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TABLEAU CYCLING AND CATALAN NUMBERS

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Abstract

We develop two combinatorial proofs of the fact that certain Young tableaux are counted by the Catalan numbers. The setting is a larger class of tableaux where labels increase along rows without attention to whether labels increase down columns. We define a new operation called *tableau cycling*. It is used to duplicate the reflection argument attributed to André in the tableaux setting, and also to prove a tableaux analog of the Chung-Feller theorem.

1. Preliminaries

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n, a Young tableau of shape λ is an arrangement of n boxes, λ_1 in the first row, λ_2 in the second row sharing left border with the first row, etc., with each box having a label $1, \ldots, n$ (each label appearing exactly once) such that labels increase across rows and down columns. Figure 1 shows a Young tableau of shape $(4, 4, 2)$ on the left.

Young tableaux consisting of two rows of equal length are among the many combinatorial objects counted by the Catalan numbers [S]. In particular, there are $\frac{1}{k+1}\binom{2k}{k}$ Young tableaux of shape (k, k). One way to verify this is to establish a bijection between these Young tableaux and certain lattice paths, and then apply a reflection argument attributed to André [LW]. We establish this result using a combinatorial argument on tableaux.

Dropping the Young tableaux requirement that labels increase down columns gives a larger class of *row-increasing tableaux*. Young tableaux are certainly row-increasing tableaux; a row-increasing tableau that is not a Young tableau is given on the right of Figure 1. All row-increasing tableaux of shape $(3,3)$ are given in Figure 6. Row-increasing tableaux of a particular shape are easily counted, since there is one way to order the labels assigned to a given row and no relation between rows. Specifically, the number of row-increasing tableaux of shape λ, with $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition of n, is given by the multinomial coefficient $\binom{n}{\lambda_1,\ldots,\lambda_k}$.

We now define an operation *tableau cycling* on row-increasing tableaux. Given a row-increasing tableau of n boxes and an integer q with $1 \le q \le n$, *cycling through* q gives another row-increasing tableau of the same shape. There are two steps to the construction:

relabeling Labels $j = 1, \ldots, q-1$ are replaced with $n+1+j-q$; the label q is replaced with $n+1-q$, and the labels $j = q+1, \ldots, n$ are replaced with $j-q$ (thus the name "cycling").

reordering Each row of the new labels is put in increasing order.

Figure 1 gives an example of the operation.

1	2	4	7
3	(6)	9	10
5	8		

$\longrightarrow$

6	7	9	1
8	(5)	3	4
10	2		

$\longrightarrow$

1	6	7	9
3	4	(5)	8
2	10		

Figure 1: *Young tableau of shape* $(4,4,2)$ *cycled through 6 to a row-increasing tableau of the same shape.*

For the remainder of the article, we will consider only tableaux with two rows. Tableau cycling, and a variant, will be used to establish the connections to the Catalan numbers and the analog of the Chung-Feller theorem. One more definition is necessary to help determine what number we will cycle through in various situations.

Given a row-increasing tableau with shape (k,k), the *subtableaux* are minimal collections of adjacent columns that together contain consecutive integers. Each row-increasing tableau of shape (k,k) can be partitioned into 1 to k subtableaux. Figure 2 shows a row-increasing tableau of shape $(6,6)$ with five subtableaux, separated by thick lines.

		•	•		•
1	3	7	8	9	12
2	4	5	6	10	11

Figure 2: *Subtableaux of a shape* $(6,6)$ *row-increasing tableau with decreasing columns marked.*

Each column of a row-increasing tableau of shape (k,k) is either increasing or decreasing; Figure 2 introduces the way we will mark decreasing columns. A Young tableau of shape (k,k) is then a row-increasing tableau with no decreasing columns.

We need one lemma that will be applied in the proofs of both theorems.

Lemma. *Given a row-increasing tableau of shape (k, k), every subtableau consists of only increasing columns or only decreasing columns.*

Proof. In the given row-increasing tableau T, suppose that column i is increasing and that column $i+1$ is decreasing. We show that column i ends one subtableau and column $i+1$ begins another. Write $t_{r,c}$ with $r = 1, 2$ and $1 \leq c \leq k$ for the labels of the tableau. The situation is shown in Figure 3. We actually prove that $t_{2,i} = 2i$ and $t_{2,i+1} = 2i + 1$.

$$\begin{array}{ccccccc} & & & \bullet & & & \\ \hline \multicolumn{1}{|c|}{t_{1,1}} & \cdots & \multicolumn{1}{|c|}{t_{1,i}} & \multicolumn{1}{c|}{t_{1,i+1}} & \cdots & \multicolumn{1}{|c|}{t_{1,k}} \\ \hline \multicolumn{1}{|c|}{t_{2,1}} & \cdots & \multicolumn{1}{|c|}{t_{2,i}} & \multicolumn{1}{c|}{t_{2,i+1}} & \cdots & \multicolumn{1}{|c|}{t_{2,k}} \\ \hline \end{array}$$

Figure 3: *Transition from an increasing column to a decreasing column.*

By assumption, $t_{1,i} < t_{2,i} < t_{2,i+1} < t_{1,i+1}$. Since the tableau is row-increasing, all labels $t_{2,1}, \ldots, t_{2,i-1}$ are less than $t_{2,i}$, and since $t_{1,i} < t_{2,i}$, all labels $t_{1,1}, \ldots, t_{1,i}$ are less than $t_{2,i}$ as well. Similarly, all labels $t_{2,i+2}, \ldots, t_{2,k}$ and $t_{1,i+1}, \ldots, t_{1,k}$ are all greater than $t_{2,i+1}$. Together, we know the following about the ordering of the labels:

$$\{2i - 1 \text{ labels}\}, t_{2,i}, \ldots, t_{2,i+1}, \{2k - 2i - 1 \text{ labels}\}$$

Since the labels are exactly the integers $1, \ldots, 2k$, this implies that $t_{2,i} = 2i$ and that the labels $1, \ldots, 2i - 1$ are above it and to its left in the tableau T. Also, $t_{2,i+1} = 2i + 1$ and the labels $2i + 2, \ldots, 2k$ are above it and to its right in T. Therefore, column i ends one subtableau and column $i + 1$ begins another.

The case of a transition from a decreasing column to an increasing column is analogous. In conclusion, no subtableau can contain both increasing and decreasing columns. $\square$

2. Counting Young Tableaux of Shape (k, k)

We are now ready to prove, entirely in the tableaux setting, that the Young tableaux of shape (k, k) are counted by the Catalan numbers.

Theorem 1. *There is a bijection between row-increasing tableaux of shape (k, k) with at least one decreasing column and all row-increasing tableaux of shape $(k + 1, k - 1)$.*

Proof. Given a row-increasing tableau of shape (k, k) with at least one decreasing column, let q be the bottom label of the leftmost decreasing column. We cycle through q with a variation: the box with q, which is relabeled $2k+1-q$, is moved to the first row, and the gap in the second row is removed. Clearly this gives a row-increasing tableau of shape $(k+1, k-1)$. An example is shown in Figure 4.

$$
\begin{array}{cccccc}
 & & \bullet & \bullet & & \bullet \\
\hline
\multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} & \multicolumn{1}{c|}{9} & \multicolumn{1}{c|}{12} \\
\hline
\multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{(5)} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{10} & \multicolumn{1}{c|}{11} \\
\hline
\end{array}
\longrightarrow
\begin{array}{|c|c|c|c|c|c|c|}
\hline
2 & 3 & 4 & 7 & (8) & 9 & 11 \\
\hline
1 & 5 & 6 & 10 & 12 & \multicolumn{2}{c}{} \\
\cline{1-5}
\end{array}
$$

Figure 4: *A row-increasing tableau of shape* $(6,6)$ *with decreasing columns is cycled to a row-increasing tableau of shape* $(7,5)$.

Given a row-increasing tableau T of shape $(k+1, k-1)$, realign the second row so that it is flush *right* under the first row. If this altered tableau has at least one decreasing column, let q be the top label of the rightmost decreasing column. If the altered tableau has no decreasing columns, let $q = 2$ (in this situation, the first row must begin with labels 1 and 2). In either case, now cycle through q, with the variation that the box with q, now relabeled $2k+1-q$, moves to the second row. With gaps removed and rows realigned, this gives a row-increasing tableau U of shape (k,k). Figure 5 gives examples of both cases.

$$
\begin{array}{ccccccc}
 & & \bullet & \bullet & \bullet & & \\
\hline
\multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{(8)} & \multicolumn{1}{c|}{9} & \multicolumn{1}{c|}{11} \\
\hline
 & & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{10} & \multicolumn{1}{c|}{12} \\
\cline{3-7}
\end{array}
\longrightarrow
\begin{array}{cccccc}
 & & \bullet & \bullet & & \bullet \\
\hline
\multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} & \multicolumn{1}{c|}{9} & \multicolumn{1}{c|}{12} \\
\hline
\multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{(5)} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{10} & \multicolumn{1}{c|}{11} \\
\hline
\end{array}
$$

$$
\begin{array}{ccccccc}
\hline
\multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{(2)} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{8} & \multicolumn{1}{c|}{11} \\
\hline
 & & \multicolumn{1}{|c|}{6} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{9} & \multicolumn{1}{c|}{10} & \multicolumn{1}{c|}{12} \\
\cline{3-7}
\end{array}
\longrightarrow
\begin{array}{cccccc}
 & & & & & \bullet \\
\hline
\multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{9} & \multicolumn{1}{c|}{12} \\
\hline
\multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} & \multicolumn{1}{c|}{10} & \multicolumn{1}{c|}{(11)} \\
\hline
\end{array}
$$

Figure 5: *Two altered row-increasing tableaux of shape* $(7,5)$ *are cycled to row-increasing tableaux of shape* $(6,6)$.

We claim that the image tableau U has at least one decreasing column. In particular, it is the one that includes $2k+1-q$, the image of the cycling number now in the second row. By the lemma, we know that the labels to the right of q in the original T form a subtableau of increasing columns and that these labels are exactly $q+1, \ldots, 2k$. Cycling through q, then, will send this subtableau to a subtableau of increasing columns in U consisting of the labels $1, \ldots, 2k-q$. (In the first example of Figure 5, this is {9, 10, 11, 12} going to {1, 2, 3, 4}; in the second example, this is {3, . . . , 12} going to {1, . . . , 10}.) The construction puts $2k+1-q$ in the second row of the next column, so the label eventually placed above it must be greater, giving at least one decreasing column. In the case where T has no decreasing columns and $q = 2$, there is exactly one decreasing column in U, specifically $2k$ over $2k-1$ in the rightmost column.

It is straightforward to see that these are inverse maps, establishing the bijection. $\square$

Since the number of row-increasing tableaux of a given size is easy to count, this gives a formula for the number of Young tableaux of shape (k, k). We write $|S|$ for the cardinality of a set S. We have $\binom{2k}{k} = |\text{row-increasing tableaux of shape } (k,k)| = |\text{Young tableaux of shape } (k,k)| + |\text{row-increasing tableaux of shape} (k,k) \text{with decreasing columns}| = |\text{Young tableaux of shape } (k,k)| = |\text{Young tableaux of shape } (k,k)| + \binom{2k}{k+1}$.

A brief computation shows that $\binom{2k}{k} - \binom{2k}{k+1} = \frac{1}{k+1}\binom{2k}{k}$, so that the number of Young tableaux of shape (k, k) is the kth Catalan number. If one converts to lattice paths at every stage of the argument, this is equivalent to the reflection method.

3. Chung-Feller for Tableaux

Examining all row-increasing tableaux of shape (k, k) (feasible for small k), one sees that the kth Catalan number not only counts the Young tableaux, but also the tableaux with one decreasing column, the tableaux with two decreasing columns, etc. See Figure 6 for a table of all row-increasing tableaux of shape $(3, 3)$, organized into columns by number of decreasing columns (i.e., number of dots; the bars and marked labels have to do with bijections used in the following proof). This result is an analog of Chung-Feller theorem [CF], which we now prove using tableau cycling.

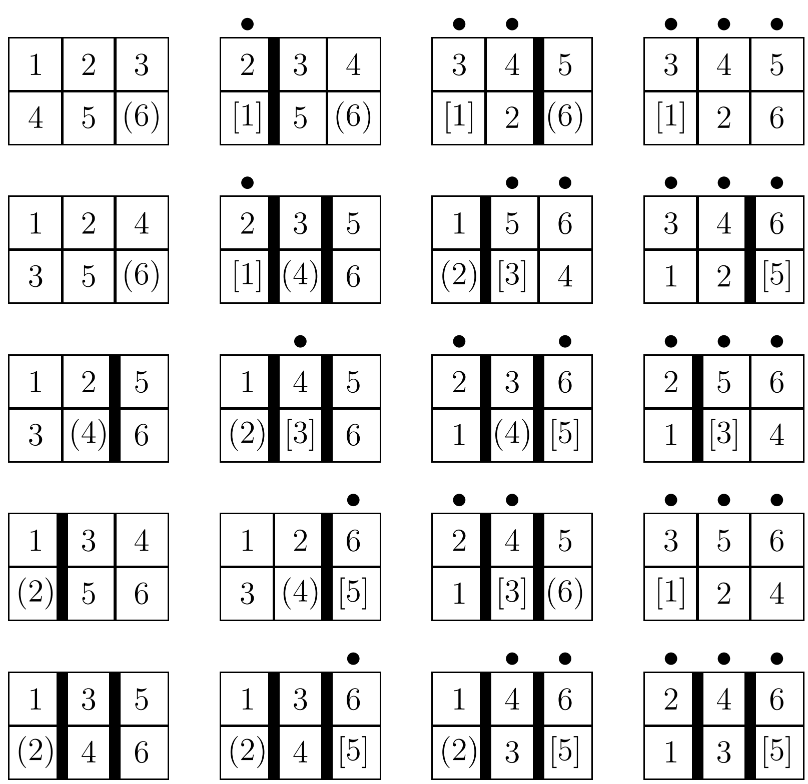

Figure 6: *Row-increasing tableaux of shape* $(3, 3)$ *with decreasing columns and subtableaux indicated. Notation for cycling numbers is explained in the proof of Theorem 2.*

Theorem 2. *Partition the row-increasing tableaux of shape (k, k) into sets D_i for $0 \leq i \leq k$ by the number of decreasing columns in each tableaux. There are bijections between each pair of $D_0, D_1, \ldots, D_k$.*

Proof. Given a row-increasing tableau T of shape (k, k) with at least one increasing column, we know by the lemma that T has at least one subtableau consisting of increasing columns. Determine the leftmost subtableau of increasing columns, its rightmost column i, and let $q = t_{2,i}$, the bottom label of that column. Cycle through q to produce a row-increasing tableau U of shape (k, k) (following the construction in the definition, not the variation of the preceding section). This is shown for all row-increasing tableaux of shape $(3, 3)$ in Figure 6: q labels are designated by parentheses, the image tableau is the adjacent tableau to the right, where the image of q is designated by brackets.

We need to show that U has exactly one more decreasing column than T. By the proof of the lemma, we know $q = 2i$. Columns $i + 1$ to k of T, with labels $2i + 1, \ldots, 2k$, become columns 1 to $k - i$ of U, with labels $1, \ldots, 2k - 2i$ and the identical structure in terms of increasing and decreasing columns as in T. Column $k-i+1$ of U will be the new decreasing column: $u_{2,k-i+1} = 2k - 2i + 1$ and there are only larger labels left for $u_{1,k-i+1}$.

The analysis of the remaining columns of U is more subtle, as the subtableaux structure may not be maintained. Columns 1 to i of T, with labels $1, \ldots, 2i$, become columns $k-i+1$ to k of U, with labels $2k-2i+1, \ldots, 2k$. From the definition of tableau cycling, we see that the labels for these boxes are as shown in Figure 7.

$t_{1,1} + q'$	$t_{1,2} + q'$	$\cdots$	$t_{1,i} + q'$
q'	$t_{2,1} + q'$	$\cdots$	$t_{2,i-1} + q'$

Figure 7: *Columns $k - i + 1$ to k of U, with $q' = 2k - 2i + 1$.*

Notice, for $1 \leq h < i$, that $t_{2,h}$, the label below $t_{1,h}$, is sent to $u_{2,k-i+1+h} = t_{2,h}+2k-2i+1$, now below $u_{1,k-i+1+h} = t_{1,h+1} + 2k - 2i + 1$. Since $t_{1,h} < t_{1,h+1}$, if h was a decreasing column of T, then $k-i+1+h$ is a decreasing column of U (with a greater decrease). So the number of decreasing columns in U is no less than in T.

However, if h was an increasing column of T, then $k - i + 1 - h$ is still an increasing column in U despite the fact that the difference between the labels is less. Recall that q was chosen from the rightmost column of the leftmost subtableau of increasing columns. Consider columns $j, j + 1$ of a subtableau of increasing columns with at least two columns. It must be the case that $t_{2,j} > t_{1,j+1}$. For if not, all labels between $t_{1,j}$ and $t_{2,j}$ would be in the columns up to j and there would be a subtableau separation between columns j and $j + 1$, a contradiction. Therefore, for column h in the subtableau of increasing columns, the corresponding column $k-i+1+h$ of U is still increasing. We conclude that U has exactly one more decreasing column than T.

Given a row-increasing tableau U of shape (k,k) with at least one decreasing column, we know by the lemma that U has at least one subtableau consisting of decreasing columns. Determine the rightmost subtableau of decreasing columns, its leftmost column i, and let $r = t_{2,i}$, the bottom label of that column. Cycle through r to produce a row-increasing tableau T of shape (k,k). This is shown for all row-increasing tableaux of shape $(3,3)$ in Figure 6; r labels are designated by brackets, the image tableau is the adjacent tableau to the left, where the image of r is designated by parentheses. A proof analogous to the preceding shows that T has exactly one less decreasing column than U. It is also clear that these are inverse maps, establishing bijections between adjacent pairs of $D_0, D_1, \dots, D_k$. The full result follows from transitivity. $\square$

Since the $\binom{2k}{k}$ row-increasing tableaux of shape (k,k) are equally distributed between the $k+1$ sets $D_0, D_1, \dots, D_k$, it follows that each set has cardinality $|D_i| = \frac{1}{k+1}\binom{2k}{k}$, the kth Catalan number. This re-proves and generalizes the result about Young tableaux (D_0 here). If one converts to lattice paths at every step of the proof, this is equivalent to a case of Callan's proof of the Chung-Feller theorem [C].

4. Future Directions and Acknowledgments

We hope that tableau cycling will be a helpful tool for establishing combinatorial verifications of formulas for the number of various Young tableaux, including those with more than 2 rows, while remaining within a tableaux context.

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