COMBINATORIAL PROOFS OF SOME SIMONS-TYPE BINOMIAL COEFFICIENT IDENTITIES

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Abstract

We provide elementary bijective proofs of some curious binomial coefficient identities which were obtained using Cauchy's integral formula.

1. Introduction

Simons [5] proved a binomial coefficient identity using repeated differentiation which can be equivalently written as

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}(1+x)^k=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k. \tag{1}$$

Alternate proofs of this identity have been given by Chapman [1] using generating functions and by Prodinger [4] using Cauchy's integral formula. See also the related paper of Hirschhorn [2]. Munarini [3] generalizes the approach of Prodinger to obtain the curious identities

$$\sum_{k=0}^{n}\binom{\beta-\alpha+n}{n-k}\binom{\beta+k}{k}(-1)^{n-k}(x+y)^k y^{n-k}=\sum_{k=0}^{n}\binom{\alpha}{n-k}\binom{\beta+k}{k}x^k y^{n-k}, \tag{2}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{\alpha}{k}\binom{2\alpha-2k}{n-2k}(-1)^k s^{2k}(x+s)^{n-2k}=\sum_{k=0}^{n}\binom{\alpha}{k}\binom{2\alpha-k}{n-k}(2s)^k x^{n-k}, \tag{3}$$

and

$$\sum_{k=0}^{n}\binom{2\beta+n+1}{n-k}\binom{\beta+k}{k}(-2s)^k y^{n-k}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{2\beta+n+1}{n-2k}\binom{\beta+k}{k}s^{2k}(y-s)^{n-2k}, \tag{4}$$

where α, β, x, y, and s are indeterminates. Note that (2) reduces to (1) when $\alpha = \beta = n$ and $y = 1$.

In this note, we provide elementary bijective proofs of identities (2)–(4). Our strategy will be to identify and define sign-changing involutions on sets of configurations which have net weight given by the alternating sums on the left, the survivors of which will have cardinality given by the positive sums on the right.

2. The First Two Identities

We first provide a combinatorial interpretation for (2). Since both sides are polynomials in the indeterminates α, β, x and y, one can take them to be positive integers with $\beta \geqslant \alpha$. We begin by understanding the unsigned quantity, $\binom{\beta-\alpha+n}{n-k}\binom{\beta+k}{k}(x+y)^k y^{n-k}$, occurring in the sum on the left-hand side of (2). Choose and color $n-k$ members of $[\beta-\alpha+n] \subseteq [\beta+n]$ using y possible shades of black in $\binom{\beta-\alpha+n}{n-k}y^{n-k}$ ways, where $0 \leqslant k \leqslant n$. (Recall that $[m] := \{1, 2, ..., m\}$ for positive integers m with $[0] := \emptyset$.) We also circle these $n-k$ elements to distinguish them further. Then choose and color k of the remaining $\beta + k$ members of $[\beta+n]$ using x shades of red and y shades of black in $\binom{\beta+k}{k}(x+y)^k$ ways. We call such a coloring of $[\beta+n]$ a configuration.

Let E and O denote those configurations with an even or odd number of circled elements, respectively. We pair members of E with members of O by identifying the smallest element in $[\beta-\alpha+n]$ painted a shade of black and either circling it or uncircling it (without changing the shade of black). Below we illustrate such a pairing when $\beta = n = 4$ and $\alpha = 2$:

$$1 \ \underline{2} \ \overline{3} \ 4 \ \underline{5} \ 6 \ 7 \ \underline{8} \quad \longleftrightarrow \quad 1 \ \underline{2} \ \overline{3} \ 4 \ \underline{5} \ 6 \ 7 \ \underline{8} \ ,$$

wherein black and red numbers are indicated by lines below and above, respectively (which we'll use throughout).

This pairs all configurations except those in which every element painted black is in $[\beta+n]-[\beta-\alpha+n]$. These configurations necessarily belong to E and have cardinality given by the right-hand side of (2). For if $n-k$ denotes the number of elements painted a shade of black, then the number of ways to paint k of the remaining $\beta+k$ elements of $[\beta+n]$ red is $\binom{\beta+k}{k}x^k$, which proves (2). Note how the above argument specializes to (1).

We now interpret (3) using colored configurations. Let α, s and x be positive integers with $\alpha \geqslant n$. Again, we start with the left-hand side. First consider the set of α consecutive pairs, $\{\{1,2\},\{3,4\},...,\{2\alpha-1,2\alpha\}\}$. Choose k of these α pairs and color the $2k$ individual members with s shades of black in $\binom{\alpha}{k}s^{2k}$ ways, where $0 \leqslant k \leqslant \lfloor n/2 \rfloor$. We also circle these k

pairs to distinguish them. Then choose and color $n-2k$ of the remaining $2\alpha-2k$ members of $[2\alpha]$ using x shades of red and s shades of black in $\binom{2\alpha-2k}{n-2k}(x+s)^{n-2k}$ ways.

Let E and O denote those configurations described above with an even or odd number of circled pairs, respectively. Identify the smallest pair $\{2i-1,2i\}$ in which both members are shades of black and either circle it or uncircle it. For example, the following is a pairing when $\alpha=n=6$:

$$\underline{1}\ \ 2\ \ \underline{3}\ \ \underline{4}\ \ 5\ \ 6\ \ \underline{7}\ \ \underline{8}\ \ \overline{9}\ \ 10\ \ 11\ \ 12 \longleftrightarrow \underline{1}\ \ 2\ \ \underline{3}\ \ \underline{4}\ \ 5\ \ 6\ \ \underline{7}\ \ \underline{8}\ \ \overline{9}\ \ 10\ \ 11\ \ 12.$$

Every configuration is matched with another of opposite sign except those which do not contain a black pair $\{2i-1,2i\}$. These configurations belong to E and the right-hand side of (3) gives their cardinality according to the number of pairs $\{2i-1,2i\}$ containing a single black member. Note that there are $\binom{2\alpha-k}{n-k}x^{n-k}$ ways to paint $n-k$ of the remaining $2\alpha-k$ members of $[2\alpha]$ red once one has painted k members black in one of $\binom{\alpha}{k}(2s)^k$ ways, where $0\leqslant k\leqslant n$.

3. The Third Identity

We provide an interpretation for (4), assuming β, s and y to be positive integers with $y\geqslant s$. We first prove the case $y=s$:

$$\sum_{k=0}^{n}\binom{2\beta+n+1}{n-k}\binom{\beta+k}{k}(-2)^k=\begin{cases}0, & \text{if } n \text{ is odd;}\\ \binom{\beta+\frac{n}{2}}{\frac{n}{2}}, & \text{if } n \text{ is even.}\end{cases} \tag{5}$$

Consider colorings of $[2\beta+n+1]$ obtained by the following four steps:

(i) Paint $n-k$ members of $[2\beta+n+1]$ red, where $0\leqslant k\leqslant n$;

(ii) Of the remaining $2\beta+k+1$ members of $[2\beta+n+1]$, paint the final $\beta+1$ white;

(iii) Paint k of the remaining $\beta+k$ members of $[2\beta+n+1]$ black, circling some subset of them;

(iv) Paint the remaining β numbers white.

If E and O denote the configurations described above with an even or odd number of black elements, respectively, then the left-hand side of (5) gives $|E|-|O|$.

Given a configuration, let m denote the median white number (i.e., the $(\beta+1)^{\underline{st}}$ number painted white). Pair configurations in E with those in O by identifying the smallest number

j satisfying (i) j is black and circled, or (ii) $j < m$ is red, and switching to the other option. For example when $\beta = 2$ and $n = 5$, we have

$$1 \;\; \underline{2} \;\; \underline{\textcircled{3}} \;\; 4 \;\; \underline{\textcircled{5}} \;\; 6 \;\; \overline{7} \;\; 8 \;\; 9 \;\; \overline{10} \quad \longleftrightarrow \quad 1 \;\; \underline{2} \;\; \overline{3} \;\; 4 \;\; \underline{\textcircled{5}} \;\; 6 \;\; \overline{7} \;\; 8 \;\; 9 \;\; \overline{10},$$

where the white numbers are unmarked. The set S of survivors of this pairing are those configurations in which all red numbers are greater than m and in which all black numbers (which are necessarily less than m) are uncircled.

Given $\lambda \in S$, let a_i denote the number of black elements between the $(i-1)^{\underline{st}}$ and $i^{\underline{th}}$ white numbers if $2 \leqslant i \leqslant \beta+1$, with a_1 being the number of black elements before the first white number. Similarly, let b_i, $1 \leqslant i \leqslant \beta$, denote the number of red elements between the $(\beta+i)^{\underline{th}}$ and the $(\beta+i+1)^{\underline{st}}$ white number, with $b_{\beta+1}$ being the number of red elements after the last white number. Note that a member of S is uniquely determined by its vectors $(a_1, ..., a_{\beta+1})$, $(b_1, ..., b_{\beta+1})$.

Let $S' \subseteq S$ consist of those configurations in which $a_i = 0$ and b_i is even for all i, $1 \leqslant i \leqslant \beta+1$. Clearly, S' is empty if n is odd, since the sum of all the entries in the two vectors must be n. If n is even, then $S' \subseteq E$ with $|S'| = \binom{\beta+\frac{n}{2}}{\frac{n}{2}}$, since there are n red numbers to be distributed in runs of even length amongst the final β white numbers.

We now use the vectors described above to define a sign-changing involution of $S - S'$ which will prove (5). Given $\lambda \in S - S'$, let i_0 be the smallest index i such that either

(I) $a_i + b_i$ is odd, or

(II) $a_i + b_i$ is even with $a_i \geqslant 1$.

If (I) occurs and a_{i_0} is odd (resp., even), let λ' be the configuration obtained from λ by replacing a_{i_0} with $a_{i_0} - 1$ and b_{i_0} with $b_{i_0} + 1$ (resp., a_{i_0} with $a_{i_0} + 1$ and b_{i_0} with $b_{i_0} - 1$), leaving the rest of the configuration undisturbed. For (II), proceed exactly as in (I) with the even and odd subcases reversed. For example when $\beta = 3$ and $n = 8$, there is the pairing

$$\begin{aligned} &1 \;\; 2 \;\; \underline{3} \;\; \underline{4} \;\; 5 \;\; \underline{6} \;\; \underline{7} \;\; 8 \;\; 9 \;\; \overline{10} \;\; \overline{11} \;\; 12 \;\; \overline{13} \;\; \overline{14} \;\; 15 \longleftrightarrow \\ &\qquad\qquad 1 \;\; 2 \;\; \underline{3} \;\; 4 \;\; \underline{5} \;\; \underline{6} \;\; 7 \;\; 8 \;\; \overline{9} \;\; \overline{10} \;\; 11 \;\; \overline{12} \;\; \overline{13} \;\; \overline{14} \;\; 15, \end{aligned}$$

since the first configuration has vectors $(a_1, a_2, a_3, a_4) = (0, 0, 2, 2)$ and $(b_1, b_2, b_3, b_4) = (0, 2, 2, 0)$, while the second has vectors $(0, 0, 1, 2)$ and $(0, 2, 3, 0)$; note that $i_0 = 3$ in this case.

If $y > s$, then we use $y - s$ shades of green, s shades of red, and s shades of black to color $[2\beta + n + 1]$:

(i) Paint $n - k$ members of $[2\beta + n + 1]$ a shade of green or red, where $0 \leqslant k \leqslant n$;

(ii) Of the remaining $2\beta + k + 1$ members of $[2\beta + n + 1]$, paint the final $\beta + 1$ white;

(iii) Paint k of the remaining $\beta + k$ members of $[2\beta + n + 1]$ a shade of black, circling some subset of them;

(iv) Paint the remaining β numbers white.

The left-hand side of (4) then gives $|E| - |O|$, where E and O are now determined by the number of black-shaded elements.

Apply the argument of the case $y = s$ only to the numbers painted black, white, or red, noting that there are now s shades of red or black. Leave all green-shaded numbers undisturbed. Match shades of red with shades of black in two places when defining the involution. The right-hand side of (4) gives the cardinality of the survivors according to the number, $n - 2k$, of green-shaded elements. For there are then $2k$ red-shaded numbers amongst the final β white numbers coming in runs of even length, which completes the proof.

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