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SETS OF LENGTHS DO NOT CHARACTERIZE NUMERICAL MONOIDS

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Abstract

We study the sets of lengths and unions of sets of lengths of numerical monoids. Our paper focuses on a numerical monoid S generated by an arithmetic progression of positive integers. First, we determine exact solutions for the length sets of S and then use these formulas to enumerate the $\mathcal{V}_n(S)$ sets. Next, we determine necessary and sufficient conditions for two such numerical monoids to have identical sequences of $\mathcal{V}_n(S)$ sets. Finally, we determine necessary and sufficient conditions for two such numerical monoids to have equal length sets.

1. Introduction

Let M be a commutative cancellative monoid with set $\mathcal{A}(M)$ of irreducible elements and M^* of nonunits. We call M *atomic* if each element of M^* has a factorization into elements from $\mathcal{A}(M)$. The behavior of such irreducible factorizations has earned much attention in the recent mathematical literature (see the monograph [11] and the references therein). The set of lengths of $x \in M^*$ is defined as

$$\mathcal{L}(x) = \{n \mid x = x_1 \cdots x_n \text{ with each } x_i \in \mathcal{A}(M)\}$$

and the set of lengths of M as

$$\mathcal{L}(M) = \{\mathcal{L}(x) \mid x \in M^*\}.$$

The study of the sets $\mathcal{L}(x)$ and $\mathcal{L}(M)$ is a fundamental topic in the theory of non-unique factorizations. An indepth study of these sets when $M = \mathcal{B}(G)$ is a block monoid can be found in [11, Section 7.3]. Let G be an abelian group and $\mathcal{F}(G)$ be the free abelian monoid on G. The block monoid $\mathcal{B}(G)$ consists of all $\prod_{g_i \in G} g_i^{n_i} \in \mathcal{F}(G)$ with the property that $\sum_{g_i \in G} n_i g_i = 0$. If G_1 and G_2 are finite abelian groups, then $\mathcal{L}(\mathcal{B}(G_1)) = \mathcal{L}(\mathcal{B}(G_2))$ does not imply that $G_1 \cong G_2$, but only two counterexamples are known ($\mathcal{L}(\mathcal{B}(\{0\})) = \mathcal{L}(\mathcal{B}(\mathbb{Z}_2))$ [11, Theorem 3.4.11.5], and $\mathcal{L}(\mathcal{B}(\mathbb{Z}_3)) = \mathcal{L}(\mathcal{B}(\mathbb{Z}_2 \oplus \mathbb{Z}_2))$ [11, Theorem 7.3.2]). In fact, $\mathcal{L}(\mathcal{B}(G_1)) = \mathcal{L}(\mathcal{B}(G_2))$ implies $G_1 \cong G_2$, provided that $\mid G_1 \mid \geq 4$ and G_1 is either cyclic or an elementary 2-group. The same is true if $G_1 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_n$ with $n \geq 3$ [10].

The notion of a set of lengths was generalized in [6] as follows. With M as above, for each $n \in \mathbb{N}$ set

$$\mathcal{W}_n(M) = \{m \in M \mid n \in \mathcal{L}(m)\}$$

and

$$\mathcal{V}_n(M) = \bigcup_{m \in \mathcal{W}_n(M)} \mathcal{L}(m).$$

We refer to the set $\mathcal{V}_n(M)$ as a *union of sets of lengths*. In [6], the basic properties of these sets are determined. Since, for atomic monoids M_1 and M_2, $\mathcal{L}(M_1) = \mathcal{L}(M_2)$ implies $\mathcal{V}_n(M_1) = \mathcal{V}_n(M_2)$ for all n [6, Proposition 1.1], we cannot conclude that $\mathcal{V}_n(\mathcal{B}(G_1)) = \mathcal{V}_n(\mathcal{B}(G_2))$ for each n implies $G_1 \cong G_2$. Moreover, the results of [6] indicate that the converse of the former statement is not true. For instance, [6, Example 2.7] shows that $\mathcal{V}_n(\mathcal{B}(\mathbb{Z}_3 \oplus \mathbb{Z}_3)) = \mathcal{V}_n(\mathcal{B}(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2))$ for all n.

In this paper, we will explore questions related to those above for numerical monoids. A numerical monoid S is an additive submonoid of $\mathbb{N} \cup \{0\}$. The elements of S are positive integers x such that

$$x = x_1 a_1 + \cdots + x_t a_t = \sum_{i=1}^{t} x_i a_i$$

for some $x_i \in \mathbb{N} \cup \{0\}$. The set $\{a_1, \ldots, a_t\}$ is the generating set of S, often denoted as $S = \langle a_1, \ldots, a_t \rangle$. Every numerical monoid S has a unique minimal set of generators. The monoid S is *primitive* if $\gcd\{s \mid s \in S\} = 1$. Every numerical monoid S is isomorphic to a unique primitive numerical monoid, so we always assume that S is primitive. A good general survey on numerical monoids and numerical semigroups can be found in [9, Chapter 10].

Our focus in this paper are numerical monoids whose minimal generating sets form arithmetic progressions. Hence we consider such monoids where

$$S = \langle a, a+k, \ldots, a+wk \rangle, \tag{1}$$

with $0 \leq w < a$ and $\gcd(a, k) = 1$. In Section 2, we find in Theorem 2.2 a formula for the length set of any element in S. In Theorem 2.7 we find a corresponding formula for the sets $\mathcal{V}_n(S)$. Suppose that S_1 and S_2 are numerical monoids of the form (1). In Section 3, we use the results of Section 2 to determine necessary and sufficient conditions such that

1. $\mathcal{V}_n(S_1) = \mathcal{V}_n(S_2)$ for all n (Theorem 3.1) and
2. $\mathcal{L}(S_1) = \mathcal{L}(S_2)$ (Theorem 3.2).

Hence, unlike the situation with block monoids, we are able to build a large class of distinct numerical monoids which have equal length sets and and an even larger class with equal sequences of $\mathcal{V}_n(S)$ sets.

Before proceeding, we will require some further notation. If M is as above and $x \in M^*$, then set $L(x) = \max \mathcal{L}(x)$ and $l(x) = \min \mathcal{L}(x)$. The quotient $\frac{L(x)}{l(x)}$ is called the *elasticity* of x and the constant

$$\rho(M) = \sup\left\{\frac{L(x)}{l(x)} \mid x \in M^*\right\}$$

is known as the *elasticity* of M. If S is a numerical monoid of form (1) above, then $\rho(S) = \frac{a+wk}{a}$ by [5, Theorem 2.1]. If

$$\mathcal{L}(x) = \{n_1, \ldots, n_t\} \tag{2}$$

with the n_i's listed in increasing order, then the delta set of x is

$$\Delta(x) = \{n_i - n_{i-1} \mid 2 \leq i \leq t\}.$$

The delta set of M is then defined as

$$\Delta(M) = \bigcup_{x \in M^*} \Delta(x).$$

By [2, Theorem 3.9], $\Delta(\langle a, a+k, \ldots, a+wk\rangle) = \{k\}$. Notice that Theorem 2.2 along with Corollary 2.3 provide a considerably shorter alternate proof of this fact.

We will also require the following generalization of $\Delta(M)$. For a fixed monoid M, suppose for each $n \in \mathbb{N}$ that $\mathcal{V}_n(M) = \{v_{1,n}, \ldots, v_{t,n}\}$ where $v_{i,n} < v_{i+1,n}$ for $1 \leq i < t$. Define the $\mathcal{V}_n$-Delta set of M to be

$$\Delta(\mathcal{V}_n) = \{v_{i,n} - v_{i-1,n} \mid 2 \leq i \leq t\}$$

and the $\mathcal{V}$-Delta set of M to be

$$\Delta_{\mathcal{V}}(M) = \bigcup_{n \in \mathbb{N}} \Delta(\mathcal{V}_n).$$

Some preliminary results concerning these sets can be found in [1].

2. Numerical Monoids Generated by an Interval

Let $S = \langle a, a+k, \ldots, a+wk\rangle$, with $0 \le w < a$ and $\gcd(a,k) = 1$, as well as $S' = \langle c, c+t, \ldots, c+vt\rangle$, where $v < c$, $\gcd(c,t) = 1$ and $S \neq S'$. Our results in Lemma 2.1 and Theorem 2.2 take advantage of the membership criteria for a numerical monoid of the form S found in [3, Lemma 7].

Lemma 2.1. *If $n \in S$, then $n = c_1 a + c_2 k$ with $c_1, c_2 \in \mathbb{N}_0$ and $0 \le c_2 < a$.*

Proof. Any $n \in S$ can be written $d_1 a + d_2 k$, with $d_1, d_2 \in \mathbb{N}$. Let $d_2 = pa + q$ with $0 \le q < a$. Now $n = d_1 a + d_2 k = a(d_1 + pk) + qk$. □

Theorem 2.2. *Suppose $n = c_1 a + c_2 k \in S$ with $0 \le c_2 < a$. Then*

$$\mathcal{L}(n) = \left\{ c_1 + kd \,\middle|\, \left\lceil \frac{c_2 - c_1 w}{a + wk} \right\rceil \le d \le 0 \right\}.$$

Proof. Suppose $l \in \mathcal{L}(n)$. Now $la \equiv n \equiv c_1 a \mod k$, and thus $\mathcal{L}(n) \subset c_1 + k\mathbb{Z}$.

We can now let $l = c_1 + kd$. We know that

$$a(c_1 + kd) \le n \le (a+wk)(c_1 + kd) \implies \left\lceil \frac{\frac{n}{a+wk} - c_1}{k} \right\rceil \le d \le \left\lfloor \frac{\frac{n}{a} - c_1}{k} \right\rfloor$$

$$\implies \min \mathcal{L}(n) \ge c_1 + k \left\lceil \frac{\frac{n}{a+wk} - c_1}{k} \right\rceil = c_1 + k \left\lceil \frac{c_2 - c_1 w}{a + wk} \right\rceil,$$

and

$$\max \mathcal{L}(n) \le c_1 + k \left\lfloor \frac{\frac{n}{a} - c_1}{k} \right\rfloor = c_1.$$

Thus, $\mathcal{L}(n) \subset \left\{ c_1 + kd \,\middle|\, \lceil \frac{c_2 - c_1 w}{a+wk} \rceil \le d \le 0 \right\}$.

Let $d \in \mathbb{Z}$ such that $\lceil \frac{c_2 - c_1 w}{a+wk} \rceil \le d \le 0$. Let $p = \frac{n - a(c_1 + dk)}{k}$, which is $\in \mathbb{Z}$. If q is the remainder upon division of p by w, then we have,

$$n = a(c_1 + dk) + kp =$$

$$= a\left(\left\lfloor \frac{p}{w} \right\rfloor + 1 + c_1 + dk - 1 - \left\lfloor \frac{p}{w} \right\rfloor \right) + k\left(w \left\lfloor \frac{p}{w} \right\rfloor + q \right) =$$

$$\left\lfloor \frac{p}{w} \right\rfloor (a + wk) + \left(a + \left(p - w\left\lfloor \frac{p}{w} \right\rfloor \right) k \right) + \left(c_1 + dk - 1 - \left\lfloor \frac{p}{w} \right\rfloor \right) a,$$

which is a factorization of n of length $c_1 + dk$. Thus $c_1 + dk \in \mathcal{L}(n)$, as desired. □

An obvious corollary to this theorem follows.

Corollary 2.3. *$\Delta(S) = \{k\}$ and hence $\Delta_{\mathcal{V}}(S) = \{k\}$.*

Proof. The first assertion follows directly from the characterization of length sets in Theorem 2.2. The second assertion follows from [1, Corollary 4]. □

Lemma 2.4. $\mathcal{W}_n(S) = \{an, an+k, \ldots, an+nwk\}$.

Proof. Let $r \in \mathcal{W}_n(S)$. Then,

$$r = \alpha_0 \cdot a + \cdots + \alpha_w \cdot (a+wk),$$

so $r = an + ck$ with $0 \le c \le nw$. Also, for any such c,

$$an + ck = (a + \left\lfloor \frac{c}{n} \right\rfloor \cdot k) \cdot (1 - \{\frac{c}{n}\})n + (a + \left\lceil \frac{c}{n} \right\rceil \cdot k) \cdot (\{\frac{c}{n}\})n.$$

□

Lemma 2.5. *Given $n, n+k \in S$, $l(n+k) = l(n)$ or $l(n)+k$ and $L(n+k) = L(n)$ or $L(n)+k$.*

Proof. Let $n = c_1 a + c_2 k$, and $n + k = c_1' a + c_2' k$ with $c_1, c_2, c_1', c_2' \in \mathbb{N}_0$ and $c_2, c_2' < a$.

Case 1: $c_2 < a-1$. Now $c_2' = c_2 + 1$ and $c_1' = c_1$. From Theorem 2.2, we have $L(n+k) = c_1 = L(n)$ and

$$l(n+k) - l(n) = k\left(\left\lceil \frac{1 + c_2 - c_1 w}{a + wk} \right\rceil - \left\lceil \frac{c_2 - c_1 w}{a+wk} \right\rceil\right),$$

which is 0 or k.

Case 2: $c_2 = a-1$. Now $c_2' = 0$ and $c_1' = c_1 + k$. From Theorem 2.2, we have $L(n+k) = c_1' = L(n) + k$ and

$$l(n+k) - l(n) = k + k\left(\left\lceil \frac{-(c_1+k)w}{a+wk} \right\rceil - \left\lceil \frac{a - 1 - c_1 w}{a+wk} \right\rceil\right).$$

The numerator in the second ceiling function is $a + wk - 1$, greater than the first, thus $l(n+k) - l(n) = 0$ or k. □

We say that $\mathcal{L}(S)$ has a *jump* at n if $n, n+k \in S$, $l(n) + k = l(n+k)$, and $L(n) + k = L(n+k)$.

Lemma 2.6. *$\mathcal{L}(S)$ has a jump if and only if* $\gcd(a, w) = 1$.

Proof. Suppose $\mathcal{L}(S)$ has a jump at n. Let $n = c_1 a + c_2 k$, and $n + k = c_1' a + c_2' k$ with $c_1, c_2, c_1', c_2' \in \mathbb{N}_0$ and $c_2, c_2' < a$. $c_1 + k = L(n) + k = L(n+k) = c_1'$. Thus $c_2 = a - 1$ and $c_2' = 0$. By Theorem 2.2,

$$0 = l(n+k) - l(n) - k = c_1 + k + k\left\lceil \frac{-(c_1+k)w}{a+wk} \right\rceil - \left(c_1 + k\left\lceil \frac{a-1-c_1 w}{a+wk} \right\rceil\right) - k.$$

The two ceiling functions are equal, although the second numerator is $a + wk - 1$ greater than the first. Thus, the second fraction is an integer so $a + wk$ divides $a - 1 - c_1 w$. Any factor dividing a and w also divides $a + wk$ but not $a - 1 - c_1 w$, thus $\gcd(a, w) = 1$.

Now suppose $\gcd(a, w) = 1$. We also have $\gcd(a + wk, w) = 1$, so choose positive c_1 such that $a + wk$ divides $a - 1 - c_1 w$. Let $n = c_1 a + (a - 1)k$. The same calculation as above shows that $\mathcal{L}(S)$ has a jump at n. □

Theorem 2.7. *For $n \in \mathbb{N}$,*

$$\mathcal{V}_n(S) = \left\{ n + kd \mid -\left\lfloor \frac{nw}{a + wk} \right\rfloor \leq d \leq \left\lfloor \frac{nw}{a} \right\rfloor \right\}.$$

Proof. From Corollary 2.3, $\mathcal{V}_n(S)$ is a sequence where all pairs of consecutive terms have a difference of k. For all $m \in \mathcal{W}_n(S)$ we have $an \leq m \leq (a + wk)n$. In addition, we have $an, (a + wk)n \in \mathcal{W}_n(S)$. From Lemma 2.5, both $l(x)$ and $L(x)$ are increasing when x is incremented by k, so by Theorem 2.2,

$$\min \mathcal{V}_n(S) = l\left(\min \mathcal{W}_n(S)\right) = l(an) = n + k \left\lceil \frac{-nw}{a + wk} \right\rceil,$$

and

$$\max \mathcal{V}_n(S) = L\left(\max \mathcal{W}_n(S)\right) = L((a + wk)n) = n + k \left\lceil \frac{nw}{a} \right\rceil.$$

Therefore,

$$\mathcal{V}_n(S) = \left\{ n - k \left\lfloor \frac{nw}{a + wk} \right\rfloor, n - k \left\lfloor \frac{nw}{a + wk} \right\rfloor + k, \ldots, n + k \left\lfloor \frac{nw}{a} \right\rfloor \right\}$$

and the result clearly follows. □

3. Equality of $\mathcal{V}_n$-Sets and Length Sets

In this section, we again let $S = \langle a, a + k, \ldots, a + wk \rangle$, with $0 \leq w < a$ and $\gcd(a, k) = 1$, as well as $S' = \langle c, c + t, \ldots, c + vt \rangle$, where $v < c$, $\gcd(c, t) = 1$ and $S \neq S'$

Theorem 3.1. *Let S and S' be as above. Then $\mathcal{V}_n(S) = \mathcal{V}_n(S')$ for every $n \in \mathbb{N}$ if and only if $k = t$ and $\frac{c}{a} = \frac{v}{w}$.*

Proof. Suppose for every $n \in \mathbb{N}$ that $\mathcal{V}_n(S) = \mathcal{V}_n(S')$. Now $\min \mathcal{V}_n(S) = \min \mathcal{V}_n(S')$ implies that

$$n - k \left\lfloor \frac{nw}{a + wk} \right\rfloor = n - t \left\lfloor \frac{nv}{c + vt} \right\rfloor.$$

Let $n = (a + wk)(c + vt)$. So, $kw(c + vt) = tv(a + wk)$ and thus $avt = cwk$. However, $\Delta(S) = \{k\}$ and $\Delta(S') = \{t\}$, so from Corollary 2.3, $\Delta_{\mathcal{V}}(S) = \{k\}$ and $\Delta_{\mathcal{V}}(S') = \{t\}$. Thus $k = t$ and therefore $\frac{c}{a} = \frac{v}{w}$.

Now suppose $k = t$ and $\frac{c}{a} = \frac{v}{w}$. Since $k = t$, $\Delta_{\mathcal{V}}(S) = \Delta_{\mathcal{V}}(S')$. By Theorem 2.2,

$$\min \mathcal{V}_n(S') = n - t\left\lfloor \frac{nv}{c+vt} \right\rfloor = n - k\left\lfloor \frac{nw}{a+wk} \right\rfloor = \min \mathcal{V}_n(S),$$

and

$$\max \mathcal{V}_n(S') = n + t\left\lfloor \frac{nv}{c} \right\rfloor = n + k\left\lfloor \frac{nw}{a} \right\rfloor = \max \mathcal{V}_n(S).$$

Therefore, $\mathcal{V}_n(S) = \mathcal{V}_n(S')\ \forall\ n \in \mathbb{N}$. □

Theorem 3.2. *If $S \neq S'$, then $\mathcal{L}(S) = \mathcal{L}(S')$ if and only if $k = t$, $\frac{c}{a} = \frac{v}{w}$, $\gcd(a, w) \geq 2$, and $\gcd(c, v) \geq 2$.*

Proof. Suppose $\mathcal{L}(S) = \mathcal{L}(S')$. By Corollary 2.3, $k = \Delta(S) = \Delta(S') = t$. Also, by [5, Theorem 2.1] $\frac{a+wk}{a} = \rho(S) = \rho(S') = \frac{c+vt}{c}$, so $\frac{w}{a} = \frac{v}{c}$. If $\gcd(w, a) = \gcd(v, c) = 1$, then $w = v, a = c$, and $S = S'$. If only one pair is relatively prime, then by Lemma 2.6, exactly one of $\mathcal{L}(S), \mathcal{L}(S')$ has a jump, so they are not congruent. Therefore, $\gcd(a, w), \gcd(c, v) \geq 2$.

Now suppose $k = t$, $\frac{c}{a} = \frac{v}{w}$, $\gcd(a, w)$, $\gcd(c, v) \geq 2$. Let $c_1 \in \mathbb{N}$. Let $H = \{J \in \mathcal{L}(S) | \max J = c_1\}$ and $H' = \{J \in \mathcal{L}(S') | \max J = c_1\}$. From Theorem 2.2, the minimal values of the elements of H and H' are $\left\{c_1 + \lceil \frac{c_2 - c_1 w}{a+wk} \rceil \,\middle|\, 0 \leq c_2 < a\right\}$ and $\left\{c_1 + \left\lceil \frac{c_2' - c_1 v}{c+vk} \right\rceil \,\middle|\, 0 \leq c_2' < c\right\}$. The elements corresponding to $c_2 = c_2' = 0$ are clearly equal. Also, because $\gcd(a, w)$, $\gcd(c, v) \geq 2$,

$$\left\lceil \frac{a - 1 - c_1 w}{a + wk} \right\rceil = \left\lceil \frac{a - c_1 w}{a + wk} \right\rceil = \left\lceil \frac{c - c_1 v}{c + vk} \right\rceil = \left\lceil \frac{c - 1 - c_1 v}{c + vk} \right\rceil.$$

Thus, the elements corresponding to $c_2 = a - 1$ and $c_2' = c - 1$ are equal. Because the delta sets are singletons and equal, we have $H = H'$ and $\mathcal{L}(S) = \mathcal{L}(S')$. □

We close with this immediate corollary.

Corollary 3.3. *If $\mathcal{V}_n(S) = \mathcal{V}_n(S')$ for every $n \in \mathbb{N}$, then $\mathcal{L}(S) = \mathcal{L}(S')$ if and only if $\gcd(a, w) \geq 2$ and $\gcd(c, v) \geq 2$.*

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