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PERFECT DELAUNAY POLYTOPES IN LOW DIMENSIONS

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Abstract

A lattice Delaunay polytope is *perfect* if its Delaunay sphere is its only circumscribed ellipsoid. A perfect Delaunay polytope naturally corresponds to a positive quadratic function on $\mathbb{Z}^n$ that can be recovered uniquely from the data consisting of its minimum and all points of $\mathbb{Z}^n$ where this minimum is achieved – a quadratic function with this uniqueness property is also called perfect. We develop a structural theory of perfect Delaunay polytopes and quadratic functions. We also describe all known perfect Delaunay polytopes in dimensions one through eight: our conjecture is that this list is complete.

1. Introduction

A point lattice is a discrete set of points in $\mathbb{R}^n$ such that the difference vectors form a subgroup of $\mathbb{R}^n$. If Λ is a point lattice in $\mathbb{R}^n$ ($n \geq 0$), then a convex polytope $P \subset \mathbb{R}^n$ is called a *lattice polytope* (or Λ-*polytope*) if all of its vertices are lattice points. Consider the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$, and a convex $\mathbb{Z}^n$-polytope P. If P can be circumscribed by an ellipsoid

$$\mathcal{E} = \{\mathbf{x} \in \mathbb{R}^n \,|\, \mathcal{Q}_{\mathcal{E}}(\mathbf{x} - \mathbf{c}_{\mathcal{E}}) \leq \rho_{\mathcal{E}}^2\}$$

with no interior $\mathbb{Z}^n$-elements so that the boundary $\mathbb{Z}^n$-elements of $\mathcal{E}$ are exactly the vertices of P, we will say that P is a *Delaunay* polytope with respect to the form $\mathcal{Q}_{\mathcal{E}}$ defined by $\mathcal{E}$;

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more informally, we will say that a lattice polytope is *Delaunay* if it can be circumscribed by an *empty ellipsoid*. *Ellipsoid* is commonly used to refer to both hypersurfaces defined by positive definite quadratic forms and solid bodies bounded by such surfaces: the meaning of our usage will be clear from the context. Typically, there is a family of empty ellipsoids that can be circumscribed about a given Delaunay polytope P, but, if there is only one, so that $\mathcal{E}$ is uniquely determined by P, we will say that P is a *perfect Delaunay polytope* in $\mathbb{Z}^n$. Perfect Delaunay polytopes are also sometimes referred to as *extreme*. Perfect Delaunay polytopes are fascinating geometrical objects – examples are the six- and seven-dimensional [27] polytopes with 27 and 56 vertices respectively, which appear in the Delaunay tilings of the root lattices E_6 and E_7 (see, e.g., [6] for a description). In this paper we describe, up to an isometries and dilations, all known perfect Delaunay polytopes in $\mathbb{R}^n$ for $n \leq 8$; we also present a study of the geometry and combinatorics of these polytopes. We suspect that the list of perfect Delaunay polytopes that we give here is complete for $n \leq 8$. Erdahl [20, 21] proved that 0 and $[0, 1]$ are the only perfect Delaunay polytopes for $n \leq 5$; Dutour [16] proved that there is only one perfect Delaunay polytope in $\mathbb{R}^6$ – the Gosset 6-polytope (Coxeter's 2_{21}), which is described in Section 6. Only two perfect Delaunay polytopes are known in $\mathbb{R}^7$: they are Gosset's 7-polytope (Coxeter's 3_{21}) and a 35-tope found by Erdahl and Rybnikov [24], which are described in Section 7. We list twenty-seven 8-dimensional perfect Delaunay polytopes, they are identified by numbers, 1 through 27; Section 8 contains a detailed description of these polytopes. There are infinite series of perfect Delaunay polytopes – the first such series was found by Erdahl and Rybnikov in 2001 [32]. This series was further generalized in [23] to a 3-parametric series (where one parameter is the dimension) of perfect Delaunay polytopes. Another infinite series has been found by Dutour [13]. Prior to 2001 only sporadic examples of perfect Delaunay polytopes had been known, besides the cases for $n \leq 7$ mentioned above, all of them were found by Deza, Grishukhin and Laurent [9] and all of their examples were constructed as sections of the Leech and Barnes-Wall lattices.

2. Definitions and Notation

Formally speaking, the subject of this paper is the study of zero-sets of positive $\mathbb{Q}$-valued quadratic functions on free $\mathbb{Z}$-modules of finite rank. Since any free $\mathbb{Z}$-module of finite rank Λ can be realized as a discrete subgroup of $\mathbb{R}^n$ for any $n \geq \mathsf{rank}\,\Lambda$, we can think of Λ geometrically as a discrete set of vectors in vector space $\mathbb{R}^n$ or as a discrete set of points in affine space $\mathbb{R}^n$. Whenever we approach a $\mathbb{Z}$-module from this point of view, we call it a *lattice*.

Definition 1. A quadratic lattice is a pair $(\Lambda, \mathbb{R})$, where Λ is a free $\mathbb{Z}$-module of finite rank and $\mathcal{Q} : \Lambda \to \mathbb{R}$ is a quadratic form.

A function F on a module is called quadratic if it can be written as $\mathtt{QForm}\,F + A$, where $\mathtt{QForm}\,F$ is a quadratic form and A is an affine function. In general, we denote the quadratic form part of a polynomial P by $\mathtt{QForm}\,P$.

Definition 2. An *affine* quadratic lattice is a pair $\mathrm{Aff}(\Lambda, Q)$, where Λ is free $\mathbb{Z}$-module of finite rank and $Q : \Lambda \to \mathbb{R}$ is a quadratic function.

Sometimes, to stress that we consider a quadratic lattice, as opposed to an affine quadratic lattice, we call the former a linear or homogeneous quadratic lattice. $\mathrm{Aff}(\Lambda, Q)$ (resp. $(\Lambda, \mathcal{Q})$) is called degenerate if $\mathsf{rank}\,\mathtt{QForm}\,Q < \mathsf{rank}\,\Lambda$ (resp. $\mathsf{rank}\,\mathcal{Q} < \mathsf{rank}\,\Lambda$). Quadratic lattices $(\Lambda, \mathcal{Q})$ and $(\Lambda_1, \mathcal{Q}_1)$ are called isomorphic if there is a $\mathbb{Z}$-module isomorphism $L : \Lambda \to \Lambda_1$ such that $\mathcal{Q}(\mathbf{x}) = \mathcal{Q}_1(L\mathbf{x})$ for any $\mathbf{x} \in \Lambda$. Affine quadratic lattices $\mathrm{Aff}(\Lambda, Q)$ and $\mathrm{Aff}(\Lambda_1, Q_1)$ are called isomorphic if there is a $\mathbb{Z}$-module isomorphism $L : \Lambda \to \Lambda_1$ and some $\mathbf{z} \in \Lambda_1$ such that $Q(\mathbf{x}) = Q_1(L\mathbf{x} - \mathbf{z})$ for any $\mathbf{x} \in \Lambda$. We allow quadratic forms and functions to take values not only in $\mathbb{Z}$, the ground ring of the modules under consideration, but also in $\mathbb{Q}$ and its extensions, unlike the classical case of (linear) quadratic modules, where forms are supposed to be valued in the ground ring of the module (as in, e.g., [34]). Since we restrict ourselves to $\mathsf{rank}\,\Lambda < \infty$, any $\mathbb{Q}$-valued function on Λ can be rescaled into a $\mathbb{Z}$-valued quadratic function.

Suppose Q_1 and Q_2 are valued in $\mathbb{R}$. Two affine quadratic lattices $\mathrm{Aff}(\Lambda_1, Q_1)$ and $\mathrm{Aff}(\Lambda_2, Q_2)$ are called equivalent up to scaling if there is a $\mathbb{Z}$-module isomorphism $L : \Lambda_1 \to \Lambda_2$, some $\mathbf{z} \in \Lambda_2$ and a real $c > 0$ such that $Q_1(\mathbf{x}) = c\,Q_2(L\mathbf{x} - \mathbf{z})$.

A quadratic form $\mathcal{Q}$ induces a symmetric bilinear form on $\Lambda \times \Lambda$ by

$$(\mathbf{x}, \mathbf{y}) \mapsto \frac{1}{2}\left\{\mathcal{Q}[\mathbf{x} + \mathbf{y}] - \mathcal{Q}[\mathbf{x}] - \mathcal{Q}[\mathbf{y}]\right\};$$

we will denote this bilinear form also by $\mathcal{Q}$ – normally, there is no confusion, since a quadratic form has one arguments, while the corresponding bilinear form has two. We call a number a positive (resp. negative) if $a \geq 0$ (resp. $a \leq 0$); same terminology is applied to functions. Thus, a form $\mathcal{Q}$ is called positive if $\mathcal{Q}[\mathbf{x}] \geq 0$ for any $\mathbf{x}$. A form $\mathcal{Q}$ is called positive definite if $\mathcal{Q}[\mathbf{x}] > 0$ for any $\mathbf{x} \neq \mathbf{0}$.

Suppose a quadratic function Q is valued in $\mathbb{R}$. A number $b \in \mathbb{R}$ is called the arithmetic minimum of an affine quadratic lattice $\mathrm{Aff}(\Lambda, Q)$ if $b = \min_{\mathbf{z} \in \Lambda} Q(\mathbf{z})$. The vectors of Λ on which the minimum of $\mathrm{Aff}(\Lambda, Q)$ is attained are called the minimal vectors of $\mathrm{Aff}(\Lambda, Q)$. The definition of the arithmetic minimum of an *affine* quadratic lattice is slightly different from that of a homogeneous quadratic lattice: in the case of a homogeneous quadratic lattice the minimum is taken over all non-zero vectors $\mathbf{z} \in \Lambda$.

Definition 3. Let $Q : \Lambda \to \mathbb{R}$ be a fixed quadratic function with positive $\mathtt{QForm}\,Q$, and let X be a quadratic function with unknown coefficients. Let $b = \min_{\mathbf{z} \in \Lambda} Q(\mathbf{z})$. The affine quadratic lattice $\mathrm{Aff}(\Lambda, Q)$ is called *perfect* if the system of equations

$$\{X(\mathbf{m}) = b \mid \mathbf{m} \text{ is a minimal vector of } Q\}$$

has the only solution $X = Q$.

When we refer to a function as perfect without specifying Λ, the meaning of Λ is clear from the context. If a concrete formula for the function is given, Λ is presumed to be $\mathbb{Z}^n$. We will often use a shorthand such as e.g. *k-lattice* (polytope, cell, etc.), instead of *k-dimensional lattice* (polytope, cell, etc.).

From now on all quadratic functions are valued in $\mathbb{Q}$ or $\mathbb{R}$. In this context a symmetric bilinear form is a scalar product on $\Lambda \otimes \mathbb{Q} \cong \mathbb{Q}^n$ or $\Lambda \otimes \mathbb{R} \cong \mathbb{R}^n$. There are two canonical ways to describe an affine quadratic lattice, one by fixing the lattice to be $\mathbb{Z}^n$ and the other by fixing the quadratic form part of the function to be, say $\sum x_i^2$. The first method is more flexible, since it allows for quadratic forms of arbitrary signature. Furthermore, in any kind of machine computations it is far more convenient to deal with the former representation.

In each dimension there are only finitely many non-isomorphic perfect affine quadratic lattices, up to scale. This follows from Voronoi's L-type reduction theory (see, e.g., [10]). Namely, in each dimension there is a strict L-type reduction domain $\mathcal{D}$, which has finitely many extreme rays. The quadratic part of a perfect quadratic function is always arithmetically equivalent to a form lying on an extreme ray of $\mathcal{D}$ (but not vice versa). This implies the finiteness.

Proposition 4. *For $n = 0$ the only perfect affine quadratic lattice is $(\mathbb{Z}^0, 0)$. For $n = 1$ the only perfect affine quadratic lattice, up to isomorphisms and scaling, is $\mathrm{Aff}(\mathbb{Z}^1, (x - \frac{1}{2})^2)$.*

Perfect quadratic functions are inhomogeneous analogs of perfect quadratic forms introduced in the middle of 19-th century by Korkine and Zolotareff [30] and later studied by Voronoi, Barnes, Conway and Sloane, Stacey, Martinet, and others (see [31] for a survey). The interest to perfect forms has been mostly fueled by the theorem, proven by Korkine and Zolotareff [30], that forms that are extreme points of the ball packing density function are perfect. We prefer not to use the term inhomogeneous form, employed in some number-theoretic literature (e.g., [28]) since a *form* is by definition a *homogeneous polynomial.*

2.1. Delaunay Tilings

The language of Delaunay tilings provides a geometric way of thinking about quadratic functions with positive quadratic part. We denote the vertex set of a polytope P by $\mathsf{vert}\, P$. A convex Λ-polytope D is called a *Delaunay polytope* in a (linear) quadratic lattice $(\Lambda, \mathcal{Q})$, where $\mathcal{Q}[\mathbf{x}] > 0$ for $\mathbf{x} \neq 0$, if there is an ellipsoid $Q(\mathbf{x}) \leq 0$, with $\mathtt{QForm}\, Q = \mathcal{Q}$, whose boundary contains $\mathsf{vert}\, D$, but no points of $\Lambda \setminus \mathsf{vert}\, D$. If $\mathsf{dim}\, D < \mathsf{rank}\, \Lambda$ such an ellipsoid is not unique; however, the intersection of any such ellipsoid with the affine span of D is unique (for fixed $(\Lambda, \mathcal{Q})$). In particular, such an ellipsoid is unique when D is of maximal dimension – in this case this ellipsoid is called the Delaunay ellipsoid (or empty ellipsoid) of D.

For any $S \subset \Lambda$, we denote the affine span of S in $\Lambda \otimes \mathbb{R}$ by $\mathsf{aff}\, S$. The affine span of S in $\Lambda \otimes \mathbb{Q}$ is denoted by $\mathsf{aff}_{\mathbb{Q}}\, S$, and the lattice spanned by all vectors $\mathbf{x} - \mathbf{y}$, where $\mathbf{x}, \mathbf{y} \in S$, by

$\mathsf{aff}_{\mathbb{Z}}\, S$. Note that when $\mathbf{0} \in S$, the linear span of S, $\mathsf{lin}\, S$, and the affine span of S, $\mathsf{aff}\, S$, are the same. Often lattices arise as sections of other lattices by affine subspaces of the ambient affine Euclidean space. $\Gamma \subset \mathbb{R}^n$ is called an *affine lattice* if $\overrightarrow{\Gamma} = \{\mathbf{x} - \mathbf{y} \mid \mathbf{x}, \mathbf{y} \in \Gamma\}$ is a $\mathbb{Z}$-module of finite rank. In such situations it is convenient to have the notion of isomorphism for affine lattices. Let $\Gamma \subset \mathbb{R}^n$ and $\Gamma' \subset \mathbb{R}^m$ be affine lattices. A map $f : \Gamma \to \Gamma'$ is called an *affine isomorphism* if there are $\mathbf{o} \in \Gamma$ and $\mathbf{o}' \in \Gamma'$ such that $\mathbf{x} - \mathbf{o} \mapsto f\mathbf{x} - \mathbf{o}'$ is a $\mathbb{Z}$-module isomorphism from $\overrightarrow{\Gamma}$ onto $\overrightarrow{\Gamma'}$. Two functions $\varphi : \Gamma \to S$, $\psi : \Gamma' \to S$ on affine lattices Γ and Γ' are called *arithmetically equivalent* if there is an affine isomorphism $f : \Gamma \to \Gamma'$ such that $\varphi(\mathbf{x}) = \psi(f\mathbf{x})$. A Λ-polyhedron P can be thought of as the indicator function, which is 1 on $P \cap \Lambda$ and 0 elsewhere on Λ. Then, arithmetic equivalence of lattice polyhedra is a special case of the arithmetic equivalence between functions. If $\Gamma = \Gamma' = \mathbb{Z}^n$, the arithmetic equivalence is the same as the equivalence with respect to $Aff(n, \mathbb{Z})$, the group of all transformations of type $\mathbf{z} \mapsto L\mathbf{z} + \mathbf{t}$, where $L \in GL(n, \mathbb{Z})$ and $\mathbf{t} \in \mathbb{Z}^n$.

It is a theorem of Delaunay [8] that for a positive definite quadratic form $\mathcal{Q} : \Lambda \longrightarrow \mathbb{R}$ the space $\mathsf{aff}\, \Lambda$ is partitioned into the relative interiors of Delaunay Λ-polytopes with respect to $\mathcal{Q}$; this partition is organized so that the intersection of any family of Delaunay polytopes is again a Delaunay polytope (we add the empty polytope $\varnothing$ to the partition) – in other words the resulting *Delaunay tiling* is *face-to-face.* This theorem also says that a Delaunay tiling for $(\Lambda, \mathcal{Q})$ is unique. In studying Delaunay tilings and L-types of lattices it is often beneficial not to restrict to positive definite forms, but to use the concept of Delaunay tiling with respect to any positive $\mathbb{Q}$-valued quadratic form. Since traditionally, in the geometric context, quadratic forms are valued in $\mathbb{R}$, we will say a few words about the case where $\mathcal{Q}$ is $\mathbb{R}$-valued.

Definition 5. The rational closure of the cone of positive-definite quadratic forms on Λ is the set of all positive $\mathbb{R}$-valued forms $\mathcal{Q}$ that satisfy the condition $\mathsf{rank}(\Lambda \cap \mathfrak{Ker}_{\mathbb{R}}\, \mathcal{Q}) = \dim \mathfrak{Ker}_{\mathbb{R}}\, \mathcal{Q}$.

It is easy to see that the rational closure of the cone of positive-definite quadratic forms on Λ is a convex cone over $\mathbb{R}$. When $\Lambda = \mathbb{Z}^n$ we denote by $\mathsf{Sym}(n, \mathbb{R})$ the space of $\mathbb{R}$-valued quadratic forms on $\mathbb{Z}^n$, and by $\mathsf{Sym}_+(n, \mathbb{R})$ cone of all positive-definite quadratic forms in $\mathsf{Sym}(n, \mathbb{R})$. Then the real closure of $\mathsf{Sym}_+(n, \mathbb{R})$ in $\mathsf{Sym}(n, \mathbb{R})$ is denoted by $\overline{\mathsf{Sym}}_+(n, \mathbb{R})$, and the rational closure of $\mathsf{Sym}_+(n, \mathbb{R})$ by $\overline{\mathsf{Sym}}_+^{\mathbb{Q}}(n, \mathbb{R})$. When we consider an arbitrary lattice Λ, rather than $\mathbb{Z}^n$, we write $\mathsf{Sym}(\Lambda, *)$ instead of $\mathsf{Sym}(n, *)$.

$\overline{\mathsf{Sym}}_+^{\mathbb{Q}}(n, \mathbb{R})$ can also be described as the real cone spanned by rank-one forms in indeterminates $(x_1, \ldots, x_n) = \mathbf{x}$ of type $(\mathbf{v} \cdot \mathbf{x})^2$ where $\mathbf{v}$ runs over $\mathbb{Z}^n$ (see, e.g., [19]).

Sometimes the condition $\mathsf{rank}(\mathbb{Z}^n \cap \mathfrak{Ker}_{\mathbb{R}}\, \mathcal{Q}) = \dim \mathfrak{Ker}_{\mathbb{R}}\, \mathcal{Q}$ is phrased as that $\mathcal{Q}$ has rational kernel, although this expression can be somewhat misleading, since $\mathfrak{Ker}_{\mathbb{R}}\, \mathcal{Q} \cap \mathbb{Z}^n$ is always a rational subspace of $\mathbb{Q}^n$. Since, $\overline{\mathsf{Sym}}_+^{\mathbb{Q}}(n, \mathbb{R}) \cap \mathsf{Sym}(n, \mathbb{Q})$ consists of all positive forms with rational coefficients, perhaps, it would be more elegant to consider only $\mathbb{Q}$-valued forms on $\mathbb{Q}^n$, but we decided to follow the tradition and embed the cone of positive $\mathbb{Q}$-valued forms into $\overline{\mathsf{Sym}}_+^{\mathbb{Q}}(n, \mathbb{R})$. Let us denote by $\widetilde{\mathsf{QP}}_+(n, \mathbb{R})$ the cone of all real quadratic polynomials on

$\mathbb{R}^n$ whose quadratic form parts belong to $\overline{\mathsf{Sym}}_+^{\mathbb{Q}}(n,\mathbb{R})$. It is easy to see that $\widetilde{\mathsf{QP}}_+(n,\mathbb{R})$ is a convex cone in the space $\mathsf{QP}(n,\mathbb{R})$ of quadratic polynomials on $\mathbb{R}^n$.

When $\mathcal{Q}:\Lambda\to\mathbb{R}$ is positive semidefinite, it defines a tiling of $\Lambda\otimes\mathbb{R}$ only when

$$\mathsf{rank}(\mathfrak{Ker}_{\mathbb{R}}\,\mathcal{Q}\cap\Lambda)=\mathsf{dim}\,\mathfrak{Ker}_{\mathbb{R}}\,\mathcal{Q},$$

i.e., when $\mathcal{Q}\in\overline{\mathsf{Sym}}_+^{\mathbb{Q}}(\Lambda,\mathbb{R})$ (see, e.g., [19]). In this case $\mathsf{vert}\,P$ should be interpreted as $\partial P\cap\Lambda$ rather than as the set of vertices of P in the sense of convex geometry in $\mathbb{R}^n$; for simplicity, we still call elements of $\mathsf{vert}\,P$ vertices.

Definition 6. Let $(\Lambda,\mathcal{Q})$ be a quadratic lattice with $\mathcal{Q}\in\overline{\mathsf{Sym}}_+^{\mathbb{Q}}(\Lambda,\mathbb{R})$. A convex polyhedron $P\subset\Lambda\otimes\mathbb{R}$ is a Delaunay polyhedron for $(\Lambda,\mathcal{Q})$ if there is a quadratic polynomial E_P on $\Lambda\otimes\mathbb{R}$, with $\texttt{QForm}\,E_P=\mathcal{Q}$, such that $E_P(\mathbf{z})=0$ for any $\mathbf{z}\in\Lambda\cap P$ and $E_P(\mathbf{z})>0$ for any $\mathbf{z}\in\Lambda\diagdown P$.

In particular, the empty polytope $\varnothing$ and the whole space $\Lambda\otimes\mathbb{R}$ are Delaunay polyhedra in $(\Lambda,0)$: polynomials $E_\varnothing=1$ and $E_{\Lambda\otimes\mathbb{R}}=0$ can serve as witnesses. It is not difficult to show that when $\mathcal{Q}:\Lambda\longrightarrow\mathbb{Q}$ is positive semidefinite, $\mathsf{aff}\,\Lambda$ is covered by Delaunay *polyhedra* of various dimensions: some of these polyhedra are polytopes and some are cylinders over Delaunay polytopes of lower dimensions. We often refer to a Delaunay polyhedron as a Delaunay cell. In the semidefinite case the relative interiors of Delaunay cells also form a face-to-face partition of $\Lambda\otimes\mathbb{R}$, but the elements of Λ can no longer be considered as 0-cells of the tiling – the tiling in this case does not have any 0-cells unless $\mathsf{rank}\,\Lambda=0$. We denote the set of all cells of the Delaunay tiling of $\Lambda\otimes\mathbb{R}$ with respect to a semidefinite form $\mathcal{Q}$ by $Del(\Lambda,\mathcal{Q})$. $Del(\Lambda,\mathcal{Q})$ has a poset structure, namely, $F\preceq C$ if and only if $F\subset\partial C$. Furthermore, since both $\varnothing$ and $\Lambda\otimes\mathbb{R}$ are in $Del(\Lambda,\mathcal{Q})$, it is a lattice. In discussions of concrete Delaunay polytopes it is often more convenient to refer to faces by their vertex sets. The partial order on $Del(\Lambda,\mathcal{Q})$ induces a partial order on the vertex (in the generalized sense explained above) sets of Delaunay cells of $Del(\Lambda,\mathcal{Q})$. We will need the notion of Delaunay tiling for degenerate quadratic lattices only in Subsection 3.2, so, for the exception of that part of the paper, the reader may safely assume that $\mathcal{Q}$ is positive definite and all Delaunay cells are polytopes.

For formal definitions and detailed information on Delaunay tilings of lattices we refer to [10]. We only remark that the Delaunay tilings for lattices are classically defined [8] with the Euclidean norm $x_1^2+\ldots+x_n^2$ (in geometry of numbers the norm of a vector is its squared length), but are most effectively studied by isomorphically mapping the lattice Λ onto $\mathbb{Z}^n$, and replacing the Euclidean norm by a positive quadratic form $\mathcal{Q}$ that makes $(\mathbb{Z}^n,\mathcal{Q})$ and $(\Lambda,\sum x_i^2)$ equivalent. This allows us to think in terms of Euclidean lattices, i.e. geometrically, but to compute in terms of quadratic forms.

Definition 7. Let $(\Lambda,\mathcal{Q})$ be a quadratic lattice with $\mathcal{Q}\in\overline{\mathsf{Sym}}_+^{\mathbb{Q}}(\Lambda,\mathbb{R})$. Suppose $P\in Del(\Lambda,\mathcal{Q})$. P is called *perfect* if its Delaunay ellipsoid (or elliptic cylinder, if $\mathsf{rank}\,\mathcal{Q}<\mathsf{rank}\,\Lambda$) with respect to $\mathcal{Q}$ is the only quadric circumscribed about P in $\Lambda\otimes\mathbb{R}$.

Indeed, the notion of perfection, that was introduced in 19th century by the Italian school of algebraic geometry, is *independent of the Delaunay property of P and nature of $\mathcal{Q}$*. More generally, let F be a finite-dimensional linear space of $\mathbb{R}$-valued functions on Λ. Then a set $R \subset \Lambda$ is called perfect with respect to F if the system of *linear* inhomogeneous equations $\{f(\mathbf{r}) = c \,|\, \mathbf{r} \in R\}$ on the coefficients of f has a unique solution in F for any $c \neq 0$.

We have a natural bijection between perfect affine quadratic lattices $\mathrm{Aff}(\Lambda, Q)$ and triples (Λ, P, ρ^2), where P is a perfect Delaunay polyhedron and $\rho^2 \geq 0$ is the squared radius of its Delaunay ellipsoid. Thus, there are only finitely many arithmetically inequivalent perfect Delaunay polyhedra in each dimension, up to rescaling. Furthermore, since for perfect Delaunay polyhedra arithmetic equivalence implies isometry, there are only finitely many nonisometric perfect Delaunay polytopes in each dimension, up to rescaling. For $n = 0$ there is only one perfect Delaunay and it is $\mathbf{0}$. For $n = 1$ the polytope $[\mathbf{0}, \mathbf{1}]$ is perfect and Delaunay in $(\mathbb{Z}^1, x^2)$, and it is unique up to arithmetic equivalence.

3. Geometric Structure of Delaunay Cells

It is easy to see that a section of the vertex set of a Delaunay polytope by a rational affine subspace is the vertex set of a Delaunay polytope in the induced sublattice. This observation suggests a recursive approach to Delaunay polytopes where each newly discovered Delaunay polytope is represented as a disjoint union of Delaunay polytopes of smaller dimensions lying in parallel subspaces. Indeed, for $n > 1$ such a representation is never unique. It has been observed that dealing with a smaller numbers of big laminae is easier than studying a large number of small laminae. In other words, one is usually working with a representation in which the number of laminae is as small as possible.

Definition 8. The lamina number $l(P)$ of a lattice polytope P in a lattice Λ is the minimal number of disjoint affine subspaces of $\Lambda \otimes \mathbb{R}$ whose intersections with $\mathsf{vert}\, P$ form a partition of $\mathsf{vert}\, P$ into proper subsets.

The natural question is what laminar constructions lead to perfect Delaunay polytopes. In particular, is it possible to construct perfect Delaunay polytopes by using non-trivial (of dimension greater than 1) lower-dimensional perfect Delaunay polytopes as some of the laminae? It turns out for $n = 6 - 8$ such constructions are rather common, although not all of these polytopes have sections that are non-trivial perfect polytopes of smaller dimensions. The only perfect Delaunay 6-polytope, Gosset's G_6, does not have non-trivial perfect sections. The only two known perfect 7-polytopes, Gosset's G_7 and the 35-tope found by Erdahl and Rybnikov, each have a section isometric to G_6. Our study showed that of the 27 known perfect Delaunay 8-polytopes 17 have a section which is isometric to G_6, of which 10 have a section isometric to the 35-tope and one has a section isometric to G_7. The remaining 10 perfect 8-polytopes do not have non-trivial perfect sections.

The lamina number $l(P)$ is closely related to the notion of lattice width. Denote by $\Lambda^*_{\mathcal{Q}} \subset \mathsf{aff}\,\Lambda$ the dual of Λ with respect to the bilinear form $\mathcal{Q}$: $\Lambda^*_{\mathcal{Q}}$ consists of all vectors of $\mathsf{aff}\,\Lambda$ whose $\mathcal{Q}$-products with vectors of Λ are integer. If B is a convex body in $\mathsf{aff}\,\Lambda$, then the width $w(B)$ of B with respect to $(\Lambda, \mathcal{Q})$ is defined as the minimal natural number w such that B lies between hyperplanes $\mathcal{Q}(\mathbf{a}^*, \mathbf{x}) = k$ and $\mathcal{Q}(\mathbf{a}^*, \mathbf{x}) = k + w$, for some $\mathbf{a}^* \in \Lambda^*$. It is widely believed (see e.g. Barvinok, 2002) that a body in $\mathsf{aff}\,\Lambda$ whose interior is empty of Λ-points cannot have lattice width exceeding $\mathsf{rank}\,\Lambda$.

Proposition 9. *If P is a Delaunay polytope, then $l(P) = w(P) + 1$.*

We do not know of any Delaunay polytopes whose lattice width exceeds 2. On the other hand, we have

Theorem 10. *If P is a perfect (need not be Delaunay) polytope of dimension $n > 1$, then $w(P) + 1 = l(P) > 2$.*

Proof. Let $(\Lambda, \mathcal{Q})$ be the lattice in which P is perfect. Since the partition into laminae must be proper, $l(P) > 1$. If $l(P) = 2$, then there are affine sublattices L_1 and L_2 of codimension 1 such that $\mathsf{vert}\,P = (\mathsf{vert}\,P \cap L_1) \sqcup (\mathsf{vert}\,P \cap L_1)$. There exists an affine function A on Λ, which is 1 on L_1 and 0 on L_2. Then the quadric $A\mathbf{x}\,(A\mathbf{x} - 1) = 0$ is circumscribed about P. If Q is the quadratic function defined by P uniquely up to scale, then $Q + A(A - 1)$ must be proportional to Q, which means that $\mathtt{QForm}\,Q$ is of rank 1. Since P is perfect, $\mathsf{dim}\,P = \mathsf{rank}\,\Lambda$. Since P is a polytope, $\mathsf{rank}\,\Lambda = \mathsf{rank}\ \mathtt{QForm}\,Q = 1$ and $\mathsf{dim}\,P = 1$. □

It turns out that all perfect Delaunay polytopes in dimensions $n = 6 - 8$ have the lamina number l equal to 3.

Theorem 11. *Each perfect Delaunay polytope P described in Sections 6–8 has $l(P) = 3$*

Proof. $\mathsf{vert}\,G_6$ has a 3-laminae partition into a vertex, a 5-half-cube, and a 5-cross-polytope [25]; another partition is into a 5-simplex, a 15-vertex polytope, and another 5 simplex (see [21]). The partition of $\mathsf{vert}\,G_7$ into the union of the vertex sets of a 6-half-cube and two 6-cross-polytopes is given in Lemma 7.1. The partition of the 35-tope Υ^7 is given in Lemma 7.2 The partitions of the vertex sets of perfect 8-polytopes into layers follow from their coordinate representation given in Section 8. □

3.1. Perfect Delaunay Polytopes with Small Number of Vertices

The smallest number of vertices that a perfect Delaunay polytope of dimension n can have is $\frac{n(n+1)}{2} + n$. We call such polytopes *vertex-minimal*. It was noticed in [25] that *any* subset of vertices of Gosset's 6-polytope is the vertex set of some lattice Delaunay polytope. It turns out that this universality property holds for all vertex-minimal perfect Delaunay polytopes.

Theorem 12. *Let P be a vertex-minimal perfect Delaunay polytope of dimension n in $[\mathbb{Z}^n, \mathcal{Q}]$. Then for any $S \subset \mathsf{vert}\, P$ there is a positive definite quadratic form $\mathcal{Q}_S$ such that $\mathsf{conv}\, S$ is a Delaunay polytope in $[\mathbb{Z}^n, \mathcal{Q}_S]$.*

Proof. Consider the linear space $\mathcal{F}_0$ of real quadratic polynomials in n indeterminates with zero constant terms. The dimension of this space is $\frac{n(n+1)}{2} + n$. Let $\mathbf{c}$ be the Delaunay center and let $\rho > 0$ be the radius of P so that the equation of the ellipsoid is $\mathcal{Q}[\mathbf{x}] - 2\mathcal{Q}(\mathbf{x}, \mathbf{c}) + \mathcal{Q}[\mathbf{c}] = \rho^2$. Let us denote $\mathcal{Q}[\mathbf{x}] - 2\mathcal{Q}(\mathbf{x}, \mathbf{c})$ by $F_P(\mathbf{x})$. Any $\mathbf{x} \in \mathbb{R}^n$ can be regarded as a linear function on $\mathcal{F}_0$ that takes $F \in \mathcal{F}_0$ to $F(\mathbf{x})$ – when we consider $\mathbf{x}$ as such we write $\mathbf{x}(F)$. We can also view F_P as a point in the affine version of $\mathcal{F}_0$. The point F_P lies at the intersection of $\frac{n(n+1)}{2} + n$ distinct affine hyperplanes $\mathbf{v}(F_P) = \rho^2 - \mathcal{Q}[\mathbf{c}]$ corresponding to the vertices of P. If a point $F \in \mathcal{F}_0$ is sufficiently close to F_P, then the quadratic function $F - (\rho^2 - \mathcal{Q}[\mathbf{c}])$ is strictly positive on $\mathbb{Z}^n \setminus \mathsf{vert}\, P$. For each $\mathbf{v} \in \mathsf{vert}\, P$ denote by $H_\mathbf{v}$ the affine hyperplane $\{X \in \mathcal{F}_0 \mid \mathbf{v}(X) = \rho^2 - \mathcal{Q}[\mathbf{c}]\}$. Let $F = F_P + \varepsilon\mathbf{n}$ where $\varepsilon > 0$ and $\mathbf{n}$ is a non-zero vector in $\mathcal{F}_0$. For each of the hyperplanes $H_\mathbf{v}$ the sign of $\mathbf{v}(\mathbf{n})$ determines whether $F(\mathbf{v}) > \rho^2 - \mathcal{Q}[\mathbf{c}]$ or $F(\mathbf{v}) < \rho^2 - \mathcal{Q}[\mathbf{c}]$, where the former corresponds to $\mathbf{v}$ lying outside of the ellipsoid $F(\mathbf{x}) = \rho^2 - \mathcal{Q}[\mathbf{c}]$ and the latter to $\mathbf{v}$ lying inside of the ellipsoid $F(\mathbf{x}) = \rho^2 - \mathcal{Q}[\mathbf{c}]$. For each $\mathbf{v} \in \mathsf{vert}\, P$ we can pick a vector $\mathbf{n}_\mathbf{v}$ that lies in each of the hyperplanes $H_\mathbf{w}$ for all $\mathbf{w} \in \mathsf{vert}\, P \setminus \mathbf{v}$, but does not lie in $H_\mathbf{v}$. We can also adjust the direction of $\mathbf{n}_\mathbf{v}$, if necessary, so that for $F = F_P + \varepsilon\mathbf{n}_\mathbf{v}$ we have $F(\mathbf{v}) > \rho^2 - \mathcal{Q}[\mathbf{c}]$. Then any perturbation $\widetilde{F}$ of F_P, such that $\widetilde{F}(\mathbf{v}) \geq \rho^2 - \mathcal{Q}[\mathbf{c}]$ for all $\mathbf{v} \in \mathsf{vert}\, P$, can be written as $F_P + \sum_{\mathbf{v} \in \mathsf{vert}\, P} \varepsilon_\mathbf{v}\mathbf{n}_\mathbf{v}$ for some $\varepsilon_\mathbf{v} \geq 0$. Suppose $\varepsilon_\mathbf{v}$'s are all restricted to 0 or 1. Then for each $\mathbf{v} \in \mathsf{vert}\, P$ we know that $\varepsilon_\mathbf{v} = 0$ if and only if $\widetilde{F}(\mathbf{v}) = \rho^2 - \mathcal{Q}[\mathbf{c}]$, and $\varepsilon_\mathbf{v} = 1$ if and only if $\widetilde{F}(\mathbf{v}) > \rho^2 - \mathcal{Q}[\mathbf{c}]$. Note that if all of $\mathbf{n}_\mathbf{v}$ are sufficiently short, then $\widetilde{F} - \rho^2 - \mathcal{Q}[\mathbf{c}]$ is strictly positive on $\mathbb{Z}^n \setminus \mathsf{vert}\, P$. Thus, for any subset S of $\mathsf{vert}\, P$ we can pick the values of parameters $\varepsilon_\mathbf{v} \in \{0, 1\}$ so that $F_S = F_P + \sum_{\mathbf{v} \in \mathsf{vert}\, P} \varepsilon_\mathbf{v}\mathbf{n}_\mathbf{v}$ defines a Delaunay ellipsoid $F_S(\mathbf{x}) = \rho^2 - \mathcal{Q}[\mathbf{c}]$ circumscribed about $\mathsf{conv}\, S$. □

3.2. Structure of Perfect Affine Lattices

Recall that a pair $\mathrm{Aff}(\Lambda, Q)$, where Λ is a lattice and $Q : \Lambda \to \mathbb{R}$ a quadratic function, is called perfect if the coefficients of Q are uniquely determined from equations
$X(\mathbf{m}) = \min\{Q(\mathbf{z}) \mid \mathbf{z} \in \Lambda\}$, where $\mathbf{m}$ runs over all minimal vectors of Q and X is an unknown quadratic function on Λ. In general, for a function F defined on a lattice Λ denote by $\mathsf{V}(F)$ (the variety of F) the set of lattice points where F is 0. Perfection is a very natural notion as illustrated by the following theorem of Erdahl [21]. If L is an affine sublattice of a lattice Γ, then $\overrightarrow{L}$ *stands for the lattice* $L - L$.

Theorem 13. $\mathrm{Aff}(\Lambda, Q)$ *is perfect if and only if* $\mathsf{V}(Q) = \{\mathbf{v} + \mathbf{z} \mid \mathbf{v} \in \mathsf{vert}\, P,\ \mathbf{z} \in \Gamma\}$*, where P is a perfect Delaunay polytope in* $\mathrm{Aff}(\overrightarrow{\Lambda \cap \mathsf{aff}\, P}, Q|_{\overrightarrow{\mathsf{aff}\, P}})$ *and Γ is a sublattice of Λ such that Λ is the direct sum of $\mathbb{Z}$-modules* $\overrightarrow{\Lambda \cap \mathsf{aff}\, P}$ *and Γ, i.e.,*

$$\Lambda = \{(\mathbf{x} - \mathbf{x}') + \mathbf{z} \mid \mathbf{x}, \mathbf{x}' \in \Lambda \cap \mathsf{aff}\, P,\ \mathbf{z} \in \Gamma\}.$$

On the basis of this characterization Erdahl and Rybnikov proved the following theorem [23].

Theorem 14. *Let P be a perfect polytope in $Del(\Lambda, \mathcal{Q})$ and let Q_P be its perfect quadratic function, i.e.,* $\mathsf{vert}\, P = \mathsf{V}(Q_P)$ *and* $\mathcal{Q} = \mathtt{QForm}\, Q_P$. *Suppose $D \in Del(\Lambda, \mathcal{Q})$ is another Delaunay cell of full dimension, which is not a Λ-translate of P. If $\mathbf{e} \notin \Lambda \otimes \mathbb{R}$, then there is a positive form $\mathcal{Q}'$ on $\Lambda \oplus \mathbb{Z}\mathbf{e}$, which is not equivalent to $\mathcal{Q}$ extended by 0 to $\mathbb{Z}\mathbf{e}$, and a perfect polyhedron P' in $Del(\Lambda \oplus \mathbb{Z}\mathbf{e}, \mathcal{Q}')$ such that $P' \cap \mathsf{aff}\, \Lambda = P$ and $P' \cap \{\mathsf{aff}\, \Lambda + \mathbf{e}\} = D + \mathbf{e}$.*

The delicate part here is the case where P and D have identical Delaunay radii. By using a refinement of Erdahl's theorem [21] it is possible to prove that under in the Theorem 14 when P and D have different radii, the resulting perfect polyhedron P' must be a *polytope* (see [23]). On the basis of Theorem 14 we can make a few useful observations.

- If P is an antisymmetric perfect *polytope* in $Del(\Lambda, \mathcal{Q})$, then there is a perfect polytope P' in $Del(\Lambda \oplus \mathbb{Z}\mathbf{e}, \mathcal{Q}')$, for some form $\mathcal{Q}$, with a section isometric to P. For example, for $P = G_6$ (Gosset's 6-polytope) there are two perfect 7-polytopes that have G_6 as a section, namely G_7 and the 35-tope. G_7 can be obtained by taking $D = -P$ in Theorem 14, while the 35-tope cannot be obtained by a direct application of Theorem 14.

- We know only one example of a Delaunay tiling formed by translates of a *centrally-symmetric* perfect Delaunay polytope: this is the tiling of $\mathbb{Z}^1$ by unit intervals. Incidently, we do not know of any Delaunay polytope, except for the n-cube, that tiles $\mathbb{R}^n$ by translation. We conjecture that for $n > 1$ there are no such examples. If this is true, then Theorem 14 gives a guaranteed construction for a new type of perfect Delaunay polyhedron in dimension $n+1$ from a perfect Delaunay polytope in dimension n. However, this construction is not uniquely defined, since in Theorem 14 there may be different choices of D.

- When P is a centrally symmetric perfect n-polytope in $Del(\Lambda, \mathcal{Q})$ and there is an antisymmetric n-polytope D in $Del(\Lambda, \mathcal{Q})$, it may happen that the center of the perfect polytope P' coincides with the center of P. Then P' has at least three n-dimensional layers, which are translates of $-D$, P, and D respectively. The only 8-dimensional polytope from our list that has a section isometric to G_7 arises from this construction. The role of D is played by a Delaunay simplex of double volume in the Delaunay tiling (of lattice E_7) defined by G_7 (see [21] for a description).

- For many a perfect 8-polytope the Delaunay tiling has a significant number of arithmetically inequivalent 8-cells. This suggests that starting from $n = 9$ the number of perfect Delaunay n-polytopes explodes. (see `http://www.liga.ens.fr/~dutour` for the enumeration) It is likely that $n = 8$ is the highest dimension in which a complete classification is within reach.

4. Symmetries of Perfect Delaunay Polytopes

Recall that the group $O(\mathbb{Z}^n, \mathcal{Q})$ of linear automorphisms of a *quadratic* lattice $(\mathbb{Z}^n, \mathcal{Q})$ is defined as the full subgroup of $O(\mathbb{R}^n, \mathcal{Q})$ that maps $\mathbb{Z}^n$ onto itself, in other words, the set-stabilizer of $\mathbb{Z}^n$ in $O(\mathbb{R}^n, \mathcal{Q})$:

$$O(\mathbb{Z}^n, \mathcal{Q}) = \{\tau \in O(\mathbb{R}^n, \mathcal{Q}) \mid \tau(\mathbb{Z}^n) = \mathbb{Z}^n\}.$$

The group $O(\mathbb{Z}^n, \mathcal{Q})$ can also be seen as the subgroup of $GL_n(\mathbb{Z})$ that consists of transformations preserving $\mathcal{Q}$:

$$O(\mathbb{Z}^n, \mathcal{Q}) = \{\tau \in GL(n, \mathbb{Z}) \mid \forall \mathbf{z} \in \mathbb{Z}^n : \mathcal{Q}[\tau\mathbf{z}] = \mathcal{Q}[\mathbf{z}]\}.$$

Denote by $Iso(\mathbb{R}^n, \mathcal{Q})$ the group of affine automorphisms of $\mathbb{R}^n$ which preserve $\mathcal{Q}$. If D is a $\mathbb{Z}^n$-polytope, then $Iso(D, \mathcal{Q})$ denotes the group of all transformations from $Iso(\mathbb{R}^n, \mathcal{Q})$ that map D onto itself. Denote by $LatIso(D, \mathcal{Q})$ the group of all transformations from $O(\mathbb{Z}^n, \mathcal{Q})$ that map D to itself. Clearly $LatIso(D, \mathcal{Q}) \leq Iso(D, \mathcal{Q})$. When $\{\mathbf{x} - \mathbf{y} \mid \mathbf{x}, \mathbf{y} \in \mathsf{vert}\, D\} = \mathbb{Z}^n$ the polytope D is called *generating*. All known perfect Delaunay polytopes are generating. Obviously, for generating polytopes $Iso(D, \mathcal{Q}) = LatIso(D, \mathcal{Q})$.

An important invariant of a perfect Delaunay polytope P in $(\mathbb{Z}^n, \mathcal{Q})$ is the dimension of the subspace of quadratic forms in n variables preserved by $O(\mathbb{Z}^n, \mathcal{Q})$. We denote this space by $\mathsf{QuadInv}[O(\mathbb{Z}^n, \mathcal{Q})] = \mathsf{QuadInv}[D]$.

The metric geometry of a Delaunay polytope D is reflected in the *norm spectrum* of D, which is just the set of all possible value for $\mathcal{Q}[\mathbf{x} - \mathbf{y}]$, where $\mathbf{x}, \mathbf{y} \in \mathsf{vert}\, D$. We denote the norm spectrum by $\mathsf{Spec}(D)$.

We have classified the isometry groups of all known perfect Delaunay polytopes for $n \leq 8$. The isometry groups of six- and seven-dimensional perfect polytopes are distinct. Among the isometry groups of the 27 8-polytopes there are 21 non-isomorphic. Polytopes in the following five groups have isomorphic groups: #2 and #5; #3 and #13; #12 and #21; #14, #19, and #25; #24 and #27. The most interesting is the case of #2 and #5: both polytopes have 72 vertices in two orbits of size 56. Their group contains $\mathcal{S}_8$ as a subgroup of index 2.

5. Perfect Affine Quadratic Lattices for $n < 6$

$\mathrm{Aff}(\mathbf{0}, 0)$ is a perfect affine lattice of rank 0. All perfect affine lattices of rank 1 are obviously equivalent, up to scaling, to $\mathrm{Aff}(\mathbb{Z}^1, (x - \frac{1}{2})^2)$. The inequality $(x - \frac{1}{2})^2 \leq \frac{1}{4}$ describes the Delaunay ellipsoid for the Delaunay polytope $[0, 1]$. The Delaunay tiling for $(\mathbb{Z}^1, \mathtt{QForm}(x - \frac{1}{2})^2) = (\mathbb{Z}^1, x^2)$ consists of points of $\mathbb{Z}^1$, which are 0-dimensional polytopes, and segments $[k, k+1]$, where $k \in \mathbb{Z}$, which are 1-dimensional polytopes of the tiling. The symmetry group of $[0, 1]$ consists of 2 elements and is generated by reflection about $\frac{1}{2}$. Surprisingly, there are no perfect affine modules of ranks 2, 3, 4, and 5. This was first proven by Erdahl [20] in 1975 (see also [21]).

6. Perfect Affine Quadratic Lattices of Rank 6

The affine lattice $\mathrm{Aff}(\mathbb{Z}^6, \mathcal{E}_6[\mathbf{x}-\mathbf{c}])$, where $\mathcal{E}_6$ is given by

$$\mathcal{E}_6(\mathbf{x}) = \mathbf{x}^t \begin{bmatrix} 4 & 3 & 3 & 3 & 3 & 5 \\ 3 & 4 & 3 & 3 & 3 & 5 \\ 3 & 3 & 4 & 3 & 3 & 5 \\ 3 & 3 & 3 & 4 & 3 & 5 \\ 3 & 3 & 3 & 3 & 4 & 5 \\ 5 & 5 & 5 & 5 & 5 & 8 \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -2 \end{bmatrix},$$

turns out to be perfect, which was first observed by Erdahl [20]. Quadratic form $\mathcal{E}_6$ is of type E_6, i.e., $(\mathbb{Z}^6, \mathcal{E}_6[\mathbf{x}])$ is equivalent, up to scaling, to $(E_6, \sum x_i^2)$, where E_6 is a well-known root lattice in $\mathbb{R}^6$ (see, e.g., [7]). Lattice $(E_6, \sum x_i^2)$ has first been constructed by Korkine and Zolotareff [30], to which they referred as the *fifth perfect form in 6 variables* and denoted it by X.

Inequality $\mathcal{E}_6[\mathbf{x}-\mathbf{c}] \le \frac{4}{3}$ defines the Delaunay ellipsoid for a Delaunay polytope in $(\mathbb{Z}^6, \mathcal{E}_6)$. The set of 27 vertices of this polytope is given by the following table ($[-1, 0^4; -1]$ means the entry 0 is repeated 4 times and all permutations of the first 5 positions are taken – the last entry is separated by semicolon and is not permuted; the other records of the table are interpreted similarly).

$x_6 = -3$	$x_6 = -2$	$x_6 = -1$	$x_6 = 0$	$x_6 = 1$
$[\mathbf{1^5}; \mathbf{-3}] \times 1$	$[\mathbf{0}, \mathbf{1^4}; \mathbf{-2}] \times 5$	$[\mathbf{1^2}, \mathbf{0^3}; \mathbf{-1}] \times 10$	$[\mathbf{0^6}] \times 1$	$[\mathbf{1}, \mathbf{0^4}; \mathbf{-1}] \times 5$
			$[\mathbf{1}, \mathbf{0^4}; \mathbf{0}] \times 5$	

This polytope is known as Gosset's 6-dimensional semiangular polytope [27], which we denote by G_6. It is a two distance set and $\mathsf{Spec}(G_6) = \{2, 4\}$. Typically, G_6 is described as a Delaunay polytope for $(E_6, \sum x_i^2)$, in which case it is commonly denoted by 2_{21} – the notation going back to Coxeter (see [6] for some history). G_6 has two orbits of facets, regular simplices and regular cross-polytopes. It does not have interior diagonals and all segments joining its vertices are either edges, or diagonals of its facets. The 1-skeleton of G_6 is a strongly regular graph known as the Schlafli graph. G_6's isometry group $Iso(G_6, \mathcal{E}_6)$, of order 51840, is the famous group of automorphisms of the 27 lines on a general cubic surface. $Iso(G_6, \mathcal{E}_6)$ is isomorphic to the semidirect product of a 2-element group generated by a reflection and a reflection-free normal subgroup that consists of $2^6 \times 3^4 \times 5 = 25920$ elements; the latter group is simple and has a number of descriptions as a group of Lie type (see [5] for more details). The isometry group of G_6 is transitive on its vertex set. For more details see [6] and [25].

6.1. Laminar Structure of Gosset's G_6

Let us denote by $J(n,s)$ the polytope formed by all $\{0,1\}$-vectors in $(\mathbb{Z}^n, \sum_{i=1}^n x_i^2)$ with the sum of the coordinates equal to s. It is known that for each s, such that $0 \le s < n$, the polytope $J(n,s)$ is isometric to a Delaunay polytopes in $(A_{n-1}, \sum_{i=1}^{n-1} x_i^2)$, where A_{n-1} is a root lattice of type A of rank $n-1$. G_6 can be represented as the union of 3 laminae that are isometric to $J(6,1)$, $J(6,2)$, and $J(6,1)$, where the regular 6-simplices $J(6,1)$ are parallel (see [21]). Another lamination of G_6 is into 3 layers that consist of a 0-simplex, a 5-halfcube, and a 5-cross-polytope (see [25]). These two laminations correspond to the subdiagrams of types A_5 and D_5, respectively, of the Coxeter diagram E_6, which represents the isometry group of G_6 (see [29]).

It was long suspected that $\mathrm{Aff}(\mathbb{Z}^6, \mathcal{E}_6[\mathbf{x}-\mathbf{c}])$ is the only perfect affine lattice of rank 6 up to scaling. Finally, Dutour [15], using his *EXT-HYP7* program, created in 2002, proved that this is the case.

7. Perfect Affine Quadratic Lattices of Rank 7

7.1. Gosset Polytope in Lattice E_7

The affine lattice $\mathrm{Aff}(\mathbb{Z}^7, \mathcal{E}_7[\mathbf{x}-\mathbf{c}])$, where $\mathcal{E}_7$ is given by

$$\mathcal{E}_7(\mathbf{x}) = \mathbf{x}^t \begin{bmatrix} 4 & 3 & 3 & 3 & 3 & 5 & 4 \\ 3 & 4 & 3 & 3 & 3 & 5 & 4 \\ 3 & 3 & 4 & 3 & 3 & 5 & 4 \\ 3 & 3 & 3 & 4 & 3 & 5 & 4 \\ 3 & 3 & 3 & 3 & 4 & 5 & 4 \\ 5 & 5 & 5 & 5 & 5 & 8 & 6 \\ 4 & 4 & 4 & 4 & 4 & 6 & 6 \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

turns out to be perfect, which was first observed by Erdahl [20]. Quadratic form $\mathcal{E}_7$ is of type E_7, that is $(\mathbb{Z}^7, \mathcal{E}_7)$ is equivalent, up to scaling, to $(E_7, \sum x_i^2)$, where E_7 is a well-known root lattice in $\mathbb{R}^7$ (see, e.g., [7]). Lattice $(E_7, \sum x_i^2)$ has first been constructed by Korkine and Zolotareff [30], to which they referred as the *sixth perfect form in 7 variables* and denoted it by Y.

Inequality $\mathcal{E}_7[\mathbf{x}-\mathbf{c}] \leq 3$ defines the Delaunay ellipsoid for a polytope in $Del(\mathbb{Z}^7, \mathcal{E}_7)$, whose vertex set is given below.

$x_7=-1$	$x_7=0$	$x_7=1$	$x_7=2$
$[\mathbf{1^5}; \mathbf{-2}; \mathbf{-1}] \times 1$	$[\mathbf{0^7}] \times 1$	$[\mathbf{0^6}; \mathbf{1}] \times 1$	$[\mathbf{-1^5}; \mathbf{2}; \mathbf{2}] \times 1$
	$[\mathbf{1}, \mathbf{0^4}; \mathbf{0}; \mathbf{0}] \times 5$	$[\mathbf{-1}, \mathbf{0^4}; \mathbf{0}; \mathbf{1}] \times 5$	
	$[\mathbf{-1}, \mathbf{0^4}; \mathbf{1}; \mathbf{0}] \times 5$	$[\mathbf{1}, \mathbf{0^4}; \mathbf{-1}; \mathbf{1}] \times 5$	
	$[\mathbf{1^2}, \mathbf{0^3}; \mathbf{-1}; \mathbf{0}] \times 10$	$[\mathbf{-1^2}, \mathbf{0^3}; \mathbf{1}; \mathbf{1}] \times 10$	
	$[\mathbf{0}, \mathbf{1^4}; \mathbf{-2}; \mathbf{0}] \times 5$	$[\mathbf{0}, \mathbf{-1^4}; \mathbf{2}; \mathbf{1}] \times 5$	
	$[\mathbf{1^5}; \mathbf{-3}; \mathbf{0}] \times 1$	$[\mathbf{-1^5}; \mathbf{3}; \mathbf{1}] \times 1$	

This centrally-symmetric polytope has 56 vertices and is known as Gosset's 7-dimensional semiregular polytope [27], which we denote by G_7. It is a 3-distance set and $\mathsf{Spec}(G_7) = \{2, 4, 6\}$. Typically, G_7 is described as a Delaunay polytope for $(E_7, \sum x_i^2)$, in which case it is commonly denoted by 3_{21} after Coxeter (see [6] for some history). The 1-skeleton of G_7 is known as the Gosset graph, which is a strongly-regular graph. G_7 has 28 interior diagonals passing through its center; in fact, Patrick du Val discovered that G_7 can be thought of as the convex hull of seven congruent 3-dimensional cubes in $\mathbb{R}^7$ with common center (attributed to du Val by Coxeter [6]). The isometry group of G_7 is transitive on vertices and is isomorphic to the semidirect product of a 2-element group generated by a reflection and a reflection-free normal subgroup of $4 \times 9! = 1451520$ elements; the latter group is simple and is isomorphic, among other groups of Lie type, to $O_7(2)$ (see [5]).

Lemma 7.1. *$l(G_7) = 3$ and $\mathsf{vert}(G_7)$ is the union of the vertex sets of two 6-cross-polytopes and a 6-half-cube.*

Proof. The fact $l(G_7) = 3$ follows from the representation of G_7 as the following subset of the 7-cube $[-1, +1]^7$. Consider all cyclic permutations of 8 vectors $(\pm1, \pm1; 0; \pm1; 0, 0, 0)$. This defines a 56-element subset V of $[-1, +1]^7$. du Val had shown that the convex hull of these 56 points is the Gosset 7-polytope with respect to the usual metric $\sum x_i^2$. Note that $\mathsf{conv}\, V \notin Del(\mathbb{Z}^7, \sum x_i^2)$, since $\mathsf{conv}\, V$ obviously contains the origin.

In each of the coordinate directions $\mathsf{conv}\, V$ has three layers defined by inequalities $x_i = -1, 0, +1$ and thus $l(\mathsf{conv}\, V) = l(G_7) = 3$. It is easy to see that the sections $x_i = -1$ and $x_i = +1$ are 6-cross-polytopes and the section $x_i = 0$ is a 6-half-cube. □

7.1.1. Laminar Structure of Gosset's G_7

G_7 can be represented as the union of 4 laminae that are isometric to $J(7, 1)$, $J(7, 2)$, $J(7, 2)$, and $J(7, 1)$ (where $J(7, 1)$ is regular simplex). The above Lemma gives a lamination of $\mathsf{vert}\, G_7$

is into 3 laminae that consists of vertex sets of a 6-cross-polytope, 6-halfcube, and another 6-cross-polytope. Yet another lamination is given in the above table, where the layers are a 0-simplex, a copy of G_6, another copy of G_6, and another 0-simplex. These three laminations correspond to the *unique* subdiagrams of types A_6, D_6, and E_6 of the Coxeter diagram E_7, which represents the isometry group of G_7 (see [29]).

7.2. The 35-tope

The only known perfect affine lattice of rank 7, that is not equivalent to $\mathrm{Aff}(\mathbb{Z}^7, \mathcal{E}_7[\mathbf{x}-\mathbf{c}])$, was constructed by Erdahl and Rybnikov in 2000 (see [25] and [32]). It is $\mathrm{Aff}(\mathbb{Z}^7, \mathcal{ER}_7[\mathbf{x}-\mathbf{c}])$, where $\mathcal{ER}_7$ is given by

$$\mathcal{ER}_7(\mathbf{x}) = \mathbf{x}^t \begin{bmatrix} 8 & 6 & 6 & 6 & 6 & 6 & 9 \\ 6 & 8 & 6 & 6 & 6 & 6 & 9 \\ 6 & 6 & 8 & 6 & 6 & 6 & 9 \\ 6 & 6 & 6 & 8 & 6 & 6 & 9 \\ 6 & 6 & 6 & 6 & 8 & 6 & 9 \\ 6 & 6 & 6 & 6 & 6 & 8 & 9 \\ 9 & 9 & 9 & 9 & 9 & 9 & 13 \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{16} \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \\ 5 \\ 5 \\ -14 \end{bmatrix}$$

The lattice $(\Lambda, \mathcal{ER}_7)$ has 12 shortest vectors and $\det \Lambda_{\mathcal{ER}_7} = 256$. The order of $O(\Lambda, \mathcal{ER}_7)$ is 2880 and the dimension of the space of invariant forms is 3. $(\Lambda, \mathcal{ER}_7)$ is not perfect, but the lattice obtained from $(\Lambda, \mathcal{ER}_7)$ by adding the centers of perfect ellipsoids is perfect with 70 shortest vectors.

Inequality $\mathcal{ER}_7[\mathbf{x}-\mathbf{c}] \leq \frac{43}{16}$ defines the Delaunay ellipsoid for a perfect Delaunay Υ^7 in $Del(\mathbb{Z}^7, \mathcal{ER}_7)$, whose vertex set is given below.

$x_7 = -4$	$x_7 = -3$	$x_7 = -1$	$x_7 = 0$	$x_7 = 1$
$[\mathbf{1^6}; \mathbf{-4}] \times 1$	$[\mathbf{0}, \mathbf{1^5}; \mathbf{-3}] \times 6$	$[\mathbf{1^2}, \mathbf{0^4}; \mathbf{-1}] \times 15$	$[\mathbf{0^7}] \times 1$	$[\mathbf{-1}, \mathbf{0^5}; \mathbf{1}] \times 6$
			$[\mathbf{1}, \mathbf{0^5}; \mathbf{0}] \times 6$	

$\mathsf{Spec}(\Upsilon^7) = \{3, 4, 5, 7, 8, 9\}$. Polytope Υ^7 generalizes to an infinite series of perfect Delaunay polytopes Υ^n ($n \geq 7$) with $\frac{n(n+3)}{2}$ vertices (see [24]):

$x_n = -(n-3)$	$x_n = -(n-4)$	$x_n = -1$
$[\mathbf{1^{n-1}}; \mathbf{-(n-3)}] \times 1$	$[\mathbf{0}, \mathbf{1^{n-2}}; \mathbf{-(n-4)}] \times (n-1)$	$[\mathbf{1^2}, \mathbf{0^{n-3}}; \mathbf{-1}] \times \frac{(n-1)(n-2)}{2}$
	$[\mathbf{1}, \mathbf{0^{n-2}}; \mathbf{0}] \times (n-1)$	

$x_n = 0$	$x_n = 1$
$[\mathbf{0^n}] \times 1$	$[\mathbf{-1, 0^{n-2}; 1}] \times (n-1)$
$[\mathbf{1, 0^{n-2}; 0}] \times (n-1)$	

Polytope Υ^7 has lamina number $l(\Upsilon^7) = 3$ and can be represented as the union of Gosset polytope G_6 and regular simplices of dimensions 2 and 4 lying in parallel subspaces of $\mathbb{R}^7$:

Lemma 7.2. *The vertex set of Υ^7 is the disjoint union of the vertex sets of a Gosset polytope G_6, a regular 5-simplex, and a 1-simplex.*

Proof. Consider the following partition

$$\mathsf{vert}\,\Upsilon^7 = S_1 \bigsqcup S_2 \bigsqcup S_3 =$$

$$\{[-1, 0^5; 1] \times 6, [0, 1^5; -3] \times 6, [1^2, 0^4; -1] \times 15\} \bigsqcup \{[0^5, 1; 0] \times 6\} \bigsqcup \{[0^7], [1^6; -4]\}.$$

Let us show that the affine rank of the first subset is 6. The first subset S_1 can be represented as $S_{11} \bigsqcup S_{12} \bigsqcup S_{13}$, where $\{S_{11} = [-1, 0^5; 1] \times 6\}$, $\{S_{12} = [0, 1^5; -3] \times 6\}$, $\{S_{13} = [1^2, 0^4; -1] \times 15\}$.

Notice that

$$[-1, 0, 0, 0, 0, 0, 1] = [0, 1, 1, 0, 0, 0, -1] + ([0, 0, 0, 1, 1, 0, -1] - [1, 1, 1, 1, 1, 0, -3])$$

Applying cyclic permutations of the first six characters to the above identity, we see that, each element of S_{11} can be written as $\mathbf{p}_1 + (\mathbf{p}_2 - \mathbf{p})$, where $\mathbf{p}_1, \mathbf{p}_2 \in S_3$ and $\mathbf{p} \in S_2$. Therefore, $\mathsf{aff}(S_{11} \cup S_{12} \cup S_{13}) = \mathsf{aff}(S_{12} \cup S_{13})$. Since both S_{12} and S_{13} lie on the hyperplane $2(x_1 + x_2 + x_3 + x_4 + x_5 + x_6) + 3x_7 = 1$, $\mathsf{dim}\,\mathsf{aff}(S_1 \cup S_2 \cup S_3) = 6$. Notice that this argument does not work if we replace Υ^7 with Υ^n for $n > 7$. It turns out that for $n > 7$ we have $\mathsf{aff}(S_{11} \cup S_{12} \cup S_{13}) \neq \mathsf{aff}(S_{12} \cup S_{13})$.

It is clear that the affine subspaces generated by S_2 and S_3 are parallel to that generated by S_1. Computing squared distances between the elements of S_1 with metric $\mathcal{ER}_7$ shows that it is a two-distance set isometric to G_6 (dilated by a factor of 2). The second and the third sets are obviously regular simplices. □

8. Perfect Affine Quadratic Lattices of Rank 8

We denote the perfect quadratic functions by $\mathcal{Q}_i^8[\mathbf{x}-\mathbf{c}]$, where $\mathbf{c} \in \mathbb{Q}^n$, and the corresponding perfect Delaunay polytopes by D_i^8. For each $\mathcal{Q}_i^8[\mathbf{x}-\mathbf{c}]$ we give

- an integer Gram matrix,
- the center $\mathbf{c}$ of the perfect ellipsoid,
- the order of the group $O(\mathbb{Z}^8, \mathcal{Q}_i^8)$ (the group's generators are available from the first author upon request), together with the size of the maximal symmetric subgroup,
- the number $s(\mathbb{Z}^8, \mathcal{Q}_i^8)$ of shortest vectors,
- the dimension of the subspace of $\mathsf{Sym}(8, \mathbb{R})$ that consists of forms $\mathcal{Q}$ such that $\mathcal{Q}[T\mathbf{z}] = \mathcal{Q}[\mathbf{z}]$ for every $T \in O(\mathbb{Z}^8, \mathcal{Q}_i^8)$, we denote this subspace by $\mathsf{QuadInv}\,[O(\mathbb{Z}^8, \mathcal{Q}_i^8)]$.

For each D_i^8 we give

- the coordinates of the vertices,
- $|Iso(D_i^8, \mathcal{Q}_i^8)|$ = Order of the isometry group of D_i^8,
- Whether D_i^8 is Centrally-symmetric or Antisymmetric,
- Maximal non-trivial perfect polytope of smaller dimension (i.e. G_6, G_7, or the 35-tope) contained in D_i^8,
- Information on certain types of Delaunay polytopes contained in D_i^8. If X is an arithmetic type of a lattice Delaunay polytope such as, e.g., $J(n, s)$, and D_i^8 contains a copy of $J(n, s)$, which is not a proper subpolytope of a $J(n', s') \subset D_i^8$, then we state that $J(n, s)$ is maximally included into D_i^8.
- the norm spectrum $\mathsf{Spec}(D_i^8)$.
- the lamina number $l(D_i^8)$ (always 3).

The coordinatization of all polytopes is chosen so that the three laminae structure is transparent. For some of the polytopes we give additional geometric information.

8.1. Delaunay Tilings of Lattices A_n and D_n

The geometric structure of 8-dimensional perfect Delaunay polytopes can be analyzed by relating the geometry of these polytopes to the geometry of Delaunay tilings of lattices A_n and D_n, which is explicit in our 8-dimensional data. In this section let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{Z}^n \subset \mathbb{E}^n$, and let $I = \{\mathbf{x} \in \mathbb{R}^n \mid \forall i : 0 \leq \mathbf{x} \cdot \mathbf{e}_i \leq 1\}$ denote the standard unit cube.

A_n can be defined as $(\mathbb{Z}^n, \sum_{i=1}^n x_i^2 + \sum_{1 \leq i < j \leq n} x_i x_j)$ or, in terms of the Euclidean space $\mathbb{E}^n$, as the lattice based on a regular n-simplex. Lattice D_n can be defined as $(\mathbb{Z}^n, \sum_{i=1}^n x_i^2 +$

$x_1x_3 + \sum_{2\leq i<j\leq n} x_ix_j$) or, in terms of the Euclidean space $\mathbb{E}^n$, as the sublattice of $\mathbb{Z}^n$ that consists of all points with even sum of the coordinates; another Euclidean construction of D_n is obtained by taking $\mathbb{Z}^n$ and adding to it the centers of all facets of the unit n-cubes with integral vertices – this is an n-dimensional generalization of what is known in crystallography as the face-centered cubic lattice, or fcc. Note that for $n = 3$, A_n and D_n coincide.

Delaunay tilings of A_n have been described by Barnes [3] and Delaunay tilings of D_n have been described by Ryshkov and Shushbaev [33]. Below we give a brief description of these tilings borrowed from Baranovskii [1].

Let $\mathbf{d} = \sum \mathbf{e}_i$. Consider the sections of the standard unit cube I by hyperplanes perpendicular to $\mathbf{d}$ and passing through the points $\frac{q}{n}\mathbf{d}$ for $q = 1, \ldots, n$. These hyperplanes induce a tiling of I by n n-polytopes $P(q)$, where each $P(q)$ is squeezed between hyperplanes $\sum x_i = q - 1$ and $\sum x_i = q$. It can be shown (see [3] or [1]) that with respect to quadratic form $\sum_{i=1}^n x_i^2 + \sum_{1\leq i<j\leq n} x_ix_j$ these polytopes are Delaunay. Thus, any Delaunay n-polytope in A_n is a translate of one of these polytopes. The 1-skeletons of the faces of $P(q)$ that are defined by $P(q) \cap \{\mathbf{x} \in \mathbb{R}^n \mid \sum x_i = q - 1\}$ and $P(q) \cap \{\mathbf{x} \in \mathbb{R}^n \mid \sum x_i = q\}$ are Johnson graphs $J(n, q - 1)$ and $J(n, q)$. We also use $J(n, q - 1)$ and $J(n, q)$ to refer to the arithmetic classes of these polytopes. Note that $J(n, q)$ and $J(n, n - q)$ are isometric with respect to any quadratic form on $\mathbb{Z}^n$, since one of them can be obtained from the other by a combination of a lattice translation and an inversion with respect to a lattice point; in the terminology of geometry of numbers such polytopes are called *homologous.*

Let us consider D_n as the sublattice of $\mathbb{Z}^n \subset \mathbb{E}^n$ that consists of all points with even sum of the coordinates. Then any Delaunay n-polytopes in D_n is homologous to one of the following:

1. a cross-polytope, with vertices in D_n, centered at $\mathbf{x} \in \mathbb{Z}^n$, where $\sum x_i \equiv 1 \mod 2$,
2. the convex hull of points of D_n that belong to the standard cube I,
3. the convex hull of points of D_n that belong to the shifted cube $I + \mathbf{e}_n$.

The polytopes in 2) and 3) are known as n-halfcubes (or semicubes). Note that for $n = 3$ a halfcube is a tetrahedron and for $n = 4$ a halfcubes is a cross-polytope; the latter fact explains why the Delaunay tiling of D_4 is formed by three homology classes of cross-polytopes.

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}_1^8[x-\mathbf{c}])$, where $\mathcal{Q}_1^8$ is given by

$$\mathcal{Q}_1^8(\mathbf{x}) = \mathbf{x}^t \begin{bmatrix} 4 & 1 & 1 & 1 & 1 & 1 & 1 & -5 \\ 1 & 4 & 1 & 1 & 1 & 1 & 1 & -5 \\ 1 & 1 & 4 & 1 & 1 & 1 & 1 & -5 \\ 1 & 1 & 1 & 4 & 1 & 1 & 1 & -5 \\ 1 & 1 & 1 & 1 & 4 & 1 & 1 & -5 \\ 1 & 1 & 1 & 1 & 1 & 4 & 1 & -5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 4 & -5 \\ -5 & -5 & -5 & -5 & -5 & -5 & -5 & 19 \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{10} \begin{bmatrix} 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 7 \\ 10 \end{bmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}_1^8)| = 20160$; $S_7 < O(\mathbb{Z}^8, \mathcal{Q}_1^8)$
- $s(\mathbb{Z}^8, \mathcal{Q}_1^8) = 14$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}_1^8] = 3$

Inequality $\mathcal{Q}_1^8[\mathbf{x} - \mathbf{c}] \le \frac{43}{10}$ defines the Delaunay ellipsoid for a perfect polytope $D_1^8 \in Del(\mathbb{Z}^8, \mathcal{Q}_1^8)$, whose vertex set ($|\mathsf{vert}\, D_1^8| = 44$) is given below.

$x_8 = 0$	$x_8 = 1$	$x_8 = 2$
$[\mathbf{0^8}] \times 1$	$[\mathbf{0^2}, \mathbf{1^5}; \mathbf{1}] \times 21$	$[\mathbf{1^7}; \mathbf{2}] \times 1$
$[\mathbf{0^6}, \mathbf{1}; \mathbf{0}] \times 7$	$[\mathbf{0}, \mathbf{1^6}; \mathbf{1}] \times 7$	$[\mathbf{1^6}, \mathbf{2}; \mathbf{2}] \times 7$

- $\mathsf{Spec}(D_1^8) = 4,\ 6,\ 7,\ 9,\ 10,\ 12,\ 13,\ 15$
- $|Iso(D_1^8, \mathcal{Q}_1^8)| = 10080$
- $l(D_1^8) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $H(2),\ \frac{1}{2}H(4),\ J(8,6)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}_2^8[x-\mathbf{c}])$, where $\mathcal{Q}_2^8$ is given by

$$\mathcal{Q}_2^8(\mathbf{x}) = \mathbf{x}^t \begin{pmatrix} 8 & 6 & 6 & 6 & 6 & 10 & 8 & 4 \\ 6 & 8 & 6 & 6 & 6 & 10 & 8 & 5 \\ 6 & 6 & 8 & 6 & 6 & 10 & 8 & 6 \\ 6 & 6 & 6 & 8 & 6 & 10 & 8 & 4 \\ 6 & 6 & 6 & 6 & 8 & 10 & 8 & 4 \\ 10 & 10 & 10 & 10 & 10 & 16 & 12 & 7 \\ 8 & 8 & 8 & 8 & 8 & 12 & 12 & 7 \\ 4 & 5 & 6 & 4 & 4 & 7 & 7 & 7 \end{pmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}_2^8)| = 80640$; $S_8 < O(\mathbb{Z}^8, \mathcal{Q}_2^8)$
- $s(\mathbb{Z}^8, \mathcal{Q}_2^8) = 16$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}_2^8] = 2$

Inequality $\mathcal{Q}_2^8[\mathbf{x}-\mathbf{c}] \leq 3$ defines the Delaunay ellipsoid for a perfect polytope $D_2^8 \in Del(\mathbb{Z}^8, \mathcal{Q}_2^8)$, whose vertex set ($|\mathsf{vert}\, D_2^8| = 72$) is given below.

$x_8 = -1$	$x_8 = 0$	$x_8 = 1$
$[-\mathbf{1^2}; \mathbf{0}; -\mathbf{1^2}; \mathbf{2^2}; -\mathbf{1}] \times 1$	$[-\mathbf{1^5}; \mathbf{2^2}; \mathbf{0}] \times 1$	$[\mathbf{0}; -\mathbf{1^2}; \mathbf{0^2}; \mathbf{1}; \mathbf{0}; \mathbf{1}] \times 1$
$[-\mathbf{1}; \mathbf{0^2}; -\mathbf{1^2}; \mathbf{1}; \mathbf{2}; -\mathbf{1}] \times 1$	$[-\mathbf{1^5}; \mathbf{3}; \mathbf{1}; \mathbf{0}] \times 1$	$[\mathbf{0^2}; -\mathbf{1}; \mathbf{0}, \mathbf{1}; \mathbf{0^2}; \mathbf{1}] \times 2$
$[-\mathbf{1}; \mathbf{0^2}; -\mathbf{1^2}; \mathbf{2}; \mathbf{1}; -\mathbf{1}] \times 1$	$[-\mathbf{1^4}, \mathbf{0}; \mathbf{2}; \mathbf{1}; \mathbf{0}] \times 5$	$[\mathbf{0^7}; \mathbf{1}] \times 1$
$[-\mathbf{1}; \mathbf{0}; \mathbf{1}; \mathbf{0^3}; \mathbf{1}; -\mathbf{1}] \times 1$	$[-\mathbf{1^2}, \mathbf{0^3}; \mathbf{1^2}; \mathbf{0}] \times 10$	$[\mathbf{1}; \mathbf{0}; -\mathbf{1}; \mathbf{0^4}; \mathbf{1}] \times 1$
$[\mathbf{0^6}; \mathbf{1}; -\mathbf{1}] \times 1$	$[-\mathbf{1}, \mathbf{0^4}; \mathbf{0}; \mathbf{1}; \mathbf{0}] \times 5$	$[\mathbf{1}; \mathbf{0^2}; \mathbf{1^2}; -\mathbf{2}; \mathbf{0}; \mathbf{1}] \times 1$
$[\mathbf{0^2}; \mathbf{1}; -\mathbf{1}, \mathbf{0}; \mathbf{0}; \mathbf{1}; -\mathbf{1}] \times 2$	$[-\mathbf{1}, \mathbf{0^4}; \mathbf{1}; \mathbf{0^2}] \times 5$	$[\mathbf{1}; \mathbf{0^2}; \mathbf{1^2}; -\mathbf{1^2}; \mathbf{1}] \times 1$
$[\mathbf{0}; \mathbf{1^2}; \mathbf{0^2}; -\mathbf{1}; \mathbf{1}; -\mathbf{1}] \times 1$	$[\mathbf{0^8}] \times 1$	$[\mathbf{1^2}; \mathbf{0}; \mathbf{1^2}; -\mathbf{2}; -\mathbf{1}; \mathbf{1}] \times 1$
	$[\mathbf{0^6}; \mathbf{1}; \mathbf{0}] \times 1$	
	$[\mathbf{0^4}, \mathbf{1}; -\mathbf{1}; \mathbf{1}; \mathbf{0}] \times 5$	
	$[\mathbf{0^4}, \mathbf{1}; \mathbf{0^3}] \times 5$	
	$[\mathbf{0^3}, \mathbf{1^2}; -\mathbf{1}; \mathbf{0^2}] \times 10$	
	$[\mathbf{0}, \mathbf{1^4}; -\mathbf{2}; \mathbf{0^2}] \times 5$	
	$[\mathbf{1^5}; -\mathbf{3}; \mathbf{0^2}] \times 1$	
	$[\mathbf{1^5}; -\mathbf{2}; -\mathbf{1}; \mathbf{0}] \times 1$	

- $\mathsf{Spec}(D_2^8) = 3,\ 4,\ 5,\ 7,\ 8,\ 9,\ 12$
- $|Iso(D_2^8, \mathcal{Q}_2^8)| = 80640$
- $l(D_2^8) = 3$
- Centrally-symmetric
- Maximally contained subpolytopes: $35-tope$, $H(3)$, $\frac{1}{2}H(6)$, $J(9,7)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}_3^8[x-\mathbf{c}])$, where $\mathcal{Q}_3^8$ is given by

$$\mathcal{Q}_3^8(\mathbf{x}) = \mathbf{x}^t \begin{pmatrix} 11 & 8 & 8 & 8 & -3 & 4 & -20 & 4 \\ 8 & 11 & 8 & 8 & -3 & 4 & -20 & 4 \\ 8 & 8 & 11 & 8 & -3 & 4 & -20 & 4 \\ 8 & 8 & 8 & 11 & -3 & 4 & -20 & 4 \\ -3 & -3 & -3 & -3 & 3 & -1 & 6 & -1 \\ 4 & 4 & 4 & 4 & -1 & 4 & -10 & 1 \\ -20 & -20 & -20 & -20 & 6 & -10 & 48 & -10 \\ 4 & 4 & 4 & 4 & -1 & 1 & -10 & 4 \end{pmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{92} \begin{pmatrix} 30 \\ 30 \\ 30 \\ 30 \\ 36 \\ -42 \\ 5 \\ -42 \end{pmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}_3^8)| = 96$; $S_4 < O(\mathbb{Z}^8, \mathcal{Q}_3^8)$
- $s(\mathbb{Z}^8, \mathcal{Q}_3^8) = 2$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}_3^8] = 12$

Inequality $\mathcal{Q}_3^8[\mathbf{x} - \mathbf{c}] \leq \frac{189}{46}$ defines the Delaunay ellipsoid for a perfect polytope $D_3^8 \in Del(\mathbb{Z}^8, \mathcal{Q}_3^8)$, whose vertex set ($|\mathsf{vert}\, D_3^8| = 47$) is given below.

$x_5 = -1$	$x_5 = 0$	$x_5 = 1$
$[\mathbf{0^4}; \mathbf{-1}; \mathbf{0^3}] \times 1$	$[\mathbf{-1}, \mathbf{0^3}; \mathbf{0}; \mathbf{-1^3}] \times 4$	$[\mathbf{0^4}; \mathbf{1}; \mathbf{-2}; \mathbf{-1^2}] \times 1$
	$[\mathbf{0^5}; \mathbf{-2}; \mathbf{-1}; \mathbf{-2}] \times 1$	$[\mathbf{0^4}; \mathbf{1}; \mathbf{-1^2}; \mathbf{-2}] \times 1$
	$[\mathbf{0^5}; \mathbf{-2}; \mathbf{-1^2}] \times 1$	$[\mathbf{0^4}; \mathbf{1}; \mathbf{-1^3}] \times 1$
	$[\mathbf{0^5}; \mathbf{-1^2}; \mathbf{-2}] \times 1$	$[\mathbf{0^3}, \mathbf{1}; \mathbf{1}; \mathbf{0^3}] \times 4$
	$[\mathbf{0^8}] \times 1$	$[\mathbf{0^2}, \mathbf{1^2}; \mathbf{1}; \mathbf{-1}; \mathbf{0}; \mathbf{-1}] \times 6$
	$[\mathbf{0^7}; \mathbf{1}] \times 1$	$[\mathbf{0}, \mathbf{1^3}; \mathbf{1}; \mathbf{0}; \mathbf{1}; \mathbf{0}] \times 4$
	$[\mathbf{0^5}; \mathbf{1}; \mathbf{0^2}] \times 1$	$[\mathbf{1^5}; \mathbf{-1}; \mathbf{1}; \mathbf{-1}] \times 1$
	$[\mathbf{0^3}, \mathbf{1}; \mathbf{0}; \mathbf{-1}; \mathbf{0^2}] \times 4$	$[\mathbf{1^5}; \mathbf{-1}; \mathbf{1}; \mathbf{0}] \times 1$
	$[\mathbf{0^3}, \mathbf{1}; \mathbf{0^3}; \mathbf{-1}] \times 4$	$[\mathbf{1^5}; \mathbf{0}; \mathbf{1}; \mathbf{-1}] \times 1$
	$[\mathbf{0^3}, \mathbf{1}; \mathbf{0^4}] \times 4$	
	$[\mathbf{0}, \mathbf{1^3}; \mathbf{0^2}; \mathbf{1}; \mathbf{0}] \times 4$	

- $\mathsf{Spec}(D_3^8) = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$
- $|Iso(D_3^8, \mathcal{Q}_3^8)| = 48$
- $l(D_3^8) = 3$
- Antisymmetric
- Maximally contained subpolytopes: G_6, $H(3)$, $\frac{1}{2}H(5)$, $J(7,5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}_4^8[x-\mathbf{c}])$, where $\mathcal{Q}_4^8$ is given by

$$\mathcal{Q}_4^8(\mathbf{x}) = \mathbf{x}^t \begin{bmatrix} 3 & 1 & 1 & 1 & -2 & 2 & 0 & -2 \\ 1 & 3 & 1 & 1 & -2 & 2 & 0 & -2 \\ 1 & 1 & 3 & 1 & -2 & 2 & 0 & -2 \\ 1 & 1 & 1 & 3 & -2 & 2 & 0 & -2 \\ -2 & -2 & -2 & -2 & 6 & -1 & 3 & 4 \\ 2 & 2 & 2 & 2 & -1 & 6 & 3 & -1 \\ 0 & 0 & 0 & 0 & 3 & 3 & 6 & 3 \\ -2 & -2 & -2 & -2 & 4 & -1 & 3 & 6 \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{12}\begin{bmatrix} 7 \\ 7 \\ 7 \\ 7 \\ 4 \\ -4 \\ 4 \\ 4 \end{bmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}_4^8)| = 1728$; $S_4 < O(\mathbb{Z}^8, \mathcal{Q}_4^8)$
- $s(\mathbb{Z}^8, \mathcal{Q}_4^8) = 8$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}_4^8] = 4$

Inequality $\mathcal{Q}_4^8[\mathbf{x}-\mathbf{c}] \leq \frac{7}{2}$ defines the Delaunay ellipsoid for a perfect polytope $D_4^8 \in Del(\mathbb{Z}^8, \mathcal{Q}_4^8)$, whose vertex set ($|\,\mathsf{vert}\, D_4^8| = 54$) is given below.

$x_5 = -1$	$x_5 = 0$	$x_5 = 1$
$[\mathbf{0^4}; -\mathbf{1}; \mathbf{0}; \mathbf{1}; \mathbf{0}] \times 1$	$[\mathbf{0^8}] \times 1$	$[\mathbf{0^2}, \mathbf{1^2}; \mathbf{1}; \mathbf{0^3}] \times 6$
	$[\mathbf{0^6}; \mathbf{1}; -\mathbf{1}] \times 1$	$[\mathbf{0}, \mathbf{1^3}; \mathbf{1}; \mathbf{0^3}] \times 4$
	$[\mathbf{0^6}; \mathbf{1}; \mathbf{0}] \times 1$	$[\mathbf{1^5}; -\mathbf{1}; \mathbf{0^2}] \times 1$
	$[\mathbf{0^5}; \mathbf{1}; \mathbf{0^2}] \times 1$	$[\mathbf{1^5}; -\mathbf{1}; \mathbf{0}; \mathbf{1}] \times 1$
	$[\mathbf{0^3}, \mathbf{1}; \mathbf{0^4}] \times 4$	$[\mathbf{1^5}; -\mathbf{1}; \mathbf{1}; \mathbf{0}] \times 1$
	$[\mathbf{0^3}, \mathbf{1}; \mathbf{0^2}; \mathbf{1}; \mathbf{0}] \times 4$	$[\mathbf{1^5}; \mathbf{0}; -\mathbf{1}; \mathbf{1}] \times 1$
	$[\mathbf{0^2}, \mathbf{1^2}; \mathbf{0}; -\mathbf{1}; \mathbf{1}; \mathbf{0}] \times 6$	$[\mathbf{1^5}; \mathbf{0^2}; \mathbf{1}] \times 1$
	$[\mathbf{0^2}, \mathbf{1^2}; \mathbf{0^3}; \mathbf{1}] \times 6$	$[\mathbf{1^3}, \mathbf{2}; \mathbf{1}; -\mathbf{1}; \mathbf{0}; \mathbf{1}] \times 4$
	$[\mathbf{0}, \mathbf{1^3}; \mathbf{0}; -\mathbf{1}; \mathbf{1}; \mathbf{0}] \times 4$	
	$[\mathbf{0}, \mathbf{1^3}; \mathbf{0^3}; \mathbf{1}] \times 4$	
	$[\mathbf{1^4}; \mathbf{0}; -\mathbf{1}; \mathbf{0}; \mathbf{1}] \times 1$	
	$[\mathbf{1^4}; \mathbf{0}; -\mathbf{1}; \mathbf{1^2}] \times 1$	

- $\mathsf{Spec}(D_4^8) = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$
- $|Iso(D_4^8, \mathcal{Q}_4^8)| = 864$
- $l(D_4^8) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $H(3)$, $\frac{1}{2}H(5)$, $J(7,5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}_5^8[x-\mathbf{c}])$, where $\mathcal{Q}_5^8$ is given by

$$\mathcal{Q}_5^8(\mathbf{x}) = \mathbf{x}^t \begin{pmatrix} 15 & 11 & 11 & 11 & 11 & 11 & 11 & 11 \\ 11 & 9 & 8 & 8 & 8 & 8 & 8 & 8 \\ 11 & 8 & 9 & 8 & 8 & 8 & 8 & 8 \\ 11 & 8 & 8 & 9 & 8 & 8 & 8 & 8 \\ 11 & 8 & 8 & 8 & 9 & 8 & 8 & 8 \\ 11 & 8 & 8 & 8 & 8 & 9 & 8 & 8 \\ 11 & 8 & 8 & 8 & 8 & 8 & 9 & 8 \\ 11 & 8 & 8 & 8 & 8 & 8 & 8 & 9 \end{pmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{8} \begin{pmatrix} -3 \\ -3 \\ -3 \\ -3 \\ -3 \\ -3 \\ -3 \\ -3 \end{pmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}_5^8)| = 161280$; $S_8 < O(\mathbb{Z}^8, \mathcal{Q}_5^8)$
- $s(\mathbb{Z}^8, \mathcal{Q}_5^8) = 56$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}_5^8] = 2$

Inequality $\mathcal{Q}_5^8[\mathbf{x}-\mathbf{c}] \leq \frac{15}{8}$ defines the Delaunay ellipsoid for a perfect polytope $D_5^8 \in Del(\mathbb{Z}^8, \mathcal{Q}_5^8)$, whose vertex set ($|\,\mathsf{vert}\, D_5^8| = 72$) is given below.

$\sum x_i = -4$	$\sum x_i = -3$	$\sum x_i = -2$
$[\mathbf{2}; -\mathbf{1^6}, \mathbf{0}] \times 7$	$[-\mathbf{1^3}, \mathbf{0^5}] \times 56$	$[-\mathbf{3}; \mathbf{0^6}, \mathbf{1}] \times 7$
$[\mathbf{3}; -\mathbf{1^7}] \times 1$		$[-\mathbf{2}; \mathbf{0^7}] \times 1$

- $\mathsf{Spec}(D_5^8) = 2,\ 3,\ 4,\ 5,\ 6$
- $|Iso(D_5^8, \mathcal{Q}_5^8)| = 80640$
- $l(D_5^8) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $H(3)$, $\frac{1}{2}H(5)$, $J(8,5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}_6^8[x-\mathbf{c}])$, where $\mathcal{Q}_6^8$ is given by

$$\mathcal{Q}_6^8(\mathbf{x}) = \mathbf{x}^t \begin{bmatrix} 8 & 6 & 6 & 6 & 6 & 6 & 9 & 5 \\ 6 & 8 & 6 & 6 & 6 & 6 & 9 & 5 \\ 6 & 6 & 8 & 6 & 6 & 6 & 9 & 5 \\ 6 & 6 & 6 & 8 & 6 & 6 & 9 & 4 \\ 6 & 6 & 6 & 6 & 8 & 6 & 9 & 6 \\ 6 & 6 & 6 & 6 & 6 & 8 & 9 & 5 \\ 9 & 9 & 9 & 9 & 9 & 9 & 13 & 7 \\ 5 & 5 & 5 & 4 & 6 & 5 & 7 & 6 \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{8} \begin{bmatrix} 3 \\ 3 \\ 3 \\ 2 \\ 4 \\ 3 \\ -8 \\ -2 \end{bmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}_6^8)| = 768$; $S_4 < O(\mathbb{Z}^8, \mathcal{Q}_6^8)$
- $s(\mathbb{Z}^8, \mathcal{Q}_6^8) = 2$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}_6^8] = 5$

Inequality $\mathcal{Q}_6^8[\mathbf{x}-\mathbf{c}] \leq \frac{11}{4}$ defines the Delaunay ellipsoid for a perfect polytope $D_6^8 \in Del(\mathbb{Z}^8, \mathcal{Q}_6^8)$, whose vertex set ($|\mathsf{vert}\, D_6^8| = 54$) is given below.

$x_8 = -1$	$x_8 = 0$	$x_8 = 1$
$[\mathbf{0^3}; \mathbf{-1}; \mathbf{1}; \mathbf{0}; \mathbf{1}; \mathbf{-1}] \times 1$	$[\mathbf{-1}, \mathbf{0^5}; \mathbf{1}; \mathbf{0}] \times 6$	$[\mathbf{0^7}; \mathbf{1}] \times 1$
$[\mathbf{0^6}; \mathbf{1}; \mathbf{-1}] \times 1$	$[\mathbf{0^8}] \times 1$	$[\mathbf{0^3}; \mathbf{1}; \mathbf{-1}; \mathbf{0^2}; \mathbf{1}] \times 1$
$[\mathbf{0^4}; \mathbf{1}; \mathbf{0^2}; \mathbf{-1}] \times 1$	$[\mathbf{0^5}, \mathbf{1}; \mathbf{0^2}] \times 6$	$[\mathbf{0^3}; \mathbf{1}; \mathbf{0^2}; \mathbf{-1}; \mathbf{1}] \times 1$
$[\mathbf{0^2}, \mathbf{1}; \mathbf{0}; \mathbf{1^2}; \mathbf{-1^2}] \times 3$	$[\mathbf{0^4}, \mathbf{1^2}; \mathbf{-1}; \mathbf{0}] \times 15$	
$[\mathbf{0}, \mathbf{1^2}; \mathbf{0}; \mathbf{1}; \mathbf{0}; \mathbf{-1^2}] \times 3$	$[\mathbf{0}, \mathbf{1^5}; \mathbf{-3}; \mathbf{0}] \times 6$	
$[\mathbf{0}, \mathbf{1^2}; \mathbf{0}; \mathbf{1^2}; \mathbf{-2}; \mathbf{-1}] \times 3$	$[\mathbf{1^6}; \mathbf{-4}; \mathbf{0}] \times 1$	
$[\mathbf{1^3}; \mathbf{0}; \mathbf{1}; \mathbf{0}; \mathbf{-2}; \mathbf{-1}] \times 1$		
$[\mathbf{1^3}; \mathbf{0}; \mathbf{2}; \mathbf{1}; \mathbf{-3}; \mathbf{-1}] \times 1$		
$[\mathbf{1^6}; \mathbf{-3}; \mathbf{-1}] \times 1$		
$[\mathbf{1^4}; \mathbf{2}; \mathbf{1}; \mathbf{-4}; \mathbf{-1}] \times 1$		

- $\mathsf{Spec}(D_6^8) = 2, 3, 4, 5, 6, 7, 8, 9, 10$
- $|Iso(D_6^8, \mathcal{Q}_6^8)| = 384$
- $l(D_6^8) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $35 - tope$, $H(3)$, $\frac{1}{2}H(5)$, $J(7,5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_7[x-\mathbf{c}])$, where $\mathcal{Q}^8_7$ is given by

$$\mathcal{Q}^8_7(\mathbf{x}) = \mathbf{x}^t \begin{bmatrix} 12 & 9 & 9 & 9 & 9 & -15 & 9 & 8 \\ 9 & 12 & 9 & 9 & 9 & -15 & 9 & 11 \\ 9 & 9 & 12 & 9 & 9 & -15 & 10 & 9 \\ 9 & 9 & 9 & 12 & 9 & -15 & 10 & 9 \\ 9 & 9 & 9 & 9 & 12 & -15 & 10 & 9 \\ -15 & -15 & -15 & -15 & -15 & 24 & -15 & -14 \\ 9 & 9 & 10 & 10 & 10 & -15 & 12 & 8 \\ 8 & 11 & 9 & 9 & 9 & -14 & 8 & 13 \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{66} \begin{bmatrix} 22 \\ 4 \\ 13 \\ 13 \\ 13 \\ 32 \\ 9 \\ 18 \end{bmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_7)| = 72$; $S_3 < O(\mathbb{Z}^8, \mathcal{Q}^8_7)$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_7) = 2$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_7] = 12$

Inequality $\mathcal{Q}^8_7[\mathbf{x}-\mathbf{c}] \leq \frac{91}{22}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_7 \in Del(\mathbb{Z}^8, \mathcal{Q}^8_7)$, whose vertex set ($|\operatorname{vert} D^8_7| = 46$) is given below.

$x_8 = -1$	$x_8 = 0$	$x_8 = 1$
$[\mathbf{0}; \mathbf{1^5}; -\mathbf{1^2}] \times 1$	$[-\mathbf{1}, \mathbf{0^4}; -\mathbf{1}; \mathbf{0^2}] \times 5$	$[\mathbf{0}; -\mathbf{1}; -\mathbf{1^2}, \mathbf{0}; -\mathbf{1}; \mathbf{1^2}] \times 3$
	$[\mathbf{0^5}; -\mathbf{1^2}; \mathbf{0}] \times 1$	$[\mathbf{0}; -\mathbf{1}; -\mathbf{1}, \mathbf{0^2}; -\mathbf{1}; \mathbf{0}; \mathbf{1}] \times 3$
	$[\mathbf{0^8}] \times 1$	$[\mathbf{0}; -\mathbf{1}; \mathbf{0^4}; \mathbf{1^2}] \times 1$
	$[\mathbf{0^6}; \mathbf{1}; \mathbf{0}] \times 1$	$[\mathbf{0^7}; \mathbf{1}] \times 1$
	$[\mathbf{0^4}, \mathbf{1}; \mathbf{0^3}] \times 5$	$[\mathbf{0^5}; \mathbf{1^3}] \times 1$
	$[\mathbf{0^3}, \mathbf{1^2}; \mathbf{1}; \mathbf{0^2}] \times 10$	$[\mathbf{1}; -\mathbf{1}; \mathbf{0^5}; \mathbf{1}] \times 1$
	$[\mathbf{0}, \mathbf{1^4}; \mathbf{2}; \mathbf{0^2}] \times 5$	$[\mathbf{1}; -\mathbf{1}; \mathbf{0^3}; \mathbf{1^3}] \times 1$
	$[\mathbf{1^5}; \mathbf{2}; -\mathbf{1}; \mathbf{0}] \times 1$	$[\mathbf{1}; \mathbf{0^4}; \mathbf{1}; \mathbf{0}; \mathbf{1}] \times 1$
	$[\mathbf{1^5}; \mathbf{3}; \mathbf{0^2}] \times 1$	
	$[\mathbf{0^2}; \mathbf{1^4}; -\mathbf{1}; \mathbf{0}] \times 1$	
	$[\mathbf{0}; \mathbf{1}; \mathbf{0^3}; \mathbf{1^2}; \mathbf{0}] \times 1$	
	$[\mathbf{1}; \mathbf{0^4}; \mathbf{1^2}; \mathbf{0}] \times 1$	

- $\mathsf{Spec}(D^8_7) = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15$
- $|Iso(D^8_7, \mathcal{Q}^8_7)| = 36$
- $l(D^8_7) = 3$
- Antisymmetric
- Maximally contained subpolytopes: G_6, $H(3)$, $\frac{1}{2}H(5)$, $J(7,5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}_8^8[x-\mathbf{c}])$, where $\mathcal{Q}_8^8$ is given by

$$\mathcal{Q}_8^8(\mathbf{x}) = \mathbf{x}^t \begin{array}{|c|c|c|c|c|c|c|c|} \hline 8 & 6 & 6 & 6 & 6 & 6 & 9 & 5 \\ \hline 6 & 8 & 6 & 6 & 6 & 6 & 9 & 6 \\ \hline 6 & 6 & 8 & 6 & 6 & 6 & 9 & 6 \\ \hline 6 & 6 & 6 & 8 & 6 & 6 & 9 & 5 \\ \hline 6 & 6 & 6 & 6 & 8 & 6 & 9 & 5 \\ \hline 6 & 6 & 6 & 6 & 6 & 8 & 9 & 7 \\ \hline 9 & 9 & 9 & 9 & 9 & 9 & 13 & 8 \\ \hline 5 & 6 & 6 & 5 & 5 & 7 & 8 & 8 \\ \hline \end{array} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{20} \begin{array}{|r|} \hline 7 \\ \hline 4 \\ \hline 4 \\ \hline 7 \\ \hline 7 \\ \hline 1 \\ \hline -16 \\ \hline 6 \\ \hline \end{array} .$$

- $|O(\mathbb{Z}^8, \mathcal{Q}_8^8)| = 384$; $S_4 < O(\mathbb{Z}^8, \mathcal{Q}_8^8)$
- $s(\mathbb{Z}^8, \mathcal{Q}_8^8) = 2$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}_8^8] = 7$

Inequality $\mathcal{Q}_8^8[\mathbf{x} - \mathbf{c}] \leq \frac{14}{5}$ defines the Delaunay ellipsoid for a perfect polytope $D_8^8 \in Del(\mathbb{Z}^8, \mathcal{Q}_8^8)$, whose vertex set ($|\mathsf{vert}\, D_8^8| = 52$) is given below.

$x_8 = -1$	$x_8 = 0$	$x_8 = 1$
$[\mathbf{0}; \mathbf{1^2}; \mathbf{0^2}; \mathbf{1}; -\mathbf{1^2}] \times 1$	$[-\mathbf{1}, \mathbf{0^5}; \mathbf{1}; \mathbf{0}] \times 6$	$[\mathbf{0}; -\mathbf{1^2}; \mathbf{0^2}; -\mathbf{1}; \mathbf{2}; \mathbf{1}] \times 1$
	$[\mathbf{0^8}] \times 1$	$[\mathbf{0}; -\mathbf{1}; \mathbf{0^3}; -\mathbf{1}; \mathbf{1^2}] \times 1$
	$[\mathbf{0^5}, \mathbf{1}; \mathbf{0^2}] \times 6$	$[\mathbf{0^2}; -\mathbf{1}; \mathbf{0^2}; -\mathbf{1}; \mathbf{1^2}] \times 1$
	$[\mathbf{0^4}, \mathbf{1^2}; -\mathbf{1}; \mathbf{0}] \times 15$	$[\mathbf{0^7}; \mathbf{1}] \times 1$
	$[\mathbf{0}, \mathbf{1^5}; -\mathbf{3}; \mathbf{0}] \times 6$	$[\mathbf{0^3}; \mathbf{0}, \mathbf{1}; -\mathbf{1}; \mathbf{0}; \mathbf{1}] \times 2$
	$[\mathbf{1^6}; -\mathbf{4}; \mathbf{0}] \times 1$	$[\mathbf{0^3}; \mathbf{0}, \mathbf{1}; \mathbf{0}; -\mathbf{1}; \mathbf{1}] \times 2$
		$[\mathbf{0^3}; \mathbf{1^2}; -\mathbf{1^2}; \mathbf{1}] \times 1$
		$[\mathbf{1}; \mathbf{0^4}; -\mathbf{1}; \mathbf{0}; \mathbf{1}] \times 1$
		$[\mathbf{1}; \mathbf{0^5}; -\mathbf{1}; \mathbf{1}] \times 1$
		$[\mathbf{1}; \mathbf{0^2}; \mathbf{0}, \mathbf{1}; -\mathbf{1^2}; \mathbf{1}] \times 2$
		$[\mathbf{1}; \mathbf{0^2}; \mathbf{1^2}; \mathbf{0}; -\mathbf{2}; \mathbf{1}] \times 1$
		$[\mathbf{1}; \mathbf{0}; \mathbf{1^3}; \mathbf{0}; -\mathbf{3}; \mathbf{1}] \times 1$
		$[\mathbf{1^2}; \mathbf{0}; \mathbf{1^2}; \mathbf{0}; -\mathbf{3}; \mathbf{1}] \times 1$

- $\mathsf{Spec}(D_8^8) = 2, 3, 4, 5, 6, 7, 8, 9, 10$
- $|Iso(D_8^8, \mathcal{Q}_8^8)| = 192$
- $l(D_8^8) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $35 - tope$, $H(3)$, $\frac{1}{2}H(5)$, $J(7,5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}_9^8[x - \mathbf{c}])$, where $\mathcal{Q}_9^8$ is given by

$$\mathcal{Q}_9^8(\mathbf{x}) = \mathbf{x}^t \begin{bmatrix} 6 & 4 & 4 & 4 & 4 & 4 & -5 & 6 \\ 4 & 6 & 4 & 4 & 4 & 4 & -5 & 6 \\ 4 & 4 & 6 & 4 & 4 & 4 & -5 & 6 \\ 4 & 4 & 4 & 6 & 4 & 4 & -5 & 6 \\ 4 & 4 & 4 & 4 & 6 & 4 & -5 & 6 \\ 4 & 4 & 4 & 4 & 4 & 6 & -5 & 6 \\ -5 & -5 & -5 & -5 & -5 & -5 & 9 & -6 \\ 6 & 6 & 6 & 6 & 6 & 6 & -6 & 9 \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{8} \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 6 \\ -4 \end{bmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}_9^8)| = 2880$; $S_6 < O(\mathbb{Z}^8, \mathcal{Q}_9^8)$
- $s(\mathbb{Z}^8, \mathcal{Q}_9^8) = 12$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}_9^8] = 5$

Inequality $\mathcal{Q}_9^8[\mathbf{x} - \mathbf{c}] \leq \frac{27}{8}$ defines the Delaunay ellipsoid for a perfect polytope $D_9^8 \in Del(\mathbb{Z}^8, \mathcal{Q}_9^8)$, whose vertex set ($|\mathsf{vert}\, D_9^8| = 58$) is given below.

$x_7 = 0$	$x_7 = 1$	$x_7 = 2$
$[\mathbf{-1}, \mathbf{0^5}; \mathbf{0}; \mathbf{1}] \times 6$	$[\mathbf{0^6}; \mathbf{1^2}] \times 1$	$[\mathbf{1^6}; \mathbf{2}; \mathbf{-2}] \times 1$
$[\mathbf{0^8}] \times 1$	$[\mathbf{0^4}, \mathbf{1^2}; \mathbf{1}; \mathbf{0}] \times 15$	
$[\mathbf{0^7}; \mathbf{1}] \times 1$	$[\mathbf{0^3}, \mathbf{1^3}; \mathbf{1}; \mathbf{-1}] \times 20$	
$[\mathbf{0^5}, \mathbf{1}; \mathbf{0^2}] \times 6$	$[\mathbf{0}, \mathbf{1^5}; \mathbf{1}; \mathbf{-2}] \times 6$	
	$[\mathbf{1^7}; \mathbf{-3}] \times 1$	

- $\mathsf{Spec}(D_9^8) = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$
- $|Iso(D_9^8, \mathcal{Q}_9^8)| = 1440$
- $l(D_9^8) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $H(3)$, $\frac{1}{2}H(5)$, $J(7, 5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{10}[x-\mathbf{c}])$, where $\mathcal{Q}^8_{10}$ is given by

$$\mathcal{Q}^8_{10}(\mathbf{x}) = \mathbf{x}^t \begin{array}{|c|c|c|c|c|c|c|c|}\hline 8&6&6&6&6&6&9&4\\\hline 6&8&6&6&6&6&9&4\\\hline 6&6&8&6&6&6&9&3\\\hline 6&6&6&8&6&6&9&3\\\hline 6&6&6&6&8&6&9&2\\\hline 6&6&6&6&6&8&9&3\\\hline 9&9&9&9&9&9&13&4\\\hline 4&4&3&3&2&3&4&5\\\hline\end{array}\ \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{26}\begin{array}{|r|}\hline 6\\\hline 6\\\hline 7\\\hline 7\\\hline 8\\\hline 7\\\hline -18\\\hline 2\\\hline\end{array}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{10})| = 576$; $S_3 < O(\mathbb{Z}^8, \mathcal{Q}^8_{10})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{10}) = 4$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{10}] = 4$

Inequality $\mathcal{Q}^8_{10}[\mathbf{x}-\mathbf{c}] \leq \frac{35}{13}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{10} \in Del(\mathbb{Z}^8, \mathcal{Q}^8_{10})$, whose vertex set ($|\mathsf{vert}\, D^8_{10}| = 55$) is given below.

$x_8 = -1$	$x_8 = 0$	$x_8 = 1$
$[\mathbf{1^2}; \mathbf{0}; \mathbf{1}; \mathbf{0}; \mathbf{1}; \mathbf{-2}; \mathbf{-1}] \times 1$	$[\mathbf{-1}, \mathbf{0^5}; \mathbf{1}; \mathbf{0}] \times 6$	$[\mathbf{-1^4}; \mathbf{0^2}; \mathbf{3}; \mathbf{1}] \times 1$
$[\mathbf{1^3}; \mathbf{0^2}; \mathbf{1}; \mathbf{-2}; \mathbf{-1}] \times 1$	$[\mathbf{0^8}] \times 1$	$[\mathbf{-1^3}; \mathbf{0^2}; \mathbf{-1}; \mathbf{3}; \mathbf{1}] \times 1$
$[\mathbf{1^4}; \mathbf{0^2}; \mathbf{-2}; \mathbf{-1}] \times 1$	$[\mathbf{0^5}, \mathbf{1}; \mathbf{0^2}] \times 6$	$[\mathbf{-1^3}; \mathbf{0^3}; \mathbf{2}; \mathbf{1}] \times 1$
$[\mathbf{1^4}; \mathbf{0}; \mathbf{1}; \mathbf{-3}; \mathbf{-1}] \times 1$	$[\mathbf{0^4}, \mathbf{1^2}; \mathbf{-1}; \mathbf{0}] \times 15$	$[\mathbf{-1^2}; \mathbf{0}; \mathbf{-1}; \mathbf{0}; \mathbf{-1}; \mathbf{3}; \mathbf{1}] \times 1$
$[\mathbf{1}, \mathbf{2}; \mathbf{1^4}; \mathbf{-4}; \mathbf{-1}] \times 2$	$[\mathbf{0}, \mathbf{1^5}; \mathbf{-3}; \mathbf{0}] \times 6$	$[\mathbf{-1^2}; \mathbf{0}; \mathbf{-1}; \mathbf{0^2}; \mathbf{2}; \mathbf{1}] \times 1$
$[\mathbf{2^2}; \mathbf{1^2}; \mathbf{0}; \mathbf{1}; \mathbf{-4}; \mathbf{-1}] \times 1$	$[\mathbf{1^6}; \mathbf{-4}; \mathbf{0}] \times 1$	$[\mathbf{-1^2}; \mathbf{0^3}; \mathbf{-1}; \mathbf{2}; \mathbf{1}] \times 1$
$[\mathbf{2^2}; \mathbf{1^4}; \mathbf{-5}; \mathbf{-1}] \times 1$		$[\mathbf{-1^2}; \mathbf{0^2}; \mathbf{1}; \mathbf{0}; \mathbf{1^2}] \times 1$
		$[\mathbf{-1}, \mathbf{0}; \mathbf{0^4}; \mathbf{1^2}] \times 2$
		$[\mathbf{-1}, \mathbf{0}; \mathbf{0^2}; \mathbf{1}; \mathbf{0^2}; \mathbf{1}] \times 2$
		$[\mathbf{0^7}; \mathbf{1}] \times 1$

- $\mathsf{Spec}(D^8_{10}) = 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9$
- $|Iso(D^8_{10}, \mathcal{Q}^8_{10})| = 288$
- $l(D^8_{10}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $35 - tope$, $H(3)$, $\frac{1}{2}H(5)$, $J(7,5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{11}[x-\mathbf{c}])$, where $\mathcal{Q}^8_{11}$ is given by

$$\mathcal{Q}^8_{11}(\mathbf{x}) = \mathbf{x}^t \begin{array}{|c|c|c|c|c|c|c|c|}\hline 24 & 18 & 18 & 18 & 18 & 18 & 27 & 15 \\ \hline 18 & 24 & 18 & 18 & 18 & 18 & 27 & 15 \\ \hline 18 & 18 & 24 & 18 & 18 & 18 & 27 & 15 \\ \hline 18 & 18 & 18 & 24 & 18 & 18 & 27 & 15 \\ \hline 18 & 18 & 18 & 18 & 24 & 18 & 27 & 15 \\ \hline 18 & 18 & 18 & 18 & 18 & 24 & 27 & 11 \\ \hline 27 & 27 & 27 & 27 & 27 & 27 & 39 & 20 \\ \hline 15 & 15 & 15 & 15 & 15 & 11 & 20 & 17 \\ \hline \end{array} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{90} \begin{array}{|r|}\hline 36 \\ \hline 36 \\ \hline 36 \\ \hline 36 \\ \hline 36 \\ \hline 22 \\ \hline -91 \\ \hline -21 \\ \hline \end{array}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{11})| = 480$; $S_5 < O(\mathbb{Z}^8, \mathcal{Q}^8_{11})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{11}) = 4$
- $\mathsf{dim\,QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{11}] = 8$

Inequality $\mathcal{Q}^8_{11}[\mathbf{x}-\mathbf{c}] \leq \frac{124}{15}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{11} \in Del(\mathbb{Z}^8, \mathcal{Q}^8_{11})$, whose vertex set ($|\mathsf{vert}\, D^8_{11}| = 44$) is given below.

$x_8 = -1$	$x_8 = 0$	$x_8 = 1$
$[\mathbf{0^6}; \mathbf{1}; -\mathbf{1}] \times 1$	$[-\mathbf{1}, \mathbf{0^5}; \mathbf{1}; \mathbf{0}] \times 6$	$[\mathbf{0^7}; \mathbf{1}] \times 1$
$[\mathbf{0}, \mathbf{1^4}; \mathbf{0}; -\mathbf{2}; -\mathbf{1}] \times 5$	$[\mathbf{0^8}] \times 1$	$[\mathbf{0^5}; \mathbf{1}; -\mathbf{1}; \mathbf{1}] \times 1$
$[\mathbf{1^6}; -\mathbf{3}; -\mathbf{1}] \times 1$	$[\mathbf{0^5}, \mathbf{1}; \mathbf{0^2}] \times 6$	
	$[\mathbf{0^4}, \mathbf{1^2}; -\mathbf{1}; \mathbf{0}] \times 15$	
	$[\mathbf{0}, \mathbf{1^5}; -\mathbf{3}; \mathbf{0}] \times 6$	
	$[\mathbf{1^6}; -\mathbf{4}; \mathbf{0}] \times 1$	

- $\mathsf{Spec}(D^8_{11}) = 8, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 23, 24, 27, 28, 31$
- $|Iso(D^8_{11}, \mathcal{Q}^8_{11})| = 240$
- $l(D^8_{11}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $35-tope$, $H(2)$, $\frac{1}{2}H(5)$, $J(7,5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{12}[x-\mathbf{c}])$, where $\mathcal{Q}^8_{12}$ is given by

$$\mathcal{Q}^8_{12}(\mathbf{x}) = \mathbf{x}^t \begin{pmatrix} 12 & 9 & 9 & 9 & 9 & -15 & 7 & 6 \\ 9 & 12 & 9 & 9 & 9 & -15 & 10 & 6 \\ 9 & 9 & 12 & 9 & 9 & -15 & 8 & 5 \\ 9 & 9 & 9 & 12 & 9 & -15 & 8 & 5 \\ 9 & 9 & 9 & 9 & 12 & -15 & 8 & 5 \\ -15 & -15 & -15 & -15 & -15 & 24 & -12 & -9 \\ 7 & 10 & 8 & 8 & 8 & -12 & 12 & 5 \\ 6 & 6 & 5 & 5 & 5 & -9 & 5 & 6 \end{pmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{31} \begin{pmatrix} 9 \\ 3 \\ 6 \\ 6 \\ 6 \\ 9 \\ 6 \\ -3 \end{pmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{12})| = 96$; $S_3 < O(\mathbb{Z}^8, \mathcal{Q}^8_{12})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{12}) = 14$
- $\mathsf{dim\,QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{12}] = 8$

Inequality $\mathcal{Q}^8_{12}[\mathbf{x}-\mathbf{c}] \le \frac{126}{31}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{12} \in Del(\mathbb{Z}^8, \mathcal{Q}^8_{12})$, whose vertex set ($|\mathsf{vert}\, D^8_{12}| = 45$) is given below.

$x_8 = -1$	$x_8 = 0$	$x_8 = 1$
$[\mathbf{0}; -\mathbf{1^4}; -\mathbf{3}; \mathbf{1}; -\mathbf{1}] \times 1$	$[-\mathbf{1}, \mathbf{0^4}; -\mathbf{1}; \mathbf{0^2}] \times 5$	$[\mathbf{0^7}; \mathbf{1}] \times 1$
$[\mathbf{0}; -\mathbf{1}; -\mathbf{1^2}, \mathbf{0}; -\mathbf{2}; \mathbf{1}; -\mathbf{1}] \times 3$	$[\mathbf{0^8}] \times 1$	$[\mathbf{0^2}; \mathbf{1^3}; \mathbf{2}; \mathbf{0}; \mathbf{1}] \times 1$
$[\mathbf{0}; -\mathbf{1}; \mathbf{0^3}; -\mathbf{1}; \mathbf{1}; -\mathbf{1}] \times 1$	$[\mathbf{0^6}; \mathbf{1}; \mathbf{0}] \times 1$	$[\mathbf{0}; \mathbf{1^4}; \mathbf{2}; -\mathbf{1}; \mathbf{1}] \times 1$
$[\mathbf{0^5}; -\mathbf{1}; \mathbf{0}; -\mathbf{1}] \times 1$	$[\mathbf{0^4}, \mathbf{1}; \mathbf{0^3}] \times 5$	$[\mathbf{1^5}; \mathbf{3}; \mathbf{0}; \mathbf{1}] \times 1$
$[\mathbf{0}; \mathbf{1}; \mathbf{0^5}; -\mathbf{1}] \times 1$	$[\mathbf{0^3}, \mathbf{1^2}; \mathbf{1}; \mathbf{0^2}] \times 10$	
$[\mathbf{1}; \mathbf{0^6}; -\mathbf{1}] \times 1$	$[\mathbf{0}, \mathbf{1^4}; \mathbf{2}; \mathbf{0^2}] \times 5$	
	$[\mathbf{1^5}; \mathbf{3}; \mathbf{0^2}] \times 1$	
	$[\mathbf{0}; -\mathbf{1^2}; \mathbf{0^2}; -\mathbf{1}; \mathbf{1}; \mathbf{0}] \times 1$	
	$[\mathbf{0}; -\mathbf{1}; \mathbf{0}; -\mathbf{1}; \mathbf{0}; -\mathbf{1}; \mathbf{1}; \mathbf{0}] \times 1$	
	$[\mathbf{0}; -\mathbf{1}; \mathbf{0^2}; -\mathbf{1^2}; \mathbf{1}; \mathbf{0}] \times 1$	
	$[\mathbf{1}; -\mathbf{1}; \mathbf{0^4}; \mathbf{1}; \mathbf{0}] \times 1$	
	$[\mathbf{1}; \mathbf{2}; \mathbf{1^3}; \mathbf{3}; -\mathbf{1}; \mathbf{0}] \times 1$	

- $\mathsf{Spec}(D^8_{12}) = 2, 3, 4, 5, 6, 7$
- $|Iso(D^8_{12}, \mathcal{Q}^8_{12})| = 48$
- $l(D^8_{12}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: G_6, $H(2)$, $\frac{1}{2}H(5)$, $J(6,4)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{13}[x-\mathbf{c}])$, where $\mathcal{Q}^8_{13}$ is given by

$$\mathcal{Q}^8_{13}(\mathbf{x}) = \mathbf{x}^t \begin{array}{|c|c|c|c|c|c|c|c|}\hline 24 & 18 & 18 & 18 & 18 & 18 & 27 & 6 \\ \hline 18 & 24 & 18 & 18 & 18 & 18 & 27 & 10 \\ \hline 18 & 18 & 24 & 18 & 18 & 18 & 27 & 10 \\ \hline 18 & 18 & 18 & 24 & 18 & 18 & 27 & 10 \\ \hline 18 & 18 & 18 & 18 & 24 & 18 & 27 & 10 \\ \hline 18 & 18 & 18 & 18 & 18 & 24 & 27 & 12 \\ \hline 27 & 27 & 27 & 27 & 27 & 27 & 39 & 12 \\ \hline 6 & 10 & 10 & 10 & 10 & 12 & 12 & 16 \\ \hline \end{array} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{74} \begin{array}{|r|}\hline 22 \\ \hline 16 \\ \hline 16 \\ \hline 16 \\ \hline 16 \\ \hline 13 \\ \hline -40 \\ \hline 9 \\ \hline \end{array}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{13})| = 96$; $S_4 < O(\mathbb{Z}^8, \mathcal{Q}^8_{13})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{13}) = 2$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{13}] = 12$

Inequality $\mathcal{Q}^8_{13}[\mathbf{x}-\mathbf{c}] \leq \frac{300}{37}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{13} \in Del(\mathbb{Z}^8, \mathcal{Q}^8_{13})$, whose vertex set ($|\operatorname{vert} D^8_{13}| = 44$) is given below.

$x_8 = -1$	$x_8 = 0$	$x_8 = 1$
$[\mathbf{0}; \mathbf{1^5}; \mathbf{-3}; \mathbf{-1}] \times 1$	$[\mathbf{-1}, \mathbf{0^5}; \mathbf{1}; \mathbf{0}] \times 6$	$[\mathbf{0}; \mathbf{-1^3}, \mathbf{0}; \mathbf{-1}; \mathbf{3}; \mathbf{1}] \times 4$
$[\mathbf{1^5}; \mathbf{2}; \mathbf{-4}; \mathbf{-1}] \times 1$	$[\mathbf{0^8}] \times 1$	$[\mathbf{0^5}; \mathbf{-1}; \mathbf{1^2}] \times 1$
	$[\mathbf{0^5}, \mathbf{1}; \mathbf{0^2}] \times 6$	$[\mathbf{0^7}; \mathbf{1}] \times 1$
	$[\mathbf{0^4}, \mathbf{1^2}; \mathbf{-1}; \mathbf{0}] \times 15$	$[\mathbf{1}; \mathbf{0^4}; \mathbf{-1}; \mathbf{0}; \mathbf{1}] \times 1$
	$[\mathbf{0}, \mathbf{1^5}; \mathbf{-3}; \mathbf{0}] \times 6$	
	$[\mathbf{1^6}; \mathbf{-4}; \mathbf{0}] \times 1$	

- $\mathsf{Spec}(D^8_{13}) = 5, 8, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 23, 24, 25, 27, 28$
- $|Iso(D^8_{13}, \mathcal{Q}^8_{13})| = 48$
- $l(D^8_{13}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $35-tope$, $H(2)$, $\frac{1}{2}H(5)$, $J(7,5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{14}[x-\mathbf{c}])$, where $\mathcal{Q}^8_{14}$ is given by

$$\mathcal{Q}^8_{14}(\mathbf{x}) = \mathbf{x}^t \begin{bmatrix} 9 & 5 & 5 & 5 & -15 & 0 & 3 & -12 \\ 5 & 9 & 5 & 5 & -15 & 0 & 3 & -12 \\ 5 & 5 & 9 & 5 & -15 & 0 & 3 & -12 \\ 5 & 5 & 5 & 9 & -15 & 0 & 3 & -12 \\ -15 & -15 & -15 & -15 & 42 & 3 & -9 & 33 \\ 0 & 0 & 0 & 0 & 3 & 6 & 0 & 3 \\ 3 & 3 & 3 & 3 & -9 & 0 & 4 & -8 \\ -12 & -12 & -12 & -12 & 33 & 3 & -8 & 28 \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{16} \begin{bmatrix} 7 \\ 7 \\ 7 \\ 7 \\ 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{14})| = 576$; $S_4 < O(\mathbb{Z}^8, \mathcal{Q}^8_{14})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{14}) = 6$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{14}] = 6$

Inequality $\mathcal{Q}^8_{14}[\mathbf{x} - \mathbf{c}] \leq \frac{41}{8}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{14} \in Del(\mathbb{Z}^8, \mathcal{Q}^8_{14})$, whose vertex set ($|\operatorname{vert} D^8_{14}| = 46$) is given below.

$x_6 = -1$	$x_6 = 0$	$x_6 = 1$
$[\mathbf{1^4}; \mathbf{0}; -\mathbf{1}; \mathbf{1}; \mathbf{2}] \times 1$	$[\mathbf{0^4}; -\mathbf{1}; \mathbf{0^2}; \mathbf{1}] \times 1$	$[-\mathbf{1}, \mathbf{0^3}; \mathbf{0}; \mathbf{1}; -\mathbf{1^2}] \times 4$
$[\mathbf{1^4}; \mathbf{0}; -\mathbf{1}; \mathbf{2^2}] \times 1$	$[\mathbf{0^8}] \times 1$	$[\mathbf{0^4}; -\mathbf{1}; \mathbf{1}; \mathbf{0}; \mathbf{1}] \times 1$
$[\mathbf{1^5}; -\mathbf{1}; \mathbf{2}; \mathbf{1}] \times 1$	$[\mathbf{0^6}; \mathbf{1}; \mathbf{0}] \times 1$	$[\mathbf{0^5}; \mathbf{1}; -\mathbf{2}; -\mathbf{1}] \times 1$
	$[\mathbf{0^3}, \mathbf{1}; \mathbf{0^4}] \times 4$	$[\mathbf{0^5}; \mathbf{1}; -\mathbf{1^2}] \times 1$
	$[\mathbf{0^2}, \mathbf{1^2}; \mathbf{0^2}; \mathbf{1^2}] \times 6$	$[\mathbf{0^5}; \mathbf{1}; \mathbf{0^2}] \times 1$
	$[\mathbf{0}, \mathbf{1^3}; \mathbf{0^3}; \mathbf{1}] \times 4$	$[\mathbf{0^5}; \mathbf{1^2}; \mathbf{0}] \times 1$
	$[\mathbf{0}, \mathbf{1^3}; \mathbf{1}; \mathbf{0^3}] \times 4$	$[\mathbf{0^4}; \mathbf{1^2}; -\mathbf{1}; -\mathbf{2}] \times 1$
	$[\mathbf{0}, \mathbf{1^3}; \mathbf{1}; \mathbf{0}; \mathbf{1}; \mathbf{0}] \times 4$	$[\mathbf{0^3}, \mathbf{1}; \mathbf{0}; \mathbf{1}; \mathbf{0^2}] \times 4$
	$[\mathbf{1^4}; \mathbf{0^2}; \mathbf{1}; \mathbf{2}] \times 1$	
	$[\mathbf{1^4}; \mathbf{0^2}; \mathbf{2^2}] \times 1$	
	$[\mathbf{1^5}; \mathbf{0^3}] \times 1$	
	$[\mathbf{1^5}; \mathbf{0}; \mathbf{2}; \mathbf{1}] \times 1$	

- $\mathsf{Spec}(D^8_{14}) = 4, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 18$
- $|Iso(D^8_{14}, \mathcal{Q}^8_{14})| = 288$
- $l(D^8_{14}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $H(3)$, $\frac{1}{2}H(5)$, $J(7, 5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{15}[x - \mathbf{c}])$, where $\mathcal{Q}^8_{15}$ is given by

$$\mathcal{Q}^8_{15}(\mathbf{x}) = \mathbf{x}^t \begin{array}{|c|c|c|c|c|c|c|c|}\hline 8&6&6&6&6&6&9&2\\\hline 6&8&6&6&6&6&9&3\\\hline 6&6&8&6&6&6&9&3\\\hline 6&6&6&8&6&6&9&3\\\hline 6&6&6&6&8&6&9&3\\\hline 6&6&6&6&6&8&9&4\\\hline 9&9&9&9&9&9&13&4\\\hline 2&3&3&3&3&4&4&4\\\hline\end{array} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{20}\begin{array}{|r|}\hline 6\\\hline 7\\\hline 7\\\hline 7\\\hline 7\\\hline 8\\\hline -20\\\hline -2\\\hline\end{array}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{15})| = 384$; $S_4 < O(\mathbb{Z}^8, \mathcal{Q}^8_{15})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{15}) = 24$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{15}] = 6$

Inequality $\mathcal{Q}^8_{15}[\mathbf{x} - \mathbf{c}] \le \frac{27}{10}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{15} \in Del(\mathbb{Z}^8, \mathcal{Q}^8_{15})$, whose vertex set ($|\mathsf{vert}\, D^8_{15}| = 45$) is given below.

$x_8 = -1$	$x_8 = 0$	$x_8 = 1$
$[\mathbf{0^5}; \mathbf{1}; \mathbf{0}; \mathbf{-1}] \times 1$	$[\mathbf{-1}, \mathbf{0^5}; \mathbf{1}; \mathbf{0}] \times 6$	$[\mathbf{0^5}; \mathbf{-1}; \mathbf{1^2}] \times 1$
$[\mathbf{0}; \mathbf{0}, \mathbf{1^3}; \mathbf{1}; \mathbf{-2}; \mathbf{-1}] \times 4$	$[\mathbf{0^8}] \times 1$	$[\mathbf{0^7}; \mathbf{1}] \times 1$
$[\mathbf{0}; \mathbf{1^5}; \mathbf{-3}; \mathbf{-1}] \times 1$	$[\mathbf{0^5}, \mathbf{1}; \mathbf{0^2}] \times 6$	$[\mathbf{1}; \mathbf{0^4}; \mathbf{-1}; \mathbf{0}; \mathbf{1}] \times 1$
$[\mathbf{1^5}; \mathbf{2}; \mathbf{-4}; \mathbf{-1}] \times 1$	$[\mathbf{0^4}, \mathbf{1^2}; \mathbf{-1}; \mathbf{0}] \times 15$	
	$[\mathbf{0}, \mathbf{1^5}; \mathbf{-3}; \mathbf{0}] \times 6$	
	$[\mathbf{1^6}; \mathbf{-4}; \mathbf{0}] \times 1$	

- $\mathsf{Spec}(D^8_{15}) = 3, 4, 5, 6, 7, 8, 9$
- $|Iso(D^8_{15}, \mathcal{Q}^8_{15})| = 192$
- $l(D^8_{15}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $35 - tope$, $H(2)$, $\frac{1}{2}H(5)$, $J(7, 5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{16}[x - \mathbf{c}])$, where $\mathcal{Q}^8_{16}$ is given by

$$\mathcal{Q}^8_{16}(\mathbf{x}) = \mathbf{x}^t \begin{bmatrix} 8 & 6 & 6 & 6 & 6 & 6 & 9 & 6 \\ 6 & 8 & 6 & 6 & 6 & 6 & 9 & 5 \\ 6 & 6 & 8 & 6 & 6 & 6 & 9 & 5 \\ 6 & 6 & 6 & 8 & 6 & 6 & 9 & 6 \\ 6 & 6 & 6 & 6 & 8 & 6 & 9 & 6 \\ 6 & 6 & 6 & 6 & 6 & 8 & 9 & 5 \\ 9 & 9 & 9 & 9 & 9 & 9 & 13 & 8 \\ 6 & 5 & 5 & 6 & 6 & 5 & 8 & 7 \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{14} \begin{bmatrix} 4 \\ 5 \\ 5 \\ 4 \\ 4 \\ 5 \\ -14 \\ 2 \end{bmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{16})| = 288$; $S_3 < O(\mathbb{Z}^8, \mathcal{Q}^8_{16})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{16}) = 24$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{16}] = 5$

Inequality $\mathcal{Q}^8_{16}[\mathbf{x} - \mathbf{c}] \leq \frac{19}{7}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{16} \in Del(\mathbb{Z}^8, \mathcal{Q}^8_{16})$, whose vertex set ($|\mathsf{vert}\, D^8_{16}| = 45$) is given below.

$x_8 = -1$	$x_8 = 0$	$x_8 = 1$
$[\mathbf{0^6}; \mathbf{1}; -\mathbf{1}] \times 1$	$[-\mathbf{1}, \mathbf{0^5}; \mathbf{1}; \mathbf{0}] \times 6$	$[\mathbf{0^7}; \mathbf{1}] \times 1$
$[\mathbf{1}; \mathbf{0^2}; \mathbf{1^2}; \mathbf{0}; -\mathbf{1^2}] \times 1$	$[\mathbf{0^8}] \times 1$	$[\mathbf{0^5}; \mathbf{1}; -\mathbf{1}; \mathbf{1}] \times 1$
	$[\mathbf{0^5}, \mathbf{1}; \mathbf{0^2}] \times 6$	$[\mathbf{0}; \mathbf{0}, \mathbf{1}; \mathbf{0^3}; -\mathbf{1}; \mathbf{1}] \times 2$
	$[\mathbf{0^4}, \mathbf{1^2}; -\mathbf{1}; \mathbf{0}] \times 15$	$[\mathbf{0}; \mathbf{1^2}; \mathbf{0^2}; \mathbf{1}; -\mathbf{2}; \mathbf{1}] \times 1$
	$[\mathbf{0}, \mathbf{1^5}; -\mathbf{3}; \mathbf{0}] \times 6$	$[\mathbf{0}; \mathbf{1^2}; \mathbf{0}; \mathbf{1^2}; -\mathbf{3}; \mathbf{1}] \times 1$
	$[\mathbf{1^6}; -\mathbf{4}; \mathbf{0}] \times 1$	$[\mathbf{0}; \mathbf{1^3}; \mathbf{0}; \mathbf{1}; -\mathbf{3}; \mathbf{1}] \times 1$
		$[\mathbf{1^3}; \mathbf{0^2}; \mathbf{1}; -\mathbf{3}; \mathbf{1}] \times 1$

- $\mathsf{Spec}(D^8_{16}) = 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9$
- $|Iso(D^8_{16}, \mathcal{Q}^8_{16})| = 144$
- $l(D^8_{16}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $35 - tope$, $H(2)$, $\frac{1}{2}H(5)$, $J(7,5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{17}[x-\mathbf{c}])$, where $\mathcal{Q}^8_{17}$ is given by

$$\mathcal{Q}^8_{17}(\mathbf{x}) = \mathbf{x}^t \begin{array}{|c|c|c|c|c|c|c|c|}\hline 8 & 6 & 6 & 6 & 6 & 6 & 9 & 4 \\ \hline 6 & 8 & 6 & 6 & 6 & 6 & 9 & 4 \\ \hline 6 & 6 & 8 & 6 & 6 & 6 & 9 & 4 \\ \hline 6 & 6 & 6 & 8 & 6 & 6 & 9 & 6 \\ \hline 6 & 6 & 6 & 6 & 8 & 6 & 9 & 4 \\ \hline 6 & 6 & 6 & 6 & 6 & 8 & 9 & 4 \\ \hline 9 & 9 & 9 & 9 & 9 & 9 & 13 & 6 \\ \hline 4 & 4 & 4 & 6 & 4 & 4 & 6 & 6 \\ \hline \end{array} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{10} \begin{array}{|r|}\hline 3 \\ \hline 3 \\ \hline 3 \\ \hline 2 \\ \hline 3 \\ \hline 3 \\ \hline -8 \\ \hline 1 \\ \hline \end{array}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{17})| = 5760$; $S_6 < O(\mathbb{Z}^8, \mathcal{Q}^8_{17})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{17}) = 2$
- $\mathsf{dim\,QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{17}] = 4$

Inequality $\mathcal{Q}^8_{17}[\mathbf{x}-\mathbf{c}] \le \frac{27}{10}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{17} \in Del(\mathbb{Z}^8, \mathcal{Q}^8_{17})$, whose vertex set ($|\,\mathsf{vert}\,D^8_{17}| = 44$) is given below.

$x_8 = -1$	$x_8 = 0$	$x_8 = 1$
$[\mathbf{0^3}; \mathbf{1}; \mathbf{0^3}; \mathbf{-1}] \times 1$	$[\mathbf{-1}, \mathbf{0^5}; \mathbf{1}; \mathbf{0}] \times 6$	$[\mathbf{0^3}; \mathbf{-1}; \mathbf{0^2}; \mathbf{1^2}] \times 1$
$[\mathbf{1^3}; \mathbf{2}; \mathbf{1^2}; \mathbf{-4}; \mathbf{-1}] \times 1$	$[\mathbf{0^8}] \times 1$	$[\mathbf{0^3}; \mathbf{-1}; \mathbf{0}; \mathbf{1}; \mathbf{0}; \mathbf{1}] \times 1$
	$[\mathbf{0^5}, \mathbf{1}; \mathbf{0^2}] \times 6$	$[\mathbf{0^3}; \mathbf{-1}; \mathbf{1}; \mathbf{0^2}; \mathbf{1}] \times 1$
	$[\mathbf{0^4}, \mathbf{1^2}; \mathbf{-1}; \mathbf{0}] \times 15$	$[\mathbf{0^7}; \mathbf{1}] \times 1$
	$[\mathbf{0}, \mathbf{1^5}; \mathbf{-3}; \mathbf{0}] \times 6$	$[\mathbf{0^2}, \mathbf{1}; \mathbf{-1}; \mathbf{0^3}; \mathbf{1}] \times 3$
	$[\mathbf{1^6}; \mathbf{-4}; \mathbf{0}] \times 1$	

- $\mathsf{Spec}(D^8_{17}) = 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9$
- $|Iso(D^8_{17}, \mathcal{Q}^8_{17})| = 2880$
- $l(D^8_{17}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $35-tope$, $H(2)$, $\frac{1}{2}H(5)$, $J(8,6)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{18}[x - \mathbf{c}])$, where $\mathcal{Q}^8_{18}$ is given by

$$\mathcal{Q}^8_{18}(\mathbf{x}) = \mathbf{x}^t \begin{bmatrix} 11 & 6 & 6 & 6 & -7 & 11 & 4 & 3 \\ 6 & 11 & 6 & 6 & -7 & 11 & 4 & 3 \\ 6 & 6 & 11 & 6 & -7 & 11 & 4 & 3 \\ 6 & 6 & 6 & 11 & -7 & 11 & 4 & 3 \\ -7 & -7 & -7 & -7 & 12 & -8 & -4 & 4 \\ 11 & 11 & 11 & 11 & -8 & 20 & 4 & 8 \\ 4 & 4 & 4 & 4 & -4 & 4 & 8 & 0 \\ 3 & 3 & 3 & 3 & 4 & 8 & 0 & 13 \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{76} \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \\ 23 \\ 25 \\ 25 \\ 10 \end{bmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{18})| = 288$; $S_4 < O(\mathbb{Z}^8, \mathcal{Q}^8_{18})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{18}) = 6$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{18}] = 8$

Inequality $\mathcal{Q}^8_{18}[\mathbf{x} - \mathbf{c}] \leq \frac{501}{76}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{18} \in Del(\mathbb{Z}^8, \mathcal{Q}^8_{18})$, whose vertex set ($|\,\mathsf{vert}\, D^8_{18}| = 44$) is given below.

$x_6 = -1$	$x_6 = 0$	$x_6 = 1$	$x_6 = 2$
$[\mathbf{0}, \mathbf{1^3}; \mathbf{1}; \mathbf{-1}; \mathbf{0^2}] \times 4$	$[\mathbf{0^4}; \mathbf{-1}; \mathbf{0^2}; \mathbf{1}] \times 1$	$[\mathbf{-1^4}; \mathbf{-2}; \mathbf{1^3}] \times 1$	$[\mathbf{-1^5}; \mathbf{2}; \mathbf{1}; \mathbf{0}] \times 1$
$[\mathbf{1^5}; \mathbf{-1^2}; \mathbf{0}] \times 1$	$[\mathbf{0^8}] \times 1$	$[\mathbf{-1^3}, \mathbf{0}; \mathbf{-1}; \mathbf{1^3}] \times 4$	$[\mathbf{-1^5}; \mathbf{2}; \mathbf{1^2}] \times 1$
$[\mathbf{1^5}; \mathbf{-1}; \mathbf{0^2}] \times 1$	$[\mathbf{0^7}; \mathbf{1}] \times 1$	$[\mathbf{-1}, \mathbf{0^3}; \mathbf{0}; \mathbf{1}; \mathbf{0^2}] \times 4$	
$[\mathbf{1^4}; \mathbf{2}; \mathbf{-1}; \mathbf{0^2}] \times 1$	$[\mathbf{0^6}; \mathbf{1}; \mathbf{0}] \times 1$	$[\mathbf{-1}, \mathbf{0^3}; \mathbf{0}; \mathbf{1^2}; \mathbf{0}] \times 4$	
	$[\mathbf{0^6}; \mathbf{1^2}] \times 1$	$[\mathbf{0^5}; \mathbf{1}; \mathbf{0^2}] \times 1$	
	$[\mathbf{0^3}, \mathbf{1}; \mathbf{0^4}] \times 4$	$[\mathbf{0^4}; \mathbf{1^2}; \mathbf{0^2}] \times 1$	
	$[\mathbf{0^2}, \mathbf{1^2}; \mathbf{1}; \mathbf{0^3}] \times 6$	$[\mathbf{0^4}; \mathbf{1^3}; \mathbf{0}] \times 1$	
	$[\mathbf{0}, \mathbf{1^3}; \mathbf{2}; \mathbf{0^2}; \mathbf{-1}] \times 4$		

- $\mathsf{Spec}(D^8_{18}) = 5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 24$
- $|Iso(D^8_{18}, \mathcal{Q}^8_{18})| = 144$
- $l(D^8_{18}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $H(2)$, $\frac{1}{2}H(5)$, $J(7,5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{19}[x-\mathbf{c}])$, where $\mathcal{Q}^8_{19}$ is given by

$$\mathcal{Q}^8_{19}(\mathbf{x}) = \mathbf{x}^t \begin{bmatrix} 8 & 5 & 5 & 5 & 1 & 1 & 8 & 4 \\ 5 & 8 & 5 & 5 & 1 & 1 & 8 & 4 \\ 5 & 5 & 8 & 5 & 1 & 1 & 8 & 4 \\ 5 & 5 & 5 & 8 & 1 & 1 & 8 & 4 \\ 1 & 1 & 1 & 1 & 7 & 3 & 2 & 0 \\ 1 & 1 & 1 & 1 & 3 & 7 & 6 & 4 \\ 8 & 8 & 8 & 8 & 2 & 6 & 15 & 8 \\ 4 & 4 & 4 & 4 & 0 & 4 & 8 & 8 \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{46} \begin{bmatrix} -11 \\ -11 \\ -11 \\ -11 \\ 29 \\ -32 \\ 49 \\ 12 \end{bmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{19})| = 576$; $S_4 < O(\mathbb{Z}^8, \mathcal{Q}^8_{19})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{19}) = 8$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{19}] = 6$

Inequality $\mathcal{Q}^8_{19}[\mathbf{x}-\mathbf{c}] \leq \frac{229}{46}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{19} \in Del(\mathbb{Z}^8, \mathcal{Q}^8_{19})$, whose vertex set ($|\mathsf{vert}\, D^8_{19}| = 49$) is given below.

$x_8 = -1$	$x_8 = 0$	$x_8 = 1$
$[\mathbf{0^6}; \mathbf{1}; \mathbf{-1}] \times 1$	$[\mathbf{-1^4}; \mathbf{1}; \mathbf{-2}; \mathbf{3}; \mathbf{0}] \times 1$	$[\mathbf{-1^4}; \mathbf{1}; \mathbf{-3}; \mathbf{3}; \mathbf{1}] \times 1$
	$[\mathbf{-1^3}, \mathbf{0}; \mathbf{1}; \mathbf{-2}; \mathbf{3}; \mathbf{0}] \times 4$	$[\mathbf{-1^4}; \mathbf{1}; \mathbf{-2}; \mathbf{3}; \mathbf{1}] \times 1$
	$[\mathbf{-1^2}, \mathbf{0^2}; \mathbf{1}; \mathbf{-1}; \mathbf{2}; \mathbf{0}] \times 6$	$[\mathbf{-1^4}; \mathbf{2}; \mathbf{-3}; \mathbf{3}; \mathbf{1}] \times 1$
	$[\mathbf{-1}, \mathbf{0^3}; \mathbf{0^2}; \mathbf{1}; \mathbf{0}] \times 4$	$[\mathbf{-1^3}, \mathbf{0}; \mathbf{1}; \mathbf{-2}; \mathbf{2}; \mathbf{1}] \times 4$
	$[\mathbf{-1}, \mathbf{0^3}; \mathbf{1}; \mathbf{-1}; \mathbf{1}; \mathbf{0}] \times 4$	$[\mathbf{-1}, \mathbf{0^3}; \mathbf{1}; \mathbf{-1}; \mathbf{1^2}] \times 4$
	$[\mathbf{0^8}] \times 1$	$[\mathbf{0^7}; \mathbf{1}] \times 1$
	$[\mathbf{0^6}; \mathbf{1}; \mathbf{0}] \times 1$	$[\mathbf{0^4}; \mathbf{1}; \mathbf{-1}; \mathbf{0}; \mathbf{1}] \times 1$
	$[\mathbf{0^5}; \mathbf{1}; \mathbf{0^2}] \times 1$	$[\mathbf{0^4}; \mathbf{1}; \mathbf{0^2}; \mathbf{1}] \times 1$
	$[\mathbf{0^4}; \mathbf{1}; \mathbf{-1}; \mathbf{1}; \mathbf{0}] \times 1$	
	$[\mathbf{0^4}; \mathbf{1}; \mathbf{0^3}] \times 1$	
	$[\mathbf{0^3}, \mathbf{1}; \mathbf{0^4}] \times 4$	
	$[\mathbf{0^2}, \mathbf{1^2}; \mathbf{0}; \mathbf{1}; \mathbf{-1}; \mathbf{0}] \times 6$	

- $\mathsf{Spec}(D^8_{19}) = 4, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 18$
- $|Iso(D^8_{19}, \mathcal{Q}^8_{19})| = 288$
- $l(D^8_{19}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $H(3)$, $\frac{1}{2}H(5)$, $J(7,5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{20}[x-\mathbf{c}])$, where $\mathcal{Q}^8_{20}$ is given by

$$\mathcal{Q}^8_{20}(\mathbf{x}) = \mathbf{x}^t \begin{array}{|c|c|c|c|c|c|c|c|}\hline 12 & 9 & 9 & 9 & 9 & -15 & 3 & 3 \\ \hline 9 & 12 & 9 & 9 & 9 & -15 & 4 & 5 \\ \hline 9 & 9 & 12 & 9 & 9 & -15 & 4 & 3 \\ \hline 9 & 9 & 9 & 12 & 9 & -15 & 4 & 5 \\ \hline 9 & 9 & 9 & 9 & 12 & -15 & 4 & 4 \\ \hline -15 & -15 & -15 & -15 & -15 & 24 & -6 & -6 \\ \hline 3 & 4 & 4 & 4 & 4 & -6 & 4 & 3 \\ \hline 3 & 5 & 3 & 5 & 4 & -6 & 3 & 5 \\ \hline \end{array} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{20} \begin{array}{|c|}\hline 6 \\ \hline 7 \\ \hline 7 \\ \hline 7 \\ \hline 7 \\ \hline 13 \\ \hline -3 \\ \hline 0 \\ \hline \end{array}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{20})| = 48$; $S_3 < O(\mathbb{Z}^8, \mathcal{Q}^8_{20})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{20}) = 4$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{20}] = 11$

Inequality $\mathcal{Q}^8_{20}[\mathbf{x}-\mathbf{c}] \leq \frac{81}{20}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{20} \in Del(\mathbb{Z}^8, \mathcal{Q}^8_{20})$, whose vertex set ($|\mathsf{vert}\, D^8_{20}| = 47$) is given below.

$x_8 = -1$	$x_8 = 0$	$x_8 = 1$
$[\mathbf{0^3; 1; 0^3; -1}] \times 1$	$[\mathbf{-1, 0^4; -1; 0^2}] \times 5$	$[\mathbf{0; -1; 0^3; -1^2; 1}] \times 1$
$[\mathbf{0; 1; 0^5; -1}] \times 1$	$[\mathbf{0^8}] \times 1$	$[\mathbf{0^3; -1; 0; -1^2; 1}] \times 1$
$[\mathbf{0; 1; 0; 1; 0; 1^2; -1}] \times 1$	$[\mathbf{0^6; 1; 0}] \times 1$	$[\mathbf{0^7; 1}] \times 1$
$[\mathbf{0; 1; 0; 1^3; 0; -1}] \times 1$	$[\mathbf{0^4, 1; 0^3}] \times 5$	$[\mathbf{0^2; 1; 0^3; -1; 1}] \times 1$
$[\mathbf{0; 1^4; 2; 0; -1}] \times 1$	$[\mathbf{0^3, 1^2; 1; 0^2}] \times 10$	$[\mathbf{0^2; 1; 0; 1^2; -1; 1}] \times 1$
$[\mathbf{1^2; 0; 1^2; 2; 0; -1}] \times 1$	$[\mathbf{0, 1^4; 2; 0^2}] \times 5$	$[\mathbf{1; 0; 1; 0^2; 1; -1; 1}] \times 1$
$[\mathbf{1^2; 0; 1^2; 2; 1; -1}] \times 1$	$[\mathbf{1^5; 3; 0^2}] \times 1$	$[\mathbf{1; 0; 1; 0^2; 1; 0; 1}] \times 1$
	$[\mathbf{0^4; 1; 0; -1; 0}] \times 1$	
	$[\mathbf{0^3; 1; 0^2; -1; 0}] \times 1$	
	$[\mathbf{0^2; 1; 0^3; -1; 0}] \times 1$	
	$[\mathbf{0; 1; 0^4; -1; 0}] \times 1$	
	$[\mathbf{0; 1^4; 2; -1; 0}] \times 1$	

- $\mathsf{Spec}(D^8_{20}) = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14$
- $|Iso(D^8_{20}, \mathcal{Q}^8_{20})| = 24$
- $l(D^8_{20}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: G_6, $H(3)$, $\frac{1}{2}H(5)$, $J(6, 4)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{21}[x-\mathbf{c}])$, where $\mathcal{Q}^8_{21}$ is given by

$$\mathcal{Q}^8_{21}(\mathbf{x}) = \mathbf{x}^t \begin{bmatrix} 12 & 9 & 9 & 9 & 9 & -15 & 6 & 5 \\ 9 & 12 & 9 & 9 & 9 & -15 & 3 & 4 \\ 9 & 9 & 12 & 9 & 9 & -15 & 4 & 3 \\ 9 & 9 & 9 & 12 & 9 & -15 & 4 & 3 \\ 9 & 9 & 9 & 9 & 12 & -15 & 4 & 3 \\ -15 & -15 & -15 & -15 & -15 & 24 & -6 & -5 \\ 6 & 3 & 4 & 4 & 4 & -6 & 6 & 4 \\ 5 & 4 & 3 & 3 & 3 & -5 & 4 & 5 \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{50} \begin{bmatrix} 19 \\ 20 \\ 17 \\ 17 \\ 17 \\ 37 \\ 3 \\ -6 \end{bmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{21})| = 96$; $S_3 < O(\mathbb{Z}^8, \mathcal{Q}^8_{21})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{21}) = 4$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{21}] = 8$

Inequality $\mathcal{Q}^8_{21}[\mathbf{x}-\mathbf{c}] \le \frac{201}{50}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{21} \in Del(\mathbb{Z}^8, \mathcal{Q}^8_{21})$, whose vertex set ($|\mathsf{vert}\, D^8_{21}| = 47$) is given below.

$x_8 = -1$	$x_8 = 0$	$x_8 = 1$
$[\mathbf{0}; \mathbf{1}; \mathbf{0^4}; \mathbf{1}; -\mathbf{1}] \times 1$	$[-\mathbf{1}, \mathbf{0^4}; -\mathbf{1}; \mathbf{0^2}] \times 5$	$[-\mathbf{1}; \mathbf{0^4}; -\mathbf{1}; \mathbf{0}; \mathbf{1}] \times 1$
$[\mathbf{0}; \mathbf{1}; \mathbf{0^2}, \mathbf{1}; \mathbf{1^2}; -\mathbf{1}] \times 3$	$[\mathbf{0^8}] \times 1$	$[\mathbf{0}; -\mathbf{1}; \mathbf{0^3}; -\mathbf{1^2}; \mathbf{1}] \times 1$
$[\mathbf{1}; \mathbf{0^6}; -\mathbf{1}] \times 1$	$[\mathbf{0^6}; \mathbf{1}; \mathbf{0}] \times 1$	$[\mathbf{0^7}; \mathbf{1}] \times 1$
$[\mathbf{1^2}; \mathbf{0^3}; \mathbf{1^2}; -\mathbf{1}] \times 1$	$[\mathbf{0^4}, \mathbf{1}; \mathbf{0^3}] \times 5$	$[\mathbf{1}; \mathbf{0}; \mathbf{1^3}; \mathbf{2}; -\mathbf{1}; \mathbf{1}] \times 1$
$[\mathbf{1^2}; \mathbf{0}, \mathbf{1^2}; \mathbf{2}; \mathbf{0}; -\mathbf{1}] \times 3$	$[\mathbf{0^3}, \mathbf{1^2}; \mathbf{1}; \mathbf{0^2}] \times 10$	
$[\mathbf{2}; \mathbf{1^4}; \mathbf{3}; \mathbf{0}; -\mathbf{1}] \times 1$	$[\mathbf{0}, \mathbf{1^4}; \mathbf{2}; \mathbf{0^2}] \times 5$	
	$[\mathbf{1^5}; \mathbf{3}; \mathbf{0^2}] \times 1$	
	$[-\mathbf{1}; \mathbf{0^4}; -\mathbf{1}; \mathbf{1}; \mathbf{0}] \times 1$	
	$[-\mathbf{1}; \mathbf{1}; \mathbf{0^4}; \mathbf{1}; \mathbf{0}] \times 1$	
	$[\mathbf{1}; \mathbf{0^5}; -\mathbf{1}; \mathbf{0}] \times 1$	
	$[\mathbf{1}; \mathbf{0}; \mathbf{1^3}; \mathbf{2}; -\mathbf{1}; \mathbf{0}] \times 1$	
	$[\mathbf{2}; \mathbf{1^4}; \mathbf{3}; -\mathbf{1}; \mathbf{0}] \times 1$	

- $\mathsf{Spec}(D^8_{21}) = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13$
- $|Iso(D^8_{21}, \mathcal{Q}^8_{21})| = 48$
- $l(D^8_{21}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: G_6, $H(3)$, $\frac{1}{2}H(5)$, $J(6,4)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{22}[x - \mathbf{c}])$, where $\mathcal{Q}^8_{22}$ is given by

$$\mathcal{Q}^8_{22}(\mathbf{x}) = \mathbf{x}^t \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 1 & 2 & 1 \\ 0 & 2 & 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{10} \begin{bmatrix} 8 \\ -9 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{22})| = 645120$; $S_7 < O(\mathbb{Z}^8, \mathcal{Q}^8_{22})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{22}) = 84$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{22}] = 2$

Inequality $\mathcal{Q}^8_{22}[\mathbf{x} - \mathbf{c}] \leq \frac{9}{5}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{22} \in Del(\mathbb{Z}^8, \mathcal{Q}^8_{22})$, whose vertex set ($|\,\mathsf{vert}\, D^8_{22}| = 79$) is given below.

$x_1 = 0$	$x_1 = 1$	$x_1 = 2$
$[\mathbf{0^8}] \times 1$	$[\mathbf{1}; \mathbf{-3}; \mathbf{1^6}] \times 1$	$[\mathbf{2}; \mathbf{-3}; \mathbf{1^6}] \times 1$
$[\mathbf{0^2}; \mathbf{0^5}, \mathbf{1}] \times 6$	$[\mathbf{1}; \mathbf{-2}; \mathbf{0^2}, \mathbf{1^4}] \times 15$	
$[\mathbf{0}; \mathbf{1}; \mathbf{-1}, \mathbf{0^5}] \times 6$	$[\mathbf{1}; \mathbf{-2}; \mathbf{0}, \mathbf{1^5}] \times 6$	
$[\mathbf{0}; \mathbf{1}; \mathbf{0^6}] \times 1$	$[\mathbf{1}; \mathbf{-1}; \mathbf{0^4}, \mathbf{1^2}] \times 15$	
	$[\mathbf{1}; \mathbf{-1}; \mathbf{0^3}, \mathbf{1^3}] \times 20$	
	$[\mathbf{1}; \mathbf{0^7}] \times 1$	
	$[\mathbf{1}; \mathbf{0}; \mathbf{0^5}, \mathbf{1}] \times 6$	

- $\mathsf{Spec}(D^8_{22}) = 2, 3, 4, 5, 6$
- $|Iso(D^8_{22}, \mathcal{Q}^8_{22})| = 322560$
- $l(D^8_{22}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $H(3)$, $\frac{1}{2}H(7)$, $J(8, 6)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{23}[x-\mathbf{c}])$, where $\mathcal{Q}^8_{23}$ is given by

$$\mathcal{Q}^8_{23}(\mathbf{x}) = \mathbf{x}^t \begin{pmatrix} 7 & 4 & 4 & 4 & 4 & 2 & 7 & 4 \\ 4 & 7 & 4 & 4 & 4 & 2 & 7 & 4 \\ 4 & 4 & 7 & 4 & 4 & 2 & 7 & 4 \\ 4 & 4 & 4 & 7 & 4 & 2 & 7 & 4 \\ 4 & 4 & 4 & 4 & 7 & 2 & 7 & 4 \\ 2 & 2 & 2 & 2 & 2 & 7 & 7 & 0 \\ 7 & 7 & 7 & 7 & 7 & 7 & 14 & 4 \\ 4 & 4 & 4 & 4 & 4 & 0 & 4 & 8 \end{pmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{20} \begin{pmatrix} -7 \\ -7 \\ -7 \\ -7 \\ -7 \\ -7 \\ 27 \\ 14 \end{pmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{23})| = 1920$; $S_5 < O(\mathbb{Z}^8, \mathcal{Q}^8_{23})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{23}) = 10$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{23}] = 4$

Inequality $\mathcal{Q}^8_{23}[\mathbf{x}-\mathbf{c}] \le \frac{49}{10}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{23} \in Del(\mathbb{Z}^8, \mathcal{Q}^8_{23})$, whose vertex set ($|\mathsf{vert}\, D^8_{23}| = 49$) is given below.

$x_8 = 0$	$x_8 = 1$	$x_8 = 2$
$[\mathbf{-1}, \mathbf{0^5}; \mathbf{1}; \mathbf{0}] \times 6$	$[\mathbf{-1^5}; \mathbf{-2}; \mathbf{4}; \mathbf{1}] \times 1$	$[\mathbf{-1^6}; \mathbf{3}; \mathbf{2}] \times 1$
$[\mathbf{0^8}] \times 1$	$[\mathbf{-1^6}; \mathbf{3}; \mathbf{1}] \times 1$	
$[\mathbf{0^6}; \mathbf{1}; \mathbf{0}] \times 1$	$[\mathbf{-1^4}, \mathbf{0}; \mathbf{-1}; \mathbf{3}; \mathbf{1}] \times 5$	
$[\mathbf{0^5}, \mathbf{1}; \mathbf{0^2}] \times 6$	$[\mathbf{-1^3}, \mathbf{0^2}; \mathbf{-1}; \mathbf{2}; \mathbf{1}] \times 10$	
	$[\mathbf{-1^2}, \mathbf{0^3}; \mathbf{0}; \mathbf{1^2}] \times 10$	
	$[\mathbf{-1}, \mathbf{0^4}; \mathbf{0}; \mathbf{1^2}] \times 5$	
	$[\mathbf{0^7}; \mathbf{1}] \times 1$	
	$[\mathbf{0^5}; \mathbf{1}; \mathbf{0}; \mathbf{1}] \times 1$	

- $\mathsf{Spec}(D^8_{23}) = 4, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16$
- $|Iso(D^8_{23}, \mathcal{Q}^8_{23})| = 960$
- $l(D^8_{23}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $H(3)$, $\frac{1}{2}H(4)$, $J(7,5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{24}[x-\mathbf{c}])$, where $\mathcal{Q}^8_{24}$ is given by

$$\mathcal{Q}^8_{24}(\mathbf{x}) = \mathbf{x}^t \begin{array}{|c|c|c|c|c|c|c|c|}\hline 24 & 18 & 18 & 18 & 18 & 18 & 27 & 13 \\ \hline 18 & 24 & 18 & 18 & 18 & 18 & 27 & 13 \\ \hline 18 & 18 & 24 & 18 & 18 & 18 & 27 & 15 \\ \hline 18 & 18 & 18 & 24 & 18 & 18 & 27 & 19 \\ \hline 18 & 18 & 18 & 18 & 24 & 18 & 27 & 15 \\ \hline 18 & 18 & 18 & 18 & 18 & 24 & 27 & 15 \\ \hline 27 & 27 & 27 & 27 & 27 & 27 & 39 & 21 \\ \hline 13 & 13 & 15 & 19 & 15 & 15 & 21 & 19 \\ \hline \end{array} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{36} \begin{array}{|r|}\hline 11 \\ \hline 11 \\ \hline 12 \\ \hline 14 \\ \hline 12 \\ \hline 12 \\ \hline -33 \\ \hline -3 \\ \hline \end{array}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{24})| = 144$; $S_3 < O(\mathbb{Z}^8, \mathcal{Q}^8_{24})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{24}) = 2$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{24}] = 9$

Inequality $\mathcal{Q}^8_{24}[\mathbf{x}-\mathbf{c}] \le \frac{97}{12}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{24} \in Del(\mathbb{Z}^8, \mathcal{Q}^8_{24})$, whose vertex set ($|\mathsf{vert}\, D^8_{24}| = 44$) is given below.

$x_8 = -1$	$x_8 = 0$	$x_8 = 1$
$[\mathbf{0^3}; \mathbf{1}; \mathbf{0^3}; \mathbf{-1}] \times 1$	$[\mathbf{-1}, \mathbf{0^5}; \mathbf{1}; \mathbf{0}] \times 6$	$[\mathbf{0^7}; \mathbf{1}] \times 1$
$[\mathbf{0^3}; \mathbf{1^3}; \mathbf{-1^2}] \times 1$	$[\mathbf{0^8}] \times 1$	$[\mathbf{0}, \mathbf{1}; \mathbf{0}; \mathbf{-1}; \mathbf{0^3}; \mathbf{1}] \times 2$
$[\mathbf{0^2}; \mathbf{1^2}; \mathbf{0}, \mathbf{1}; \mathbf{-1^2}] \times 2$	$[\mathbf{0^5}, \mathbf{1}; \mathbf{0^2}] \times 6$	
$[\mathbf{0^2}; \mathbf{1^4}; \mathbf{-2}; \mathbf{-1}] \times 1$	$[\mathbf{0^4}, \mathbf{1^2}; \mathbf{-1}; \mathbf{0}] \times 15$	
$[\mathbf{1^3}; \mathbf{2}; \mathbf{1^2}; \mathbf{-4}; \mathbf{-1}] \times 1$	$[\mathbf{0}, \mathbf{1^5}; \mathbf{-3}; \mathbf{0}] \times 6$	
	$[\mathbf{1^6}; \mathbf{-4}; \mathbf{0}] \times 1$	

- $\mathsf{Spec}(D^8_{24}) = 5, 8, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 23, 24, 25, 27$
- $|Iso(D^8_{24}, \mathcal{Q}^8_{24})| = 72$
- $l(D^8_{24}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $35-tope$, $H(2)$, $\frac{1}{2}H(5)$, $J(7,5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{25}[x-\mathbf{c}])$, where $\mathcal{Q}^8_{25}$ is given by

$$\mathcal{Q}^8_{25}(\mathbf{x}) = \mathbf{x}^t \begin{pmatrix} 10 & 6 & 6 & 6 & 5 & -13 & 2 & 0 \\ 6 & 10 & 6 & 6 & 5 & -13 & 2 & 0 \\ 6 & 6 & 10 & 6 & 5 & -13 & 2 & 0 \\ 6 & 6 & 6 & 10 & 5 & -13 & 2 & 0 \\ 5 & 5 & 5 & 5 & 10 & -5 & 0 & 0 \\ -13 & -13 & -13 & -13 & -5 & 31 & 0 & 5 \\ 2 & 2 & 2 & 2 & 0 & 0 & 8 & 5 \\ 0 & 0 & 0 & 0 & 0 & 5 & 5 & 10 \end{pmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{64} \begin{pmatrix} 21 \\ 21 \\ 21 \\ 21 \\ 0 \\ 20 \\ -4 \\ 24 \end{pmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{25})| = 576$; $S_4 < O(\mathbb{Z}^8, \mathcal{Q}^8_{25})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{25}) = 8$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{25}] = 6$

Inequality $\mathcal{Q}^8_{25}[\mathbf{x}-\mathbf{c}] \leq \frac{207}{32}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{25} \in Del(\mathbb{Z}^8, \mathcal{Q}^8_{25})$, whose vertex set ($|\,\mathsf{vert}\, D^8_{25}| = 44$) is given below.

$x_6 = -2$	$x_6 = -1$	$x_6 = 0$	$x_6 = 1$
$[\mathbf{-2}, \mathbf{-1^3}; \mathbf{2}; \mathbf{-2}; \mathbf{1^2}] \times 4$	$[\mathbf{-1^2}, \mathbf{0^2}; \mathbf{1}; \mathbf{-1}; \mathbf{0}; \mathbf{1}] \times 6$	$[\mathbf{0^8}] \times 1$	$[\mathbf{0}, \mathbf{1^3}; \mathbf{-1}; \mathbf{1}; \mathbf{0^2}] \times 4$
$[\mathbf{-1^4}; \mathbf{1}; \mathbf{-2}; \mathbf{1^2}] \times 1$		$[\mathbf{0^7}; \mathbf{1}] \times 1$	$[\mathbf{0}, \mathbf{1^3}; \mathbf{0}; \mathbf{1}; \mathbf{0^2}] \times 4$
$[\mathbf{-1^4}; \mathbf{2}; \mathbf{-2}; \mathbf{1^2}] \times 1$		$[\mathbf{0^6}; \mathbf{1}; \mathbf{0}] \times 1$	$[\mathbf{1^4}; \mathbf{-1}; \mathbf{1}; \mathbf{-1}; \mathbf{0}] \times 1$
		$[\mathbf{0^4}, \mathbf{1}; \mathbf{0^3}] \times 5$	$[\mathbf{1^4}; \mathbf{-1}; \mathbf{1}; \mathbf{-1}; \mathbf{1}] \times 1$
		$[\mathbf{0^4}, \mathbf{1}; \mathbf{0^2}; \mathbf{1}] \times 5$	$[\mathbf{1^4}; \mathbf{-1}; \mathbf{1}; \mathbf{0^2}] \times 1$
		$[\mathbf{0^4}; \mathbf{1}; \mathbf{0}; \mathbf{1}; \mathbf{0}] \times 1$	

$x_6 = 2$	$x_6 = 3$
$[\mathbf{1^3}, \mathbf{2}; \mathbf{-1}; \mathbf{2}; \mathbf{-1}; \mathbf{0}] \times 4$	$[\mathbf{2^4}; \mathbf{-2}; \mathbf{3}; \mathbf{-2}; \mathbf{0}] \times 1$
	$[\mathbf{2^4}; \mathbf{-2}; \mathbf{3}; \mathbf{-1^2}] \times 1$
	$[\mathbf{2^4}; \mathbf{-2}; \mathbf{3}; \mathbf{-1}; \mathbf{0}] \times 1$

- $\mathsf{Spec}(D^8_{25}) = 5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21$
- $|Iso(D^8_{25}, \mathcal{Q}^8_{25})| = 288$
- $l(D^8_{25}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: $H(2)$, $\frac{1}{2}H(4)$, $J(7,5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{26}[x - \mathbf{c}])$, where $\mathcal{Q}^8_{26}$ is given by

$$\mathcal{Q}^8_{26}(\mathbf{x}) = \mathbf{x}^t \begin{bmatrix} 12 & 9 & 9 & 9 & 9 & -15 & 6 & 3 \\ 9 & 12 & 9 & 9 & 9 & -15 & 4 & 5 \\ 9 & 9 & 12 & 9 & 9 & -15 & 7 & 5 \\ 9 & 9 & 9 & 12 & 9 & -15 & 6 & 3 \\ 9 & 9 & 9 & 9 & 12 & -15 & 7 & 5 \\ -15 & -15 & -15 & -15 & -15 & 24 & -9 & -6 \\ 6 & 4 & 7 & 6 & 7 & -9 & 8 & 4 \\ 3 & 5 & 5 & 3 & 5 & -6 & 4 & 6 \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{26})| = 2592$; $S_3 < O(\mathbb{Z}^8, \mathcal{Q}^8_{26})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{26}) = 18$
- $\mathsf{dim\,QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{26}] = 2$

Inequality $\mathcal{Q}^8_{26}[\mathbf{x} - \mathbf{c}] \leq 4$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{26} \in Del(\mathbb{Z}^8, \mathcal{Q}^8_{26})$, whose vertex set ($|\,\mathsf{vert}\, D^8_{26}| = 45$) is given below.

$x_8 = -1$	$x_8 = 0$	$x_8 = 1$
$[\mathbf{0}; \mathbf{1}; \mathbf{0^2}; \mathbf{1^3}; -\mathbf{1}] \times 1$	$[-\mathbf{1}, \mathbf{0^4}; -\mathbf{1}; \mathbf{0^2}] \times 5$	$[\mathbf{0}; -\mathbf{1^2}; \mathbf{0}; -\mathbf{1}; -\mathbf{2}; \mathbf{0}; \mathbf{1}] \times 1$
$[\mathbf{0}; \mathbf{1^2}; \mathbf{0^2}; \mathbf{1^2}; -\mathbf{1}] \times 1$	$[\mathbf{0^8}] \times 1$	$[\mathbf{0}; -\mathbf{1}; \mathbf{0^3}; -\mathbf{1^2}; \mathbf{1}] \times 1$
$[\mathbf{0}; \mathbf{1^2}; \mathbf{0}; \mathbf{1^2}; \mathbf{0}; -\mathbf{1}] \times 1$	$[\mathbf{0^6}; \mathbf{1}; \mathbf{0}] \times 1$	$[\mathbf{0^7}; \mathbf{1}] \times 1$
$[\mathbf{0}; \mathbf{1^4}; \mathbf{2}; \mathbf{0}; -\mathbf{1}] \times 1$	$[\mathbf{0^4}, \mathbf{1}; \mathbf{0^3}] \times 5$	$[\mathbf{1}; -\mathbf{1}; \mathbf{0}; \mathbf{1}; \mathbf{0^2}; -\mathbf{1}; \mathbf{1}] \times 1$
$[\mathbf{0}; \mathbf{2}; \mathbf{0^3}; \mathbf{1^2}; -\mathbf{1}] \times 1$	$[\mathbf{0^3}, \mathbf{1^2}; \mathbf{1}; \mathbf{0^2}] \times 10$	$[\mathbf{1}; \mathbf{0^2}; \mathbf{1}; \mathbf{0}; \mathbf{1}; \mathbf{0}; \mathbf{1}] \times 1$
$[\mathbf{1^3}; \mathbf{0}; \mathbf{1}; \mathbf{2}; \mathbf{0}; -\mathbf{1}] \times 1$	$[\mathbf{0}, \mathbf{1^4}; \mathbf{2}; \mathbf{0^2}] \times 5$	$[\mathbf{1}; \mathbf{0}; \mathbf{1^3}; \mathbf{2}; -\mathbf{1}; \mathbf{1}] \times 1$
	$[\mathbf{1^5}; \mathbf{3}; \mathbf{0^2}] \times 1$	
	$[\mathbf{0^2}; \mathbf{1^4}; -\mathbf{1}; \mathbf{0}] \times 1$	
	$[\mathbf{0}; \mathbf{1}; -\mathbf{1}; \mathbf{0^3}; \mathbf{1}; \mathbf{0}] \times 1$	
	$[\mathbf{0}; \mathbf{1}; \mathbf{0^2}; -\mathbf{1}; \mathbf{0}; \mathbf{1}; \mathbf{0}] \times 1$	
	$[\mathbf{1}; \mathbf{0}; \mathbf{1}; \mathbf{0}; \mathbf{1^2}; -\mathbf{1}; \mathbf{0}] \times 1$	
	$[\mathbf{1}; \mathbf{0}; \mathbf{1^3}; \mathbf{2}; -\mathbf{1}; \mathbf{0}] \times 1$	

- $\mathsf{Spec}(D^8_{26}) = 2,\ 3,\ 4,\ 5,\ 6$
- $|Iso(D^8_{26}, \mathcal{Q}^8_{26})| = 1296$
- $l(D^8_{26}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: G_6, $H(2)$, $\frac{1}{2}H(5)$, $J(7, 5)$

Perfect affine quadratic lattice $\mathrm{Aff}(\mathbb{Z}^8, \mathcal{Q}^8_{27}[x-\mathbf{c}])$, where $\mathcal{Q}^8_{27}$ is given by

$$\mathcal{Q}^8_{27}(\mathbf{x}) = \mathbf{x}^t \begin{bmatrix} 12 & 9 & 9 & 9 & 9 & -15 & 7 & 3 \\ 9 & 12 & 9 & 9 & 9 & -15 & 6 & 2 \\ 9 & 9 & 12 & 9 & 9 & -15 & 6 & 3 \\ 9 & 9 & 9 & 12 & 9 & -15 & 6 & 3 \\ 9 & 9 & 9 & 9 & 12 & -15 & 8 & 5 \\ -15 & -15 & -15 & -15 & -15 & 24 & -9 & -5 \\ 7 & 6 & 6 & 6 & 8 & -9 & 11 & 3 \\ 3 & 2 & 3 & 3 & 5 & -5 & 3 & 5 \end{bmatrix} \mathbf{x} \quad \text{and} \quad \mathbf{c} = \frac{1}{12} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \\ -2 \\ 1 \\ 3 \\ 3 \end{bmatrix}.$$

- $|O(\mathbb{Z}^8, \mathcal{Q}^8_{27})| = 144$; $S_3 < O(\mathbb{Z}^8, \mathcal{Q}^8_{27})$
- $s(\mathbb{Z}^8, \mathcal{Q}^8_{27}) = 24$
- $\dim \mathsf{QuadInv}[\mathbb{Z}^8, \mathcal{Q}^8_{27}] = 8$

Inequality $\mathcal{Q}^8_{27}[\mathbf{x} - \mathbf{c}] \leq \frac{17}{4}$ defines the Delaunay ellipsoid for a perfect polytope $D^8_{27} \in Del(\mathbb{Z}^8, \mathcal{Q}^8_{27})$, whose vertex set ($|\operatorname{vert} D^8_{27}| = 44$) is given below.

$x_8 = -1$	$x_8 = 0$	$x_8 = 1$
$[\mathbf{0^4}; \mathbf{1}; \mathbf{0^2}; \mathbf{-1}] \times 1$	$[\mathbf{-1^5}; \mathbf{-3}; \mathbf{1}; \mathbf{0}] \times 1$	$[\mathbf{-1}; \mathbf{0}; \mathbf{-1^2}; \mathbf{-2}; \mathbf{-3}; \mathbf{1^2}] \times 1$
	$[\mathbf{-1}, \mathbf{0^4}; \mathbf{-1}; \mathbf{0^2}] \times 5$	$[\mathbf{-1}; \mathbf{0^3}; \mathbf{-1^2}; \mathbf{1^2}] \times 1$
	$[\mathbf{0^8}] \times 1$	$[\mathbf{0^4}; \mathbf{-2}; \mathbf{-1}; \mathbf{1^2}] \times 1$
	$[\mathbf{0^6}; \mathbf{1}; \mathbf{0}] \times 1$	$[\mathbf{0^4}; \mathbf{-1^2}; \mathbf{0}; \mathbf{1}] \times 1$
	$[\mathbf{0^4}, \mathbf{1}; \mathbf{0^3}] \times 5$	$[\mathbf{0^7}; \mathbf{1}] \times 1$
	$[\mathbf{0^3}, \mathbf{1^2}; \mathbf{1}; \mathbf{0^2}] \times 10$	$[\mathbf{0}; \mathbf{1}; \mathbf{0^2}; \mathbf{-1}; \mathbf{0^2}; \mathbf{1}] \times 1$
	$[\mathbf{0}, \mathbf{1^4}; \mathbf{2}; \mathbf{0^2}] \times 5$	$[\mathbf{0}; \mathbf{1}; \mathbf{0}, \mathbf{1}; \mathbf{0}; \mathbf{1}; \mathbf{0}; \mathbf{1}] \times 2$
	$[\mathbf{1^5}; \mathbf{3}; \mathbf{0^2}] \times 1$	$[\mathbf{1^2}; \mathbf{0^3}; \mathbf{1}; \mathbf{0}; \mathbf{1}] \times 1$
	$[\mathbf{-1^2}; \mathbf{0^2}; \mathbf{-1}; \mathbf{-2}; \mathbf{1}; \mathbf{0}] \times 1$	
	$[\mathbf{-1}; \mathbf{0}; \mathbf{-1}; \mathbf{0}; \mathbf{-1}; \mathbf{-2}; \mathbf{1}; \mathbf{0}] \times 1$	
	$[\mathbf{-1}; \mathbf{0^2}; \mathbf{-1^2}; \mathbf{-2}; \mathbf{1}; \mathbf{0}] \times 1$	
	$[\mathbf{-1}; \mathbf{0^3}; \mathbf{-1^2}; \mathbf{1}; \mathbf{0}] \times 1$	
	$[\mathbf{1^4}; \mathbf{2}; \mathbf{3}; \mathbf{-1}; \mathbf{0}] \times 1$	

- $\mathsf{Spec}(D^8_{27}) = 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$
- $|Iso(D^8_{27}, \mathcal{Q}^8_{27})| = 72$
- $l(D^8_{27}) = 3$
- Antisymmetric
- Maximally contained subpolytopes: G_6, $H(2)$, $\frac{1}{2}H(5)$, $J(7,5)$

9. Computational Methods

In this section we explain how we obtained the perfect Delaunay polytopes presented here. The six- and seven-dimensional polytopes, as well as some of the 8-dimensional ones have been constructed by hands. However, most of the 8-dimensional polytopes on our list have been found by a computer search. The search algorithm was based on the *adjacency method for the hypermetric cone*: below we give an outline of this method. The overall description can be found in (Dutour, 2005).

An *affine basis* of an n-dimensional lattice polytope P is a family $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ of vertices of P such that for every vertex $\mathbf{v}$ of P there exists a unique vector $b = (b_0, \dots, b_n)^t \in \mathbb{Z}^{n+1}$, such that

$$\sum_{i=0}^{n} b_i \mathbf{v}_i = \mathbf{v} \text{ and } \sum_{i=0}^{n} b_i = 1 \ .$$

There exist Delaunay polytopes without affine bases (see [17]) and we cannot exclude the possibility that there exist perfect Delaunay polytopes without affine bases. The adjacency method presented here can only be used for finding perfect Delaunay polytopes that have affine bases.

A symmetric matrix $D = (d_{ij})_{0 \le i,j \le n} \in \mathsf{Sym}(n+1, \mathbb{R})$ is called an $(n+1)$-*hypermetric* if it satisfies the following *hypermetric inequalities*:

$$\frac{1}{2}\,\mathsf{trace}(bb^t D) = \sum_{0 \le i < j \le n} b_i b_j d_{ij} \le 0 \text{ for any } b = (b_i)_{0 \le i \le n} \in \mathbb{Z}^{n+1} \text{ with } \sum_{i=0}^{n} b_i = 1 \ . \qquad (1)$$

Hypermetrics form a cone HYP_{n+1} in $\mathsf{Sym}(n+1, \mathbb{R})$ called the *hypermetric cone*. Since $\mathsf{Sym}(n+1, \mathbb{R})$ is isomorphic to $\mathbb{R}^N$, where $N = \binom{n+1}{2}$, we can identify hypermetrics with *distance vectors* $(d_{ij})_{0 \le i < j \le n} \in \mathbb{R}^N$, satisfying (1). We will denote the left hand side of (1) by $H_b[d]$. Although HYP_{n+1} is defined by an infinite set of inequalities, only finitely many of them are independent (see, e.g., [10]), which implies that the cone is polyhedral. It is easy to see that if P is an n-dimensional lattice Delaunay polytopes and $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ an affine basis of P, then the distance vector $d = (d_{ij}) = (\|\mathbf{v}_i - \mathbf{v}_j\|^2)_{0 \le i < j \le n}$, is a hypermetric. Moreover, one has $H_b[d] = 0$ if and only if $b_0\mathbf{v}_0 + \cdots + b_n\mathbf{v}_n$ is a vertex of P. The *rank* of the Delaunay polytope P is the dimension of the minimal face of HYP_{n+1} containing d. The rank does not depend on the chosen affine basis. A Delaunay polytope is perfect if and only if its rank is equal to 1, i.e., d generates an *extreme ray* of HYP_{n+1}. Two extreme rays are called *adjacent* if they belong to the same 2-face of the cone.

The adjacency method is an iterative search procedure and we will now describe one iteration of the method. Suppose we know a perfect n-dimensional Delaunay polytope P, which happens to admit an affine basis. We first compute an affine basis $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$ of P. Then for every vertex $\mathbf{v}$ of P we compute the vector $b^{\mathbf{v}}$ such that $b_0^{\mathbf{v}}\mathbf{v}_0 + \cdots + b_n^{\mathbf{v}}\mathbf{v}_n = \mathbf{v}$. The set of equalities $H_{b^{\mathbf{v}}}[d] = 0$ determine d up to a constant factor. Next we compute the set of extreme rays $\{\mathbf{R}_+ d_r\}_{1\le r\le M}$ of HYP_{n+1}, which are adjacent to the extreme ray $\mathbb{R}_+ d$ (see [14]). The list of adjacent extreme rays gives some new perfect Delaunay polytopes of dimensions not exceeding n – these polytopes are called *adjacent* to P.

When a Delaunay polytope has an affine basis, it often has a lot of such bases; for example, G_6, G_7, Υ^7 (the 35-tope) have, respectively, 26, 374, 8430 orbits of affine bases. *A priori*, the arithmetic types of adjacent Delaunay polytopes depend on the choice of the affine basis $\{\mathbf{v}_0, \dots, \mathbf{v}_n\}$. In our computations we used only one affine basis for each perfect Delaunay polytope. Furthermore, in general, extreme rays of HYP_{n+1} correspond to perfect Delaunay polytopes of dimensions 1 through n and the extreme rays corresponding to lower dimensional perfect Delaunay polytopes have very high incidence numbers, which makes their computation particularly difficult. This is one of the reasons why we cannot claim that the list of presented perfect Delaunay polytopes is complete. The starting point of the enumeration was the Delaunay polytope Υ^n. In dimension 7 (i.e. for HYP_8), we found only G_7 as other Delaunay polytope. By doing the computation in dimension 8, we found twenty four perfect Delaunay polytopes. Three more perfect Delaunay polytopes of dimension 8 were obtained from running our algorithm in dimension 9.

The analysis of the geometric and combinatorial properties of the discovered polytopes was done with the software package `polyhedral` [17] based on the computer algebra system GAP. The search of sections and subpolytopes was done via exhaustive enumeration schemes that used symmetries to reduce the complexity of the computation.

After this paper was submitted for publication, Dutour and Rybnikov [12] found a better method for discovering perfect Delaunay polytopes. This new method does not depend on the assumption of the existence of affine basis. When the method is “run in dimension n”, it attempts to discover all perfect Delaunay polytopes in this dimension. All computations have been redone with this new method for $n \le 8$. No new polytopes have been discovered; however, the three above-mentioned polytopes that were previously discovered by running the HYP_{n+1} method in dimension $n = 9$, were found by the new method running in dimension $n = 8$.

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