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DISJUNCTIVE RADO NUMBERS FOR $x_1 + x_2 + c = x_3$

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Abstract

Given two equations E_1 and E_2, the disjunctive Rado number for E_1 and E_2 is the least integer n, provided that it exists, such that for every coloring of the set $\{1, 2, \ldots, n\}$ with two colors there exists a monochromatic solution to either E_1 or E_2. If no such integer n exists, then the disjunctive Rado number for E_1 and E_2 is infinite. Let $R(c, k)$ represent the disjunctive Rado number for the equations $x_1 + x_2 + c = x_3$ and $x_1 + x_2 + k = x_3$. In this paper the values of $R(c, k)$ are found for all natural numbers c and k where $c \leq k$. It is shown that

$$R(c,k) = \begin{cases} 4c+5 & \text{if} \quad c \leq k \leq c+1 \\ 3c+4 & \text{if} \quad c+2 \leq k \leq 3c+2 \\ k+2 & \text{if} \quad 3c+3 \leq k \leq 4c+2 \\ 4c+5 & \text{if} \quad 4c+3 \leq k. \end{cases}$$

1. Introduction

Let $\mathbb{N}$ represent the set of natural numbers and let $[a, b]$ denote the set $\{n \in \mathbb{N}, a \leq n \leq b\}$. A function $\Delta : [1, n] \to [0, t-1]$ is referred to as a *t-coloring* of the set $[1, n]$. Given a t-coloring Δ and a system L of linear equations or inequalities in m variables, a solution $(x_1, x_2, \ldots, x_m)$ to the system L is monochromatic if and only if

$$\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m).$$

In 1916, I. Schur [24] proved that for every $t \geq 2$, there exists a least integer $n = S(t)$ such that for every t-coloring of the set $[1, n]$, there exists a monochromatic solution to $x_1 + x_2 = x_3$. The integers $S(t)$ are called Schur numbers. It is known that $S(2) = 5$, $S(3) = 14$ and $S(4) = 45$, but no other Schur numbers are known [25]. In 1933, R. Rado generalized the concept of Schur numbers to arbitrary systems of linear equations. Rado found necessary and sufficient conditions to determine if an arbitrary system of linear equations admits a monochromatic solution under every t-coloring of the natural numbers [6, 17, 18, 19]. For a given system L of linear equations, the least integer n, provided that it exists, such that for every t-coloring of the set $[1, n]$ there exists a monochromatic solution to L is called the t-color Rado number (or t-color generalized Schur number) for the system L. If such an integer n does not exist, then the t-color Rado number for the system L is infinite. In recent years the exact Rado numbers for several families of equations and inequalities have been found [4, 9, 10, 12, 13, 14, 23]. In [5] it was determined that the 2-color Rado number for the equation $E(c) : x_1 + x_2 + c = x_3$ is $4c + 5$ for every integer $c \geq 0$.

Recently several other problems related to Schur numbers and Rado numbers have been considered [1, 2, 3, 7, 8, 16, 20, 21, 22]. Specifically, the concept of disjunctive Rado numbers (or disjunctive generalized Schur numbers) has recently been introduced [11, 15]. Given a set L of linear equations, the least integer n, provided that it exists, such that for every 2-coloring of the set $[1, n]$ there exists a monochromatic solution to at least one equation in L is called the disjunctive Rado number for the set L. If such an integer n does not exist, then the disjunctive Rado number for the set L is infinite. Given a set of linear equations, it is clear that the disjunctive Rado number for this set is less than or equal to the 2-color Rado number for each equation in the set.

In this paper, the disjunctive Rado numbers are determined for the set consisting of the two equations

$$E(c) : x_1 + x_2 + c = x_3 \text{ and } E(k) : x_1 + x_2 + k = x_3$$

for all natural numbers c and k where $c \leq k$. This disjunctive Rado number will be denoted by $R(c, k)$.

2. Main Result

Theorem For all natural numbers c and k where $c \leq k$,

$$R(c,k) = \begin{cases} 4c+5 & \text{if} \quad c \leq k \leq c+1 \\ 3c+4 & \text{if} \quad c+2 \leq k \leq 3c+2 \\ k+2 & \text{if} \quad 3c+3 \leq k \leq 4c+2 \\ 4c+5 & \text{if} \quad 4c+3 \leq k. \end{cases}$$

Proof. It should be noted that the third interval in the expression of $R(c,k)$ could be expanded to include the values of $k = 3c+2$ and $k = 4c+3$ without changing the expression.

The lower bounds can be established by exhibiting a coloring that avoids a monochromatic solution to both $E(c)$ and $E(k)$ for each of the intervals in the expression of $R(c,k)$. Consider the coloring $\Delta' : [1, 4c+4] \to [0,1]$ defined by

$$\Delta'(x) = \begin{cases} 0 & 1 \le x \le c+1 \\ 1 & c+2 \le x \le 3c+3 \\ 0 & 3c+4 \le x \le 4c+4. \end{cases}$$

It is easy to check that the coloring Δ' avoids a monochromatic solution to $E(c)$, so every restriction of Δ' to a smaller domain does as well. We leave it to the reader to show that Δ' also avoids a monochromatic solution to $E(k)$ when $c \le k \le c+1$ or $4c+3 \le k$, that $\Delta' |_{[1,3c+3]}$ avoids a monochromatic solution to $E(k)$ when $c+2 \le k \le 3c+2$ and that $\Delta' |_{[1,k+1]}$ avoids a monochromatic solution to $E(k)$ when $3c+3 \le k \le 4c+2$.

We shall now establish upper bounds for $R(c,k)$. As was mentioned in the introduction, every 2-coloring of the set $[1, 4c+5]$ contains a monochromatic solution to $E(c)$, so for the cases $k \in [c, c+1]$ and $k \ge 4c+3$, the upper bound of $4c+5$ is already established. Hence we must consider only two cases.

Case 1: Assume that $k \in [c+2, 3c+2]$. We will establish that

$$R(c,k) \le 3c+4.$$

Assume by way of a contradiction that there exists a coloring $\Delta : [1, 3c+4] \to [0,1]$ that does not admit a monochromatic solution to either $E(c)$ or $E(k)$. Without loss of generality we may assume that $\Delta(1) = 0$, and so $\Delta(c+2) = 1$ to avoid a monochromatic solution to $E(c)$. Let $s \le c+2$ be the least integer such that $\Delta(s) = 1$. Thus it must be the case that $\Delta(2s+c) = 0$. We now establish that for every $n \in [0, 2c+4-2s]$ we have $\Delta(s+n) = 1$ and $\Delta(2s+c+n) = 0$. To prove this we will use induction on n, with the case $n = 0$ already established. We assume $\Delta(s+n_0) = 1$ and $\Delta(2s+c+n_0) = 0$ for some $n_0 \in [0, 2c+3-2s]$. Now, $\Delta(s-1) = 0$ and $\Delta(2s+c+n_0) = 0$, so $\Delta(s+n_0+1) = 1$ or else $(s-1, s+n_0+1, 2s+c+n_0)$ would be a monchromatic solution to $E(c)$. Also, since $\Delta(s) = 1$, we must have $\Delta(2s+c+n_0+1) = 0$ or else $(s, s+n_0+1, 2s+c+n_0+1)$ would be a monchromatic solution to $E(c)$.

Now, by the inductive result we have that $[1, s-1] \cup [2s+c, 3c+4]$ contains only elements of color 0. For any $k \in [c+2, 3c+2]$ there exist integers x_1 and $x_2 \in [1, s-1]$ and $x_3 \in [2s+c, 3c+4]$ such that $x_1 + x_2 + k = x_3$. This is a contradiction.

Case 2: Assume that $k \in [3c+3, 4c+2]$. We will show that

$$R(c,k) \le k+2$$

by showing that every coloring $\Delta : [1, k+2] \to [0, 1]$ contains a monochromatic solution to either $E(c)$ or $E(k)$.

Let a coloring $\Delta : [1, k+2] \to [0, 1]$ be given. Without loss of generality we may assume that $\Delta(1) = 0$. Then we may assume that $\Delta(c+2) = 1$ and $\Delta(k+2) = 1$ in order to avoid monochromatic solution to $E(c)$ and $E(k)$ respectively. Now, if $\Delta(3c+4) = 1$, then $(c+2, c+2, 3c+4)$ is a monochromatic solution to $E(c)$, so we may assume that $\Delta(3c+4) = 0$. If $\Delta(2c+3) = 0$, then $(1, 2c+3, 3c+4)$ is a monochromatic solution to $E(c)$, so we may assume that $\Delta(2c+3) = 1$. If $\Delta(k-3c-1) = 1$, then $(k-3c-1, 2c+3, k+2)$ is a monochromatic solution to $E(c)$, so we may assume that $\Delta(k-3c-1) = 0$. Finally, if $\Delta(k-2c) = 0$, then $(1, k-3c-1, k-2c)$ is a monochromatic solution to $E(c)$, and if $\Delta(k-2c) = 1$, then $(c+2, k-2c, k+2)$ is a monochromatic solution to $E(c)$. Therefore, every coloring $\Delta : [1, k+2] \to [0, 1]$ contains a monochromatic solution to either $E(c)$ or $E(k)$. Hence,

$$R(c, k) \leq k + 2$$

when $k \in [3c+3, 4c+2]$ and the proof of the Theorem is complete.

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