RECURRENCE FORMULAE FOR MULTI-POLY-BERNOULLI NUMBERS

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Abstract

In this paper we establish recurrence formulae for multi-poly-Bernoulli numbers.

–Dedicated to Professor Ryuichi Tanaka on the occasion of his sixtieth birthday

1. Introduction and Background

Let us briefly recall poly-Bernoulli numbers.

For an integer $k \in \mathbb{Z}$, put

$$Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

The formal power series $Li_k(z)$ is the k-th polylogarithm if $k \geq 1$, and a rational function if $k \leq 0$. When $k = 1$, $Li_1(z) = -\log(1-z)$. The formal power series $Li_k(z)$ can be used to introduce poly-Bernoulli numbers. The rational numbers $B_n^{(k)}$ $(n = 0, 1, 2, \ldots)$ are said to be *poly-Bernoulli numbers* if they satisfy

$$\frac{Li_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}.$$

In addition, for any $n \geq 0$, $B_n^{(1)}$ is the classical Bernoulli number, B_n. It was shown in [1] that special values of certain zeta functions at non-positive integers can be described in terms of poly-Bernoulli numbers. Furthermore, Kaneko [8] presented the following recurrences for poly-Bernoulli numbers.

Theorem 1 (Kaneko) (1) *For any $k \in \mathbb{Z}$ and $n \geq 0$,*

$$B_n^{(k)} = \frac{1}{n+1}\left\{B_n^{(k-1)} - \sum_{m=1}^{n-1}\binom{n}{m-1}B_m^{(k)}\right\}.$$

(2) *For any $k \geq 1$ and $n \geq 0$,*

$$B_n^{(k)} = \sum_{m=0}^{n}(-1)^m\binom{n}{m}B_{n-m}^{(k-1)}\left\{\sum_{l=0}^{m}\frac{(-1)^l}{n-l+1}\binom{m}{l}B_l^{(1)}\right\}.$$

In this paper we consider generalized poly-Bernoulli numbers, which we refer to as *multi-poly-Bernoulli numbers*. Kim-Kim [9] introduced these numbers and proved that special values of certain zeta functions at non-positive integers can be described in terms of these numbers. In [5], we established a closed formula, and a duality property for special multi-poly-Bernoulli numbers. Bernoulli numbers satisfy certain recurrence relationships, which are used in many computations involving Bernoulli numbers. Obtaining a recurrence formula for multi-poly-Bernoulli numbers therefore seems to be a natural and important problem. The objective of this paper is thus to establish some recurrence formulae for multi-poly-Bernoulli numbers. It will be apparent that these recurrence formulae (Theorems 6, 7) are similar to those in the theorem above.

2. Multi-poly-Bernoulli Numbers

We first define a generalization of $Li_k(z)$. Let r be an integer with a value greater than one.

Definition 2 Let $k_1, k_2, \ldots, k_r$ be integers. Define

$$Li_{k_1,k_2,\ldots,k_r}(z) = \sum_{\substack{m_1,m_2,\ldots,m_r\in\mathbb{Z}\\ 0<m_1<m_2<\cdots<m_r}} \frac{z^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}}.$$

Next let us establish the following fundamental result.

Lemma 3

$$\frac{d}{dz}Li_{k_1,k_2,\ldots,k_r}(z) = \begin{cases} \frac{1}{z}Li_{k_1,k_2,\ldots,k_{r-1},k_r-1}(z) & (k_r \neq 1) \\ \frac{1}{1-z}Li_{k_1,k_2,\ldots,k_{r-1}}(z) & (k_r = 1) \end{cases}. \quad (1)$$

The former case may be proven by means of a simple calculation. The latter case follows from

$$\begin{aligned}\sum_{0<m_1<\cdots<m_{r-1}<m_r}\frac{z^{m_r-1}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}} &= \sum_{0<m_1<\cdots<m_{r-1}}\sum_{m_r=m_{r-1}+1}^{\infty}\frac{z^{m_r-1}}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}}\\ &= \sum_{0<m_1<\cdots<m_{r-1}}\frac{1}{m_1^{k_1}\cdots m_{r-1}^{k_{r-1}}}\cdot\frac{z^{m_{r-1}}}{1-z}.\end{aligned}$$

This lemma will be required later.

Let us now introduce a generalization of poly-Bernoulli numbers, making use of $Li_{k_1,k_2,\ldots,k_r}(z)$.

Definition 4 Multi-poly-Bernoulli numbers $B_n^{(k_1,k_2,\ldots,k_r)}$ $(n = 0, 1, 2, \ldots)$ are defined for each integer $k_1, k_2, \ldots, k_r$ by the generating series

$$\frac{Li_{k_1,k_2,\ldots,k_r}(1-e^{-t})}{(1-e^{-t})^r} = \sum_{n=0}^{\infty} B_n^{(k_1,k_2,\ldots,k_r)} \frac{t^n}{n!}. \tag{2}$$

By definition, the left-hand side of (2) is

$$\frac{1}{1^{k_1}2^{k_2}\cdots r^{k_r}} + \sum_{\substack{0<m_1<\cdots<m_r \\ m_r \neq r}} \frac{(1-e^{-t})^{m_r-r}}{m_1^{k_1}\cdots m_r^{k_r}}.$$

Hence we have,

Proposition 5

$$B_0^{(k_1,k_2,\ldots,k_r)} = \frac{1}{1^{k_1}2^{k_2}\cdots r^{k_r}}.$$

The following tables show the values of $B_n^{(k_1,k_2)}$ and $B_n^{(k_1,k_2,k_3)}$ for small n, k_i.

$\backslash n$	0	1	2	3	4	5	6	7
$B_n^{(1,1)}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{1}{4}$	$\frac{1}{20}$	$-\frac{1}{12}$	$\frac{5}{84}$	$\frac{1}{12}$
$B_n^{(1,0)}$	1	$\frac{3}{2}$	$\frac{13}{6}$	3	$\frac{119}{30}$	5	$\frac{253}{42}$	7
$B_n^{(0,1)}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	$\frac{29}{30}$	$\frac{31}{30}$	$\frac{43}{42}$	$\frac{41}{42}$	$\frac{29}{30}$
$B_n^{(0,0)}$	1	2	4	8	16	32	64	128
$B_n^{(0,-1)}$	2	6	18	54	162	486	1458	2574
$B_n^{(-1,0)}$	1	3	9	27	81	243	729	1287
$B_n^{(-1,-1)}$	2	9	39	165	687	2829	11505	46965

$\backslash n$	0	1	2	3	4	5	6	7
$B_n^{(1,1,1)}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{19}{60}$	$\frac{1}{8}$	$\frac{8}{63}$	$\frac{5}{24}$
$B_n^{(1,1,0)}$	$\frac{1}{2}$	1	$\frac{23}{12}$	$\frac{7}{2}$	$\frac{121}{20}$	$\frac{59}{6}$	$\frac{1255}{84}$	$\frac{127}{6}$
$B_n^{(1,0,1)}$	$\frac{1}{3}$	$\frac{5}{8}$	$\frac{133}{120}$	$\frac{221}{120}$	$\frac{2383}{840}$	$\frac{673}{168}$	$\frac{4321}{840}$	$\frac{145}{24}$
$B_n^{(0,1,1)}$	$\frac{1}{6}$	$\frac{7}{24}$	$\frac{19}{40}$	$\frac{17}{24}$	$\frac{53}{56}$	$\frac{185}{168}$	$\frac{303}{280}$	$\frac{109}{120}$
$B_n^{(1,0,0)}$	1	$\frac{5}{2}$	$\frac{37}{6}$	15	$\frac{1079}{30}$	85	$\frac{8317}{42}$	455
$B_n^{(0,1,0)}$	$\frac{1}{2}$	$\frac{7}{6}$	$\frac{8}{3}$	$\frac{179}{30}$	$\frac{147}{10}$	$\frac{1177}{42}$	$\frac{1238}{21}$	$\frac{3659}{30}$
$B_n^{(0,0,1)}$	$\frac{1}{3}$	$\frac{3}{4}$	$\frac{33}{20}$	$\frac{71}{20}$	$\frac{1047}{140}$	$\frac{433}{28}$	$\frac{4411}{140}$	$\frac{1271}{20}$
$B_n^{(0,0,0)}$	1	3	9	27	81	243	729	2187
$B_n^{(0,0,-1)}$	3	12	48	192	768	3072	12288	49152
$B_n^{(0,-1,0)}$	2	8	32	128	512	2048	8192	32768
$B_n^{(-1,0,0)}$	1	4	16	64	256	1024	4096	16384
$B_n^{(0,-1,-1)}$	6	32	168	872	4488	22952	116808	592232
$B_n^{(-1,0,-1)}$	3	16	84	436	2244	11476	58404	296116
$B_n^{(-1,-1,0)}$	2	11	59	311	1619	8351	42779	217991
$B_n^{(-1,-1,-1)}$	6	44	306	2054	13446	86414	547686	3434174

The remainder of this paper deals with recurrences for multi-poly-Bernoulli numbers.

3. Recurrence Formulae

We present two kinds of recurrence formulae for multi-poly-Bernoulli numbers. The first formula is as follows.

Theorem 6(1) *If $k_r \neq 1$ and $n \geq 1$, then*

$$B_n^{(k_1,k_2,\ldots,k_r)} = \frac{1}{n+r}\left\{ B_n^{(k_1,\ldots,k_{r-1},k_r-1)} - \sum_{m=1}^{n-1} \binom{n}{m-1} B_m^{(k_1,k_2,\ldots,k_r)} \right\}.$$

(2) *If $k_r = 1$ and $n \geq 1$, then*

$$B_n^{(k_1,\ldots,k_{r-1},1)} =$$

$$\frac{1}{n+r}\left[B_n^{(k_1,\ldots,k_{r-1})} - \sum_{m=0}^{n-1} (-1)^{n-m} \left\{ r\binom{n}{m} + \binom{n}{m-1} \right\} B_m^{(k_1,\ldots,k_{r-1},1)} \right].$$

Proof. (1) Differentiate both sides of

$$Li_{k_1,k_2,\ldots,k_r}(1-e^{-t}) = (1-e^{-t})^r \sum_{n=0}^{\infty} B_n^{(k_1,k_2,\ldots,k_r)} \frac{t^n}{n!}.$$

By (1), we have

$$\text{L.H.S.} = \frac{e^{-t}}{1-e^{-t}} Li_{k_1,\ldots,k_{r-1},k_r-1}(1-e^{-t}).$$

On the other hand,

$$\begin{aligned}\text{R.H.S.} &= r(1-e^{-t})^{r-1}e^{-t}\sum_{n=0}^{\infty} B_n^{(k_1,k_2,\ldots,k_r)}\frac{t^n}{n!}\\ &\quad +(1-e^{-t})^r\sum_{n=1}^{\infty} B_n^{(k_1,k_2,\ldots,k_r)}\frac{t^{n-1}}{(n-1)!}.\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{Li_{k_1,k_2,\ldots,k_r-1}(1-e^{-t})}{(1-e^{-t})^r}\\ &= r\sum_{n=0}^{\infty} B_n^{(k_1,k_2,\ldots,k_r)}\frac{t^n}{n!} + (e^t-1)\sum_{n=1}^{\infty} B_n^{(k_1,k_2,\ldots,k_r)}\frac{t^{n-1}}{(n-1)!}\\ &= r\sum_{n=0}^{\infty} B_n^{(k_1,k_2,\ldots,k_r)}\frac{t^n}{n!} + \sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{m=1}^{\infty} B_m^{(k_1,k_2,\ldots,k_r)}\frac{t^{m-1}}{(m-1)!}\\ &= r\sum_{n=0}^{\infty} B_n^{(k_1,k_2,\ldots,k_r)}\frac{t^n}{n!} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_m^{(k_1,k_2,\ldots,k_r)}\frac{t^{n+m-1}}{n!(m-1)!}.\end{aligned}$$

Here we put $l = n+m-1$. The right-hand side of the last equation is then equal to

$$\begin{aligned}&r\sum_{n=0}^{\infty} B_n^{(k_1,k_2,\ldots,k_r)}\frac{t^n}{n!} + \sum_{m=1}^{\infty}\sum_{l=m}^{\infty} B_m^{(k_1,k_2,\ldots,k_r)}\frac{l!}{(l-(m-1))!(m-1)!}\cdot\frac{t^l}{l!}\\ &= r\sum_{n=0}^{\infty} B_n^{(k_1,k_2,\ldots,k_r)}\frac{t^n}{n!} + \sum_{l=1}^{\infty}\sum_{m=1}^{l} B_m^{(k_1,k_2,\ldots,k_r)}\binom{l}{m-1}\frac{t^l}{l!}\\ &= r\sum_{n=0}^{\infty} B_n^{(k_1,k_2,\ldots,k_r)}\frac{t^n}{n!} + \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\binom{n}{m-1}B_m^{(k_1,k_2,\ldots,k_r)}\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of both sides, for each $n \geq 1$,

$$\begin{aligned}B_n^{(k_1,\ldots,k_{r-1},k_r-1)} &= rB_n^{(k_1,k_2,\ldots,k_r)} + \sum_{m=1}^{n}\binom{n}{m-1}B_m^{(k_1,k_2,\ldots,k_r)}\\ &= (n+r)B_n^{(k_1,k_2,\ldots,k_r)} + \sum_{m=1}^{n-1}\binom{n}{m-1}B_m^{(k_1,k_2,\ldots,k_r)}.\end{aligned}$$

This implies the result.

(2) Differentiate both sides of

$$Li_{k_1,\ldots,k_{r-1},1}(1-e^{-t}) = (1-e^{-t})^r\sum_{n=0}^{\infty} B_n^{(k_1,\ldots,k_{r-1},1)}\frac{t^n}{n!}.$$

$$\begin{aligned}
\text{L.H.S.} &= \frac{e^{-t}}{1-(1-e^{-t})} Li_{k_1,\dots,k_{r-1}}(1-e^{-t}) \\
&= Li_{k_1,\dots,k_{r-1}}(1-e^{-t}), \\
\text{R.H.S.} &= r(1-e^{-t})^{r-1} e^{-t} \sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_{r-1},1)} \frac{t^n}{n!} + (1-e^{-t})^r \sum_{n=1}^{\infty} B_n^{(k_1,\dots,k_{r-1},1)} \frac{t^{n-1}}{(n-1)!}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_{r-1})} \frac{t^n}{n!} \\
&\quad = r e^{-t} \sum_{n=0}^{\infty} B_n^{(k_1,\dots,k_{r-1},1)} \frac{t^n}{n!} + (1-e^{-t}) \sum_{n=1}^{\infty} B_n^{(k_1,\dots,k_{r-1},1)} \frac{t^{n-1}}{(n-1)!} \\
&\quad = r \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \sum_{m=0}^{\infty} B_m^{(k_1,\dots,k_{r-1},1)} \frac{t^m}{m!} - \sum_{n=1}^{\infty} \frac{(-t)^n}{n!} \sum_{m=0}^{\infty} B_{m+1}^{(k_1,\cdots,k_{r-1},1)} \frac{t^m}{m!} \\
&\quad = r \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n B_m^{(k_1,\dots,k_{r-1},1)} \frac{t^{n+m}}{n!m!} - \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} (-1)^n B_{m+1}^{(k_1,\dots,k_{r-1},1)} \frac{t^{n+m}}{n!m!}.
\end{aligned}$$

Here we put $l = n + m$. The right-hand side of the last equation then becomes

$$\begin{aligned}
& r \sum_{m=0}^{\infty} \sum_{l=m}^{\infty} (-1)^{l-m} B_m^{(k_1,\dots,k_{r-1},1)} \frac{l!}{(l-m)!m!} \cdot \frac{t^l}{l!} \\
&\quad - \sum_{m=0}^{\infty} \sum_{l=m+1}^{\infty} (-1)^{l-m} B_{m+1}^{(k_1,\dots,k_{r-1},1)} \frac{l!}{(l-m)!m!} \cdot \frac{t^l}{l!} \\
&= r \sum_{l=0}^{\infty} \sum_{m=0}^{l} (-1)^{l-m} B_m^{(k_1,\dots,k_{r-1},1)} \binom{l}{m} \frac{t^l}{l!} \\
&\quad - \sum_{l=1}^{\infty} \sum_{m=0}^{l-1} (-1)^{l-m} B_{m+1}^{(k_1,\dots,k_{r-1},1)} \binom{l}{m} \frac{t^l}{l!} \\
&= r B_0^{(k_1,\dots,k_{r-1},1)} \\
&\quad + \sum_{n=1}^{\infty} \left\{ \sum_{m=0}^{n} (-1)^{n-m} r B_m^{(k_1,\dots,k_{r-1},1)} \binom{n}{m} - \sum_{m=0}^{n-1} (-1)^{n-m} B_{m+1}^{(k_1,\dots,k_{r-1},1)} \binom{n}{m} \right\} \frac{t^n}{n!}.
\end{aligned}$$

Comparing both sides, for each $n \geq 1$,

$$\begin{aligned}
& B_n^{(k_1,\dots,k_{r-1})} \\
&= \sum_{m=0}^{n} (-1)^{n-m} r \binom{n}{m} B_m^{(k_1,\dots,k_{r-1},1)} + \sum_{m=1}^{n} (-1)^{n-m} \binom{n}{m-1} B_m^{(k_1,\dots,k_{r-1},1)} \\
&= (n+r) B_n^{(k_1,\dots,k_{r-1},1)} + \sum_{m=0}^{n-1} (-1)^{n-m} \left\{ r \binom{n}{m} + \binom{n}{m-1} \right\} B_m^{(k_1,\dots,k_{r-1},1)}.
\end{aligned}$$

These computations imply the result.

We obtain the second formula using the integral representation of $Li_{k_1,k_2,\ldots,k_r}(1-e^{-t})$.

Theorem 7 (1) *If $k_r \neq 1$, then for any $n \geq 0$,*

$$\begin{aligned}
&B_n^{(k_1,k_2,\ldots,k_r)}\\
&= (-1)^{r-1}\frac{n!}{(n+r-1)!}\\
&\times\sum_{m=0}^{n}\left\{\sum_{p=0}^{n-m}\sum_{\substack{j_1+\cdots+j_{r-1}\\=n-m+r-1-p}}\frac{(-1)^{n-m+r-1-p}(n-m+r-1-p)!}{j_1!\cdots j_{r-1}!}\binom{n-m+r-1}{p}B_p^{(k_1,\ldots,k_{r-1},k_r-1)}\right\}\\
&\qquad\times(-1)^m\binom{n+r-1}{m}\sum_{l=0}^{m}\frac{(-1)^l}{n-l+r}\binom{m}{l}\left\{\sum_{\substack{i_1+\cdots+i_r\\=l}}B_{i_1}^{(1)}\cdots B_{i_r}^{(1)}\frac{l!}{i_1!\cdots i_r!}\right\}.
\end{aligned}$$

(2) *If $k_r = 1$, then for any $n \geq 0$,*

$$\begin{aligned}
&B_n^{(k_1,k_2,\ldots,k_r)}\\
&= (-1)^{r-1}\frac{n!}{(n+r-1)!}\\
&\times\sum_{m=0}^{n}\left\{\sum_{p=0}^{n-m}\sum_{\substack{j_1+\cdots+j_{r-1}\\=n-m+r-1-p}}\frac{(-1)^{n-m+r-1-p}(n-m+r-1-p)!}{j_1!\cdots j_{r-1}!}\binom{n-m+r-1}{p}B_p^{(k_1,\ldots,k_{r-1})}\right\}\\
&\qquad\times\binom{n+r-1}{m}\frac{1}{n-m+r}\left\{\sum_{\substack{i_1+\cdots+i_r\\=m}}B_{i_1}^{(1)}\cdots B_{i_r}^{(1)}\frac{m!}{i_1!\cdots i_r!}\right\}.
\end{aligned}$$

Proof. (1) Since

$$\frac{d}{dt}Li_{k_1,k_2,\ldots,k_r}(1-e^{-t}) = \frac{e^{-t}}{1-e^{-t}}Li_{k_1,\ldots,k_{r-1},k_r-1}(1-e^{-t}),$$

we have

$$Li_{k_1,k_2,\ldots,k_r}(1-e^{-t}) = \int_0^t \frac{e^{-s}}{1-e^{-s}}Li_{k_1,\ldots,k_{r-1},k_r-1}(1-e^{-s})ds.$$

By this equation,

$$\begin{aligned}
&\sum_{n=0}^{\infty}B_n^{(k_1,k_2,\ldots,k_r)}\frac{t^n}{n!}\\
&\quad= \frac{1}{(1-e^{-t})^r}\int_0^t \frac{e^{-s}}{1-e^{-s}}Li_{k_1,\ldots,k_{r-1},k_r-1}(1-e^{-s})ds\\
&\quad= \left(\frac{e^t}{e^t-1}\right)^r\int_0^t e^{-s}(1-e^{-s})^{r-1}\cdot\frac{Li_{k_1,\ldots,k_{r-1},k_r-1}(1-e^{-s})}{(1-e^{-s})^r}ds\\
&\quad= \left(\sum_{n=0}^{\infty}B_n^{(1)}\frac{t^{n-1}}{n!}\right)^r\int_0^t\sum_{n=0}^{\infty}\frac{(-s)^n}{n!}\left(-\sum_{n=1}^{\infty}\frac{(-s)^n}{n!}\right)^{r-1}\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_{r-1},k_r-1)}\frac{s^n}{n!}ds
\end{aligned}$$

$$= (-1)^{r-1}\left(\sum_{n=0}^{\infty} B_n^{(1)}\frac{t^{n-1}}{n!}\right)^r$$
$$\times \int_0^t \sum_{n=0}^{\infty}\frac{(-s)^n}{n!}\sum_{n=r-1}^{\infty}\sum_{\substack{j_1+\cdots+j_{r-1}\\=n}}\frac{(-1)^n s^n}{j_1!\cdots j_{r-1}!}\sum_{n=0}^{\infty}B_n^{(k_1,\ldots,k_{r-1},k_r-1)}\frac{s^n}{n!}ds.$$

We write I_1 for the integral part of the last equation.

$$I_1 = \int_0^t \sum_{n=0}^{\infty}\frac{(-s)^n}{n!}\sum_{m=0}^{\infty}\sum_{n=r-1}^{\infty}\sum_{\substack{j_1+\cdots+j_{r-1}\\=n}}\frac{(-1)^n n!}{j_1!\cdots j_{r-1}!}B_m^{(k_1,\ldots,k_{r-1},k_r-1)}\frac{s^{n+m}}{n!m!}ds.$$

Putting $l = n+m$,

$$I_1 = \int_0^t \sum_{n=0}^{\infty}\frac{(-s)^n}{n!}\sum_{m=0}^{\infty}\sum_{l=m+r-1}^{\infty}\sum_{\substack{j_1+\cdots+j_{r-1}\\=l-m}}\frac{(-1)^{l-m}(l-m)!}{j_1!\cdots j_{r-1}!}B_m^{(k_1,\ldots,k_{r-1},k_r-1)}\frac{l!}{(l-m)!m!}\cdot\frac{s^l}{l!}ds$$

$$= \int_0^t \sum_{n=0}^{\infty}\frac{(-s)^n}{n!}\sum_{l=r-1}^{\infty}\sum_{m=0}^{l-r+1}\sum_{\substack{j_1+\cdots+j_{r-1}\\=l-m}}\frac{(-1)^{l-m}(l-m)!}{j_1!\cdots j_{r-1}!}B_m^{(k_1,\ldots,k_{r-1},k_r-1)}\binom{l}{m}\frac{s^l}{l!}ds$$

$$= \int_0^t \sum_{l=r-1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{l-r+1}(-1)^n\sum_{\substack{j_1+\cdots+j_{r-1}\\=l-m}}\frac{(-1)^{l-m}(l-m)!}{j_1!\cdots j_{r-1}!}B_m^{(k_1,\ldots,k_{r-1},k_r-1)}\binom{l}{m}\frac{s^{n+l}}{n!l!}ds.$$

We put $a = l+n$. Then

$$I_1 = \int_0^t \sum_{l=r-1}^{\infty}\sum_{a=l}^{\infty}\sum_{m=0}^{l-r+1}(-1)^{a-l}\sum_{\substack{j_1+\cdots+j_{r-1}\\=l-m}}\frac{(-1)^{l-m}(l-m)!}{j_1!\cdots j_{r-1}!}B_m^{(k_1,\ldots,k_{r-1},k_r-1)}\binom{l}{m}\frac{a!}{(a-l)!l!}\cdot\frac{s^a}{a!}ds$$

$$= \int_0^t \sum_{a=r-1}^{\infty}\sum_{l=r-1}^{a}\sum_{m=0}^{l-r+1}(-1)^{a-l}\sum_{\substack{j_1+\cdots+j_{r-1}\\=l-m}}\frac{(-1)^{l-m}(l-m)!}{j_1!\cdots j_{r-1}!}B_m^{(k_1,\ldots,k_{r-1},k_r-1)}\binom{l}{m}\binom{a}{l}\frac{s^a}{a!}ds$$

$$= \sum_{a=r-1}^{\infty}\sum_{l=r-1}^{a}\sum_{m=0}^{l-r+1}(-1)^{a-l}\sum_{\substack{j_1+\cdots+j_{r-1}\\=l-m}}\frac{(-1)^{l-m}(l-m)!}{j_1!\cdots j_{r-1}!}B_m^{(k_1,\ldots,k_{r-1},k_r-1)}\binom{l}{m}\binom{a}{l}\frac{t^{a+1}}{(a+1)!}.$$

Hence,

$$(-1)^{r-1}\left(\sum_{n=0}^{\infty}B_n^{(1)}\frac{t^{n-1}}{n!}\right)^r I_1$$
$$= (-1)^{r-1}\sum_{n=0}^{\infty}\sum_{\substack{i_1+\cdots+i_r\\=n}}B_{i_1}^{(1)}\cdots B_{i_r}^{(1)}\frac{t^{n-r}}{i_1!\cdots i_r!}$$
$$\times \sum_{a=r-1}^{\infty}\sum_{l=r-1}^{a}\sum_{m=0}^{l-r+1}(-1)^{a-l}\sum_{\substack{j_1+\cdots+j_{r-1}\\=l-m}}\frac{(-1)^{l-m}(l-m)!}{j_1!\cdots j_{r-1}!}B_m^{(k_1,\ldots,k_{r-1},k_r-1)}\binom{l}{m}\binom{a}{l}\frac{t^{a+1}}{(a+1)!}$$

$$= (-1)^{r-1} \sum_{a=r-1}^{\infty} \sum_{n=0}^{\infty} \sum_{l=r-1}^{a} \sum_{m=0}^{l-r+1} (-1)^{a-l} \sum_{\substack{i_1+\cdots+i_r \\ =n}} B_{i_1}^{(1)} \cdots B_{i_r}^{(1)} \frac{n!}{i_1! \cdots i_r!}$$
$$\times \sum_{\substack{j_1+\cdots+j_{r-1} \\ =l-m}} \frac{(-1)^{l-m}(l-m)!}{j_1! \cdots j_{r-1}!} B_m^{(k_1,\ldots,k_{r-1},k_r-1)} \binom{l}{m} \binom{a}{l} \frac{t^{n+a+1}}{n!(a+1)!} \cdot t^{-r}$$

$$= (-1)^{r-1} \sum_{a=r-1}^{\infty} \sum_{b=a+1}^{\infty} \sum_{l=r-1}^{a} \sum_{m=0}^{l-r+1} (-1)^{a-l} \sum_{\substack{i_1+\cdots+i_r \\ =b-a-1}} B_{i_1}^{(1)} \cdots B_{i_r}^{(1)} \frac{(b-a-1)!}{i_1! \cdots i_r!}$$
$$\times \sum_{\substack{j_1+\cdots+j_{r-1} \\ =l-m}} \frac{(-1)^{l-m}(l-m)!}{j_1! \cdots j_{r-1}!} B_m^{(k_1,\ldots,k_{r-1},k_r-1)} \binom{l}{m} \binom{a}{l} \frac{b!}{(b-(a+1))!(a+1)!} \cdot \frac{t^b}{b!} \cdot t^{-r}$$
$$(\text{put } b = n+a+1)$$

$$= (-1)^{r-1} \sum_{b=r}^{\infty} \sum_{a=r-1}^{b-1} \sum_{l=r-1}^{a} \sum_{m=0}^{l-r+1} (-1)^{a-l} \sum_{\substack{i_1+\cdots+i_r \\ =b-a-1}} B_{i_1}^{(1)} \cdots B_{i_r}^{(1)} \frac{(b-a-1)!}{i_1! \cdots i_r!}$$
$$\times \sum_{\substack{j_1+\cdots+j_{r-1} \\ =l-m}} \frac{(-1)^{l-m}(l-m)!}{j_1! \cdots j_{r-1}!} B_m^{(k_1,\ldots,k_{r-1},k_r-1)} \binom{l}{m} \binom{a}{l} \binom{b}{a+1} \frac{t^{b-r}}{b!}$$

$$= (-1)^{r-1} \sum_{n=0}^{\infty} \sum_{a=r-1}^{n+r-1} \sum_{l=r-1}^{a} \sum_{m=0}^{l-r+1} (-1)^{a-l} \sum_{\substack{i_1+\cdots+i_r \\ =n-a+r-1}} B_{i_1}^{(1)} \cdots B_{i_r}^{(1)} \frac{(n-a+r-1)!}{i_1! \cdots i_r!}$$
$$\times \sum_{\substack{j_1+\cdots+j_{r-1} \\ =l-m}} \frac{(-1)^{l-m}(l-m)!}{j_1! \cdots j_{r-1}!} B_m^{(k_1,\ldots,k_{r-1},k_r-1)} \binom{l}{m} \binom{a}{l} \binom{n+r}{a+1} \frac{t^n}{(n+r)!}$$
$$(\text{put } n = b-r)$$

$$= \sum_{n=0}^{\infty} \Bigg\{ (-1)^{r-1} \frac{n!}{(n+r)!} \sum_{a=r-1}^{n+r-1} \left(\sum_{\substack{i_1+\cdots+i_r \\ =n-a+r-1}} B_{i_1}^{(1)} \cdots B_{i_r}^{(1)} \frac{(n-a+r-1)!}{i_1! \cdots i_r!} \right)$$
$$\times \sum_{l=r-1}^{a} (-1)^{a-l} \binom{n+r}{a+1} \binom{a}{l} \sum_{m=0}^{l-r+1} \sum_{\substack{j_1+\cdots+j_{r-1} \\ =l-m}} \frac{(-1)^{l-m}(l-m)!}{j_1! \cdots j_{r-1}!} \binom{l}{m} B_m^{(k_1,\ldots,k_{r-1},k_r-1)} \Bigg\} \frac{t^n}{n!}.$$

If we compare the coefficients of both sides,

$$
\begin{aligned}
&B_n^{(k_1,k_2,\dots,k_r)}\\
&= (-1)^{r-1}\frac{n!}{(n+r)!}\sum_{a=r-1}^{n+r-1}\left(\sum_{\substack{i_1+\cdots+i_r\\=n-a+r-1}} B_{i_1}^{(1)}\cdots B_{i_r}^{(1)}\frac{(n-a+r-1)!}{i_1!\cdots i_r!}\right)\\
&\times\sum_{l=r-1}^{a}(-1)^{a-l}\binom{n+r}{a+1}\binom{a}{l}\sum_{m=0}^{l-r+1}\sum_{\substack{j_1+\cdots+j_{r-1}\\=l-m}}\frac{(-1)^{l-m}(l-m)!}{j_1!\cdots j_{r-1}!}\binom{l}{m}B_m^{(k_1,\dots,k_{r-1},k_r-1)}
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{r-1}\frac{n!}{(n+r-1)!}\sum_{l=r-1}^{n+r-1}\sum_{m=0}^{l-r+1}\sum_{\substack{j_1+\cdots+j_{r-1}\\=l-m}}\frac{(-1)^{l-m}(l-m)!}{j_1!\cdots j_{r-1}!}\binom{l}{m}B_m^{(k_1,\dots,k_{r-1},k_r-1)}\\
&\times\sum_{a=l}^{n+r-1}\frac{(-1)^{a-l}}{a+1}\binom{n+r-1}{a}\binom{a}{l}\left(\sum_{\substack{i_1+\cdots+i_r\\=n-a+r-1}} B_{i_1}^{(1)}\cdots B_{i_r}^{(1)}\frac{(n-a+r-1)!}{i_1!\cdots i_r!}\right)\\
&\qquad\left(\text{by } \binom{n+r}{a+1}=\frac{n+r}{a+1}\binom{n+r-1}{a} \text{ and } \sum_{a=r-1}^{n+r-1}\sum_{l=r-1}^{a}=\sum_{l=r-1}^{n+r-1}\sum_{a=l}^{n+r-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{r-1}\frac{n!}{(n+r-1)!}\\
&\times\sum_{l=r-1}^{n+r-1}(-1)^{l}\binom{n+r-1}{l}\left\{\sum_{m=0}^{l-r+1}\sum_{\substack{j_1+\cdots+j_{r-1}\\=l-m}}\frac{(-1)^{l-m}(l-m)!}{j_1!\cdots j_{r-1}!}\binom{l}{m}B_m^{(k_1,\dots,k_{r-1},k_r-1)}\right\}\\
&\times\sum_{a=l}^{n+r-1}\frac{(-1)^{a}}{a+1}\binom{n+r-1-l}{n+r-1-a}\left\{\sum_{\substack{i_1+\cdots+i_r\\=n-a+r-1}} B_{i_1}^{(1)}\cdots B_{i_r}^{(1)}\frac{(n-a+r-1)!}{i_1!\cdots i_r!}\right\}\\
&\qquad\left(\text{by } \binom{n+r-1}{a}\binom{a}{l}=\binom{n+r-1}{l}\binom{n+r-1-l}{n+r-1-a}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{r-1}\frac{n!}{(n+r-1)!}\sum_{l'=0}^{n}(-1)^{l'+r-1}\binom{n+r-1}{l'+r-1}\\
&\times\left\{\sum_{m=0}^{l'}\sum_{\substack{j_1+\cdots+j_{r-1}\\=l'+r-1-m}}\frac{(-1)^{l'+r-1-m}(l'+r-1-m)!}{j_1!\cdots j_{r-1}!}\binom{l'+r-1}{m}B_m^{(k_1,\dots,k_{r-1},k_r-1)}\right\}\\
&\times\sum_{a=l'+r-1}^{n+r-1}\frac{(-1)^{a}}{a+1}\binom{n-l'}{n+r-1-a}\left\{\sum_{\substack{i_1+\cdots+i_r\\=n-a+r-1}} B_{i_1}^{(1)}\cdots B_{i_r}^{(1)}\frac{(n-a+r-1)!}{i_1!\cdots i_r!}\right\}\\
&\qquad(\text{put } l'=l-r+1)
\end{aligned}
$$

$$= \frac{n!}{(n+r-1)!} \sum_{l'=0}^{n} (-1)^{l'} \binom{n+r-1}{l'+r-1}$$
$$\times \left\{ \sum_{m=0}^{l'} \sum_{\substack{j_1+\cdots+j_{r-1} \\ =l'+r-1-m}} \frac{(-1)^{l'+r-1-m}(l'+r-1-m)!}{j_1!\cdots j_{r-1}!} \binom{l'+r-1}{m} B_m^{(k_1,\ldots,k_{r-1},k_r-1)} \right\}$$
$$\times \sum_{a'=l'}^{n} \frac{(-1)^{a'+r-1}}{a'+r} \binom{n-l'}{n-a'} \left\{ \sum_{\substack{i_1+\cdots+i_r \\ =n-a'}} B_{i_1}^{(1)} \cdots B_{i_r}^{(1)} \frac{(n-a')!}{i_1!\cdots i_r!} \right\}$$

(put $a' = a - r + 1$)

$$= \frac{n!}{(n+r-1)!} \sum_{l''=0}^{n} (-1)^{n-l''} \binom{n+r-1}{n+r-1-l''}$$
$$\times \left\{ \sum_{m=0}^{n-l''} \sum_{\substack{j_1+\cdots+j_{r-1} \\ =n-l''+r-1-m}} \frac{(-1)^{n-l''+r-1-m}(n-l''+r-1-m)!}{j_1!\cdots j_{r-1}!} \binom{n-l''+r-1}{m} B_m^{(k_1,\ldots,k_{r-1},k_r-1)} \right\}$$
$$\times \sum_{a'=n-l''}^{n} \frac{(-1)^{a'+r-1}}{a'+r} \binom{l''}{n-a'} \left\{ \sum_{\substack{i_1+\cdots+i_r \\ =n-a'}} B_{i_1}^{(1)} \cdots B_{i_r}^{(1)} \frac{(n-a')!}{i_1!\cdots i_r!} \right\}$$

(put $l'' = n - l'$)

$$= \frac{n!}{(n+r-1)!} \sum_{l''=0}^{n} (-1)^{n-l''} \binom{n+r-1}{l''}$$
$$\times \left\{ \sum_{m=0}^{n-l''} \sum_{\substack{j_1+\cdots+j_{r-1} \\ =n-l''+r-1-m}} \frac{(-1)^{n-l''+r-1-m}(n-l''+r-1-m)!}{j_1!\cdots j_{r-1}!} \binom{n-l''+r-1}{m} B_m^{(k_1,\ldots,k_{r-1},k_r-1)} \right\}$$
$$\times \sum_{a''=0}^{l''} \frac{(-1)^{n+r-1-a''}}{n-a''+r} \binom{l''}{a''} \left\{ \sum_{\substack{i_1+\cdots+i_r \\ =a''}} B_{i_1}^{(1)} \cdots B_{i_r}^{(1)} \frac{a''!}{i_1!\cdots i_r!} \right\}$$

(put $a'' = n - a'$)

$$= (-1)^{r-1}\frac{n!}{(n+r-1)!}\sum_{l''=0}^{n}(-1)^{l''}\binom{n+r-1}{l''}$$
$$\times\left\{\sum_{m=0}^{n-l''}\sum_{\substack{j_1+\cdots+j_{r-1}\\=n-l''+r-1-m}}\frac{(-1)^{n-l''+r-1-m}(n-l''+r-1-m)!}{j_1!\cdots j_{r-1}!}\binom{n-l''+r-1}{m}B_m^{(k_1,\dots,k_{r-1},k_r-1)}\right\}$$
$$\times\sum_{a''=0}^{l''}\frac{(-1)^{a''}}{n-a''+r}\binom{l''}{a''}\left\{\sum_{\substack{i_1+\cdots+i_r\\=a''}}B_{i_1}^{(1)}\cdots B_{i_r}^{(1)}\frac{a''!}{i_1!\cdots i_r!}\right\}.$$
$$(\text{put } a''=n-a')$$

This shows the claim.

(2) Since

$$\frac{d}{dt}Li_{k_1,k_2,\dots,k_{r-1},1}(1-e^{-t}) = Li_{k_1,\dots,k_{r-1}}(1-e^{-t}),$$

we have

$$Li_{k_1,\dots,k_{r-1},1}(1-e^{-t}) = \int_0^t Li_{k_1,\dots,k_{r-1}}(1-e^{-s})ds.$$

Hence

$$\sum_{n=0}^{\infty}B_n^{(k_1,\dots,k_{r-1},1)}\frac{t^n}{n!}$$
$$= \left(\frac{e^t}{e^t-1}\right)^r\int_0^t(1-e^{-s})^{r-1}\cdot\frac{Li_{k_1,\dots,k_{r-1}}(1-e^{-s})}{(1-e^{-s})^{r-1}}ds$$
$$= \left(\sum_{n=0}^{\infty}B_n^{(1)}\frac{t^{n-1}}{n!}\right)^r\int_0^t\left(-\sum_{n=1}^{\infty}\frac{(-s)^n}{n!}\right)^{r-1}\sum_{n=0}^{\infty}B_n^{(k_1,\dots,k_{r-1})}\frac{s^n}{n!}ds$$
$$= (-1)^{r-1}\left(\sum_{n=0}^{\infty}B_n^{(1)}\frac{t^{n-1}}{n!}\right)^r\int_0^t\sum_{n=r-1}^{\infty}\sum_{\substack{j_1+\cdots+j_{r-1}\\=n}}\frac{(-1)^n s^n}{j_1!\cdots j_{r-1}!}\sum_{m=0}^{\infty}B_m^{(k_1,\dots,k_{r-1})}\frac{s^m}{m!}ds.$$

Denote by I_2 the integral part of the last equation.

$$I_2 = \int_0^t\sum_{m=0}^{\infty}\sum_{n=r-1}^{\infty}\sum_{\substack{j_1+\cdots+j_{r-1}\\=n}}\frac{(-1)^n n!}{j_1!\cdots j_{r-1}!}B_m^{(k_1,\dots,k_{r-1})}\frac{s^{n+m}}{n!m!}ds$$
$$= \int_0^t\sum_{m=0}^{\infty}\sum_{l=m+r-1}^{\infty}\sum_{\substack{j_1+\cdots+j_{r-1}\\=l-m}}\frac{(-1)^{l-m}(l-m)!}{j_1!\cdots j_{r-1}!}B_n^{(k_1,\dots,k_{r-1})}\frac{l!}{(l-m)!m!}\cdot\frac{s^l}{l!}ds$$
$$(\text{put } l=n+m)$$
$$= \int_0^t\sum_{l=r-1}^{\infty}\sum_{m=0}^{l-r+1}\sum_{\substack{j_1+\cdots+j_{r-1}\\=l-m}}\frac{(-1)^{l-m}(l-m)!}{j_1!\cdots j_{r-1}!}B_m^{(k_1,\dots,k_{r-1})}\binom{l}{m}\frac{s^l}{l!}ds$$

$$= \sum_{l=r-1}^{\infty} \sum_{m=0}^{l-r+1} \sum_{\substack{j_1+\cdots+j_{r-1}\\=l-m}} \frac{(-1)^{l-m}(l-m)!}{j_1!\cdots j_{r-1}!} B_m^{(k_1,\ldots,k_{r-1})} \binom{l}{m} \frac{t^{l+1}}{(l+1)!}.$$

Thus,

$$\begin{aligned}
&(-1)^{r-1}\left(\sum_{n=0}^{\infty} B_n^{(1)} \frac{t^{n-1}}{n!}\right)^r I_2 \\
&= (-1)^{r-1} \sum_{n=0}^{\infty} \sum_{\substack{i_1+\cdots+i_r\\=n}} B_{i_1}^{(1)}\cdots B_{i_r}^{(1)} \frac{t^{n-r}}{i_1!\cdots i_r!} \\
&\quad\times \sum_{l=r-1}^{\infty} \sum_{m=0}^{l-r+1} \sum_{\substack{j_1+\cdots+j_{r-1}\\=l-m}} \frac{(-1)^{l-m}(l-m)!}{j_1!\cdots j_{r-1}!} B_m^{(k_1,\ldots,k_{r-1})} \binom{l}{m} \frac{t^{l+1}}{(l+1)!}
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{r-1} \sum_{l=r-1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{l-r+1} \sum_{\substack{i_1+\cdots+i_r\\=n}} B_{i_1}^{(1)}\cdots B_{i_r}^{(1)} \frac{n!}{i_1!\cdots i_r!} \sum_{\substack{j_1+\cdots+j_{r-1}\\=l-m}} \frac{(-1)^{l-m}(l-m)!}{j_1!\cdots j_{r-1}!} \\
&\quad\times B_m^{(k_1,\ldots,k_{r-1})} \binom{l}{m} \frac{t^{n+l+1}}{n!(l+1)!} \cdot t^{-r}
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{r-1} \sum_{l=r-1}^{\infty} \sum_{a=l+1}^{\infty} \sum_{m=0}^{l-r+1} \sum_{\substack{i_1+\cdots+i_r\\=a-l-1}} B_{i_1}^{(1)}\cdots B_{i_r}^{(1)} \frac{(a-l-1)!}{i_1!\cdots i_r!} \sum_{\substack{j_1+\cdots+j_{r-1}\\=l-m}} \frac{(-1)^{l-m}(l-m)!}{j_1!\cdots j_{r-1}!} \\
&\quad\times B_m^{(k_1,\ldots,k_{r-1})} \binom{l}{m} \frac{a!}{(a-(l+1))!(l+1)!} \cdot \frac{t^a}{a!} \cdot t^{-r} \\
&\qquad\qquad (\text{put } a = n+l+1)
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{r-1} \sum_{a=r}^{\infty} \sum_{l=r-1}^{a-1} \sum_{m=0}^{l-r+1} \sum_{\substack{i_1+\cdots+i_r\\=a-l-1}} B_{i_1}^{(1)}\cdots B_{i_r}^{(1)} \frac{(a-l-1)!}{i_1!\cdots i_r!} \sum_{\substack{j_1+\cdots+j_{r-1}\\=l-m}} \frac{(-1)^{l-m}(l-m)!}{j_1!\cdots j_{r-1}!} \\
&\quad\times B_m^{(k_1,\ldots,k_{r-1})} \binom{l}{m} \binom{a}{l+1} \frac{t^{a-r}}{a!}
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{r-1} \sum_{n=0}^{\infty} \sum_{l=r-1}^{n+r-1} \sum_{m=0}^{l-r+1} \sum_{\substack{i_1+\cdots+i_r\\=n-l+r-1}} B_{i_1}^{(1)}\cdots B_{i_r}^{(1)} \frac{n!}{i_1!\cdots i_r!} \sum_{\substack{j_1+\cdots+j_{r-1}\\=l-m}} \frac{(-1)^{l-m}(l-m)!}{j_1!\cdots j_{r-1}!} \\
&\quad\times B_m^{(k_1,\ldots,k_{r-1})} \binom{l}{m} \binom{n+r}{l+1} \frac{t^n}{(n+r)!} \\
&\qquad\qquad (\text{put } n = a-r)
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{(-1)^{r-1} n!}{(n+r)!} \sum_{l=r-1}^{n+r-1} \binom{n+r}{l+1} \sum_{\substack{i_1+\cdots+i_r \\ =n-l+r-1}} B_{i_1}^{(1)} \cdots B_{i_r}^{(1)} \frac{(n-l+r-1)!}{i_1! \cdots i_r!} \right.$$
$$\left. \times \sum_{m=0}^{l-r+1} \left(\begin{array}{c} l \\ m \end{array} \right) \sum_{\substack{j_1+\cdots+j_{r-1} \\ =l-m}} \frac{(-1)^{l-m}(l-m)!}{j_1! \cdots j_{r-1}!} B_m^{(k_1,\dots,k_{r-1})} \right\} \frac{t^n}{n!}.$$

If we compare the coefficients of both sides of the equation,

$$\begin{aligned} B_n^{(k_1,k_2,\dots,k_r)} &= \frac{(-1)^{r-1} n!}{(n+r)!} \sum_{l=r-1}^{n+r-1} \binom{n+r}{l+1} \left(\sum_{\substack{i_1+\cdots+i_r \\ =n-l+r-1}} B_{i_1}^{(1)} \cdots B_{i_r}^{(1)} \frac{(n-l+r-1)!}{i_1! \cdots i_r!} \right) \\ &\quad \times \sum_{m=0}^{l-r+1} \left(\sum_{\substack{j_1+\cdots+j_{r-1} \\ =l-m}} \frac{(-1)^{l-m}(l-m)!}{j_1! \cdots j_{r-1}!} \right) \left(\begin{array}{c} l \\ m \end{array} \right) B_m^{(k_1,\dots,k_{r-1})} \end{aligned}$$

$$\begin{aligned} &= \frac{(-1)^{r-1} n!}{(n+r-1)!} \sum_{l=r-1}^{n+r-1} \frac{1}{l+1} \binom{n+r-1}{l} \left(\sum_{\substack{i_1+\cdots+i_r \\ =n+r-1-l}} B_{i_1}^{(1)} \cdots B_{i_r}^{(1)} \frac{(n-l+r-1)!}{i_1! \cdots i_r!} \right) \\ &\quad \times \sum_{m=0}^{l-r+1} \left(\sum_{\substack{j_1+\cdots+j_{r-1} \\ =l-m}} \frac{(-1)^{l-m}(l-m)!}{j_1! \cdots j_{r-1}!} \right) \left(\begin{array}{c} l \\ m \end{array} \right) B_m^{(k_1,\dots,k_{r-1})} \\ &\qquad \left(\text{by } \binom{n+r}{l+1} = \frac{n+r}{l+1} \binom{n+r-1}{l} \right) \end{aligned}$$

$$\begin{aligned} &= \frac{(-1)^{r-1} n!}{(n+r-1)!} \sum_{a=0}^{n} \frac{1}{a+r} \binom{n+r-1}{a+r-1} \left(\sum_{\substack{i_1+\cdots+i_r \\ =n-a}} B_{i_1}^{(1)} \cdots B_{i_r}^{(1)} \frac{(n-a)!}{i_1! \cdots i_r!} \right) \\ &\quad \times \sum_{m=0}^{a} \left(\sum_{\substack{j_1+\cdots+j_{r-1} \\ =a+r-1-m}} \frac{(-1)^{a+r-1-m}(a+r-1-m)!}{j_1! \cdots j_{r-1}!} \right) \left(\begin{array}{c} a+r-1 \\ m \end{array} \right) B_m^{(k_1,\dots,k_{r-1})} \\ &\qquad (\text{put } a = l-r+1) \end{aligned}$$

$$\begin{aligned} &= \frac{(-1)^{r-1} n!}{(n+r-1)!} \sum_{l'=0}^{n} \frac{1}{n-l'+r} \binom{n+r-1}{n+r-1-l'} \left(\sum_{\substack{i_1+\cdots+i_r \\ =l'}} B_{i_1}^{(1)} \cdots B_{i_r}^{(1)} \frac{l'!}{i_1! \cdots i_r!} \right) \\ &\quad \times \sum_{m=0}^{n-l'} \left(\sum_{\substack{j_1+\cdots+j_{r-1} \\ =n-l'+r-1-m}} \frac{(-1)^{n-l'+r-1-m}(n-l'+r-1-m)!}{j_1! \cdots j_{r-1}!} \right) \binom{n-l'+r-1}{m} B_m^{(k_1,\dots,k_{r-1})}. \\ &\qquad (\text{put } l' = n-a) \end{aligned}$$

This shows the claim.

Remark 8 Some recurrences exist for the original Bernoulli numbers $B_n^{(1)} = B_n$ $(n \geq 0)$, the case in which the upper index is fixed with the value (1). The problem of finding a recurrence among $B_n^{(k_1,k_2,\ldots,k_r)}$ $(n \geq 0)$ for fixed $(k_1, k_2, \ldots, k_r)$, is therefore intriguing.

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