_{2k+2r+2}\mathfrak{q}_{2k+2r+4}.$$

Let $\alpha^{2k} = A$, $\beta^{2k} = B$, $\alpha^{2r} = U$ and $\beta^{2r} = V$. The above expression reduces to $\frac{-M}{12\,(\alpha-\beta)^2}$, where M is

$$(\alpha-\beta)^2\left[p(\alpha,\beta)A^2U^4 + q(\alpha,\beta)(AUV)^2 + r(\alpha,\beta)ABU^3V - \right. \\ \left. 24\,s\,AB(UV)^2 + p(\beta,\alpha)B^2V^4 + q(\beta,\alpha)(BUV)^2 + r(\beta,\alpha)ABUV^3\right] \\ - 96\,\left[t(\alpha)U^2 - 2wUV + t(\beta)V^2\right],$$

with $t(\alpha) = \alpha^2(\alpha^4-1)$,

$$p(\alpha,\beta) = t(\alpha)(5\,\alpha^5 + \alpha^4\,\beta - 12\,\alpha^4 + 5\,\alpha + \beta),$$

$$q(\alpha,\beta) = t(\alpha\beta)(\alpha+5\,\beta)/\beta^2,$$

$$r(\alpha,\beta) = (\alpha^2+\beta^2)(\alpha^6\,\beta^2-1)(5\,\alpha+\beta),$$

$s = \alpha^3\,\beta^3((\alpha\beta)^2-1)$ and $w = \alpha\beta((\alpha\beta)^2-1)$. Replacing α and β with $1+\sqrt{2}$ and $1-\sqrt{2}$ we get

$$M = u_+\,A\,B\,U^3\,V + u_-\,A\,B\,U\,V^3 - u_+\,U^2 - u_-\,V^2,$$

where $u_\pm = 384(24 \pm 17\,\sqrt{2})$. Hence, $M = 0$ so that the difference is zero since $A\,B = 1$ and $U\,V = 1$. □

Theorem 6. *a) For every $j \geqslant 0$ and $k \geqslant 0$ the following equality holds:*

$$\sum_{i=0}^{2\,j+1} (-1)^i \cdot Q^2_{2k+2i+1} = -32 \cdot c_j \cdot a_{k+j+1} \cdot Q_{2k+2j+2}, \tag{6.3}$$

b) For every $j \geqslant 1$ and $k \geqslant 1$ the following equality holds:

$$2 \cdot P^2_{2j+1} + \sum_{i=0}^{2\,j} (-1)^i \cdot Q^2_{2k+2i+1} = d_j \cdot Q_{2k+2j} \cdot Q_{2k+2j+2}. \tag{6.4}$$

Proof of a). The proof is by induction on j. For $j = 0$ the relation (6.3) is

$$Q^2_{2k+3} - Q^2_{2k+1} = 32a_{k+1}Q_{2k+2}$$

(i. e., the relation (6.1)) which is true by Lemma 10.

Assume that the relation (6.3) is true for $j = r$. Then

$$\begin{aligned}\sum_{i=0}^{2(r+1)+1} (-1)^i\,Q^2_{2k+2i+1} &= \sum_{i=0}^{2r+1} (-1)^i\,Q^2_{2k+2i+1} + Q^2_{2k+4r+5} - Q^2_{2k+4r+7}\\ &= -32\,c_r\,a_{k+r+1}\,Q_{2k+2r+2} + Q^2_{2k+4r+5} - Q^2_{2k+4r+7}\\ &= -32\,c_{r+1}\,a_{k+r+2}\,Q_{2k+2r+4},\end{aligned}$$

where the last step uses Lemma 12. Hence, (6.3) is true for $j = r+1$ and the proof is completed. □

Proof of b). The proof is again by induction on j. For $j = 0$ the relation (6.4) is

$$2\,P_1^2 + Q^2_{2k+1} = d_0\,Q_{2k}\,Q_{2k+2}$$

which is true by Lemma 11 since $d_0 = 1$ and $P_1 = 2$.

Assume that the relation (6.4) is true for $j = r$. Then

$$2\,P_{2r+3}^2 + \sum_{i=0}^{2(r+1)} (-1)^i\,Q_{2k+2i+1}^2 = 2(P_{2r+3}^2 - P_{2r+1}^2)$$
$$+ \left(2\,P_{2r+1}^2 + \sum_{i=0}^{2r} (-1)^i\,Q_{2k+2i+1}^2\right) - Q_{2k+4r+3}^2 + Q_{2k+4r+5}^2 =$$
$$2(P_{2r+3}^2 - P_{2r+1}^2) + d_r\,Q_{2k+2r}\,Q_{2k+2r+2} - Q_{2k+4r+3}^2 + Q_{2k+4r+5}^2$$
$$= d_{r+1}\,Q_{2k+2r+2}\,Q_{2k+2r+4},$$

where the last step uses Lemma 13. Hence, (6.4) is true for $m = r+1$ and the proof is completed. □

7. Alternating Pell-Lucas products

Lemma 14. *For every $k \geqslant 0$ we have*

$$Q_{2k}\,Q_{2k+1} - Q_{2k+2}\,Q_{2k+3} + 8\,P_{2k+3}\,Q_{2k} = 80. \tag{7.1}$$

Proof. The difference of the left hand side and the right hand side in (7.1) is

$$\mathfrak{q}_{2k}\,\mathfrak{q}_{2k+1} - \mathfrak{q}_{2k+2}\,\mathfrak{q}_{2k+3} + 16\,\mathfrak{p}_1^{2k+3}\,\mathfrak{q}_{2k} - 80.$$

Let $\alpha^{2k} = A$ and $\beta^{2k} = B$. The above expression reduces to $\frac{M}{\alpha-\beta}$ where M is

$$p(\alpha)\,A^2 - q\,A\,B - p(\beta)\,B^2 - 80(\alpha - \beta),$$

with $p(\alpha) = \alpha(\beta\,\alpha^4 - \alpha^5 + 16\,\alpha^2 + \alpha - \beta)$ and

$$q = (\alpha - \beta)(\alpha^3\,\beta^2 + \alpha^2\,\beta^3 - 16\,\alpha^2 - 16\,\alpha\,\beta - 16\,\beta^2 - \alpha - \beta).$$

If we replace α and β with $1+\sqrt{2}$ and $1-\sqrt{2}$ we get $p(\alpha) = p(\beta) = 0$ and $q = -160\sqrt{2}$ and $80(\alpha - \beta) = 160\sqrt{2}$. Hence, $M = 0$ because $A\,B = 1$. □

Lemma 15. *For all $k \geqslant 0$ and $r \geqslant 0$ we have*

$$8\,c_{r+1} \cdot (P_{2k+2r+5} \cdot Q_{2k+2r+2} - 10) + Q_{2k+4r+4} \cdot Q_{2k+4r+5} =$$
$$8\,c_r \cdot (P_{2k+2r+3} \cdot Q_{2k+2r} - 10) + Q_{2k+4r+6} \cdot Q_{2k+4r+7}.$$

Proof. The difference L of the left hand side and the right hand side is

$$8\,\mathfrak{p}_4^{4r+8}\left[2\,\mathfrak{p}_1^{2k+2r+5}\,\mathfrak{q}_{2k+2r+2} - 10\right] + \mathfrak{q}_{2k+4r+4}\,\mathfrak{q}_{2k+4r+5}$$
$$- \mathfrak{q}_{2k+4r+6}\,\mathfrak{q}_{2k+4r+7} - 8\,\mathfrak{p}_4^{4r+4}\left[2\,\mathfrak{p}_1^{2k+2r+3}\,\mathfrak{q}_{2k+2r} - 10\right]$$

Let $\alpha^{2k} = A$, $\beta^{2k} = B$, $\alpha^{2r} = U$ and $\beta^{2r} = V$. The above expression reduces to

$$\frac{M}{(\alpha+\beta)(\alpha-\beta)^2(\alpha^2+\beta^2)},$$

where M is

$$p(\alpha,\beta)A^2U^4 - 16\,q(\alpha,\beta)(AUV)^2 + 16\,r(\alpha,\beta)ABU^3V - s\,AB(UV)^2 + \\ p(\beta,\alpha)B^2V^4 - 16\,q(\beta,\alpha)(BUV)^2 - 16\,r(\beta,\alpha)ABUV^3 - t(\alpha)U^2 + t(\beta)V^2,$$

with

$$p(\alpha,\beta) = \alpha^6\,(\alpha^4-1)(\alpha^6\beta - \alpha^7 + \alpha^3\beta^4 - \alpha^2\beta^5 + 16\,\alpha^4 + 16),$$

$$q(\alpha,\beta) = \alpha^3\beta^4((\alpha\beta)^4 - 1),$$

$$s = \alpha^4\beta^4((\alpha\beta)^2 - 1)(\alpha^2+\beta^2)(\alpha^2-\beta^2)^2,$$

$$r(\alpha,\beta) = \alpha^4(\alpha-\beta)(\alpha^6\,\beta^2 - 1)(\alpha^2 + \alpha\beta + \beta^2)$$

and $t(\alpha) = 80\,\alpha^4(\alpha-\beta)\ (\alpha^4-1)$. Replacing α and β with $1+\sqrt{2}$ and $1-\sqrt{2}$ we get

$$M = u_+\,A\,B\,U^3\,V + u_-\,A\,B\,U\,V^3 - u_+\,U^2 - u_-\,V^2,$$

where $u_\pm = 1280(99 \pm 70\sqrt{2})$. Hence, $M = 0$ so that the difference is zero since $A\,B = 1$ and $U\,V = 1$. □

Lemma 16. *For all $k \geqslant 0$ and $r \geqslant 0$ we have*

$$64\,c_r + d_r \cdot Q_{2k+2r} \cdot Q_{2k+2r+1} + Q_{2k+4r+4} \cdot Q_{2k+4r+5} = \\ d_{r+1} \cdot Q_{2k+2r+2} \cdot Q_{2k+2r+3} + Q_{2k+4r+2} \cdot Q_{2k+4r+3}.$$

Proof. The difference L of the left hand side and the right hand side is

$$64\,\mathfrak{p}_4^{4r+4} + \left(\mathfrak{r}_{12}^{5;1}\,\alpha^{4r} + \mathfrak{r}_{12}^{1;5}\,\beta^{4r}\right)\,\mathfrak{q}_{2k+2r}\,\mathfrak{q}_{2k+2r+1} - \mathfrak{q}_{2k+4r+2}\,\mathfrak{q}_{2k+4r+3} \\ + \mathfrak{q}_{2k+4r+4}\,\mathfrak{q}_{2k+4r+5} - \left(\mathfrak{r}_{12}^{5;1}\,\alpha^{4r+4} + \mathfrak{r}_{12}^{1;5}\,\beta^{4r+4}\right)\,\mathfrak{q}_{2k+2r+2}\,\mathfrak{q}_{2k+2r+3}.$$

Let $\alpha^{2k} = A$, $\beta^{2k} = B$, $\alpha^{2r} = U$ and $\beta^{2r} = V$. The above expression reduces to $\frac{M}{12(\alpha^4-\beta^4)}$, where M is

$$p(\alpha,\beta)A^2U^4 - q(\alpha,\beta)(AUV)^2 + r(\alpha,\beta)ABU^3V + 12\,s\,AB(UV)^2 - \\ p(\beta,\alpha)B^2V^4 - q(\beta,\alpha)(BUV)^2 - r(\beta,\alpha)ABUV^3 + t(\alpha)U^2 - t(\beta)V^2,$$

with

$$p(\alpha,\beta) = \alpha\,(\alpha^4-1)(\beta^4-\alpha^4)(5\alpha^5 + \alpha^4\beta - 12\alpha^4 + 5\,\alpha + \beta),$$

$$s = \alpha^2\beta^2((\alpha\beta)^2-1)(\beta^4-\alpha^4)(\alpha+\beta),$$

$$r(\alpha,\beta) = (\beta^4-\alpha^4)(\alpha+\beta)(5\alpha+\beta)(\alpha^6\,\beta^2-1),$$

$$q(\alpha,\beta) = \alpha((\alpha\beta)^4-1)(\beta^4-\alpha^4)(\alpha+5\beta)$$

and $t(\alpha) = 768\,\alpha^4$. Replacing α and β with $1+\sqrt{2}$ and $1-\sqrt{2}$ we get

$$M = -u_+\,A\,B\,U^3\,V + u_-\,A\,B\,U\,V^3 + u_+\,U^2 - u_-\,V^2,$$

where $u_{\pm} = 768(17 \pm 12\sqrt{2})$. Hence, $M = 0$ so that the difference is zero since $A\,B = 1$ and $U\,V = 1$. □

Theorem 7. *a) For every $j \geqslant 0$ and $k \geqslant 0$ the following equality holds:*

$$\sum_{i=0}^{2j+1} (-1)^i \cdot Q_{2k+2i} \cdot Q_{2k+2i+1} = -8 \cdot c_j \cdot (P_{2k+2j+3} \cdot Q_{2k+2j} - 10), \tag{7.2}$$

b) For every $j \geqslant 0$ and $k \geqslant 0$ the following equality holds:

$$\beta_j + \sum_{i=0}^{2j} (-1)^i \cdot Q_{2k+2i} \cdot Q_{2k+2i+1} = d_j \cdot Q_{2k+2j} \cdot Q_{2k+2j+1}, \tag{7.3}$$

where the sequence β_j is determined by the conditions: $\beta_0 = 0$ and $\beta_{j+1} = \beta_j + 64 \cdot c_j$ for every $j \geqslant 0$.

Proof of a). The proof is by induction on j. For $j = 0$ the relation (7.2) is

$$Q_{2k}\,Q_{2k+1} - Q_{2k+2}\,Q_{2k+3} = -8\,P_{2k+3}\,Q_{2k} + 80$$

(i. e., the relation (7.1)) which is true by Lemma 14.

Assume that the relation (7.2) is true for $j = r$. Then

$$\begin{aligned}\sum_{i=0}^{2(r+1)+1} (-1)^i\,Q_{2k+2i}\,Q_{2k+2i+1} &= \sum_{i=0}^{2r+1} (-1)^i\,Q_{2k+2i}\,Q_{2k+2i+1} + \\ &\quad Q_{2k+4r+4}\,Q_{2k+4r+5} - Q_{2k+4r+6}\,Q_{2k+4r+7} = -8c_r(P_{2k+2r+3}\,Q_{2k+2r} - 10) + \\ &\qquad Q_{2k+4r+4}\,Q_{2k+4r+5} - Q_{2k+4r+6}\,Q_{2k+4r+7} = -8c_{r+1}(P_{2k+2r+5}\,Q_{2k+2r+2} - 10),\end{aligned}$$

where the last step uses Lemma 15. Hence, (7.2) is true for $j = r + 1$ and the proof is completed. □

Proof of b). The proof is once again by induction on j. For $j = 0$ the relation (7.3) is $\beta_0 + Q_{2k}\,Q_{2k+1} = d_0\,Q_{2k}\,Q_{2k+1}$ which is true since $d_0 = 1$ and $\beta_0 = 0$.

Assume that the relation (7.3) is true for $j = r$. Then

$$\begin{aligned}\beta_{r+1} + \sum_{i=0}^{2(r+1)} (-1)^i\,Q_{2k+2i}\,Q_{2k+2i+1} &= \beta_{r+1} - \beta_r + \beta_r + \\ &\sum_{i=0}^{2r} (-1)^i\,Q_{2k+2i}\,Q_{2k+2i+1} - Q_{2k+4r+2}\,Q_{2k+4r+3} + Q_{2k+4r+4}\,Q_{2k+4r+5} = \\ &64\,c_r + d_r\,Q_{2k+2r}\,Q_{2k+2r+1} - Q_{2k+4r+2}\,Q_{2k+4r+3} + Q_{2k+4r+4}\,Q_{2k+4r+5} \\ &= d_{r+1}\,Q_{2k+2r+2}\,Q_{2k+2r+3},\end{aligned}$$

where the last step uses Lemma 16. Hence, (7.3) is true for $j = r + 1$ and the proof is completed. □

We mention also the following equalities that have been discovered while attempting to prove Theorem 7.

Theorem 8. *For every $k \geqslant 0$ we have*

$$Q_{2k} \cdot Q_{2k+1} = Q_{2k+3} \cdot Q_{2k-2} - 80 = 2 \cdot (P_{2k+2} \cdot P_{2k-1} - 6).$$

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