MULTIPLE CONVOLUTION FORMULAE ON CLASSICAL COMBINATORIAL NUMBERS

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Received: 05/10/07, Revised: 10/24/07, Accepted: 11/06/07, Published: 11/28/07

Abstract

We establish a general summation theorem, which expresses multiple convolutions in closed forms. As examples, several interesting formulae regarding the numbers of Fibonacci, Lucas and Pell are given.

0. Introduction

For $m, n \in \mathbb{N}_0$ with $\mathbb{N}_0$ being the set of non-negative integers, let $\sigma_n(m)$ be the set of $(n+1)$-compositions of $m-n$ given by

$$\sigma_n(m) = \{\mathbf{k} := (k_0, k_1, \cdots, k_n) \mid k_0 + k_1 + \cdots + k_n = m - n \text{ with each } k_\imath \in \mathbb{N}_0\}.$$

Consider a sequence $\{w_n\}_{n \in \mathbb{N}_0}$ associated with the ordinary generating function

$$W(x) = \sum_{n \geq 0} w_n x^n \quad \rightleftharpoons \quad w_n = [x^n]\, W(x)$$

where $[x^n]W(x)$ stands for the coefficient of x^n in the formal power series $W(x)$. This paper will investigate the Ω-function with an extra indeterminate λ defined by the following

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multiple convolution:

$$\Omega_m(\lambda, w) := \sum_{n=0}^{m} \lambda^{m-n} \sum_{\mathbf{k}\in\sigma_n(m)} \prod_{i=0}^{n} w_{k_i}. \tag{1}$$

Then the main result of this paper may be stated as follows.

Theorem 1 (Multiple convolution formula)**.**

$$\Omega_m(\lambda, w) = [x^m]\frac{W(\lambda x)}{1 - xW(\lambda x)}.$$

Proof. This theorem contains many summation formulae as special cases even though its proof is almost a routine matter. In fact, we have

$$\begin{aligned}\Omega_m(\lambda, w) &= \sum_{n=0}^{m}[x^{m-n}]\, W^{n+1}(\lambda x) = [x^m]\sum_{n=0}^{m} x^n\, W^{n+1}(\lambda x)\\ &= [x^m]\frac{W(\lambda x)\big\{1 - x^{m+1}W^{m+1}(\lambda x)\big\}}{1 - xW(\lambda x)} = [x^m]\frac{W(\lambda x)}{1 - x\, W(\lambda x)}\end{aligned}$$

which confirms the formula stated in the theorem. □

The purpose of this short paper is to show several interesting closed formulae from Theorem 1 on multiple summations. The sequences of which we are concerned have rational functions as their generating functions. In particular, the convolution formulae involving Fibonacci numbers, Lucas numbers, and Pell numbers will be examined.

1. Fibonacci Numbers

Among the classical combinatorial sequences, Fibonacci numbers [4, §6.6] are well-known. They are defined through the recurrence relation

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n \geq 2 \tag{2a}$$

and the ordinary generating function

$$F(x) \;:=\; \sum_{n\geq 0} F_n x^n \;=\; \frac{x}{1 - x - x^2}. \tag{2b}$$

Then we have the following decomposition into partial fractions:

$$\frac{F(\lambda x)}{1 - xF(\lambda x)} = \frac{\lambda x}{1 - \lambda x - \lambda x^2 - \lambda^2 x^2} = \frac{\lambda}{\sqrt{4\lambda + 5\lambda^2}}\Big\{\frac{1}{1 - x\alpha} - \frac{1}{1 - x\gamma}\Big\}$$

where α and γ are given respectively by

$$\alpha = \frac{\lambda + \sqrt{4\lambda + 5\lambda^2}}{2} \quad \text{and} \quad \gamma = \frac{\lambda - \sqrt{4\lambda + 5\lambda^2}}{2}. \tag{3}$$

According to Theorem 1, we get the multiple convolution formula

$$\Omega_m(\lambda, F) = \frac{\lambda}{\sqrt{4\lambda + 5\lambda^2}}\big(\alpha^m - \gamma^m\big)$$

which may explicitly be restated as the following proposition.

Proposition 2 (Multiple convolution formula on Fibonacci numbers)**.**

$$\sum_{n=0}^{m} \lambda^{m-n} \sum_{\mathbf{k}\in\sigma_n(m)} \prod_{i=0}^{n} F_{k_i} = \frac{\lambda}{\sqrt{4\lambda + 5\lambda^2}}\big(\alpha^m - \gamma^m\big).$$

For $\lambda \in \mathbb{N}$, the first nine examples are tabulated as follows:

λ	$\Omega_m(\lambda, F)$
1	$\frac{1}{3}\big\{2^m - (-1)^m\big\}$
2	$\frac{1}{\sqrt{7}}\big\{(1+\sqrt{7})^m - (1-\sqrt{7})^m\big\}$
3	$\sqrt{\frac{3}{19}}\big\{(\frac{3+\sqrt{57}}{2})^m - (\frac{3-\sqrt{57}}{2})^m\big\}$
4	$\frac{1}{\sqrt{6}}\big\{(2+2\sqrt{6})^m - (2-2\sqrt{6})^m\big\}$
5	$\sqrt{\frac{5}{29}}\big\{(\frac{5+\sqrt{145}}{2})^m - (\frac{5-\sqrt{145}}{2})^m\big\}$
6	$\sqrt{\frac{3}{17}}\big\{(3+\sqrt{51})^m - (3-\sqrt{51})^m\big\}$
7	$\sqrt{\frac{7}{39}}\big\{(\frac{7+\sqrt{273}}{2})^m - (\frac{7-\sqrt{273}}{2})^m\big\}$
8	$\sqrt{\frac{2}{11}}\big\{(4+2\sqrt{22})^m - (4-2\sqrt{22})^m\big\}$
9	$\frac{3^{m+1}}{7}\big\{5^m - (-2)^m\big\}$

In particular when $m = 5$, we can explicitly write down, from Proposition 2 specified with $\lambda = 1$ and $\lambda = 9$, the following two numerical equalities

$$F_5 + 2F_1F_3 + F_2^2 + F_1^3 = 11 \quad \text{and} \quad 9^5F_5 + 2\times 9^4F_1F_3 + 9^4F_2^2 + 9^3F_1^3 = 328779.$$

Observing that

$$F(x) + F(-x) = \frac{x}{1-x-x^2} - \frac{x}{1+x-x^2} = \frac{2x^2}{1-3x^2+x^4}$$

we derive the generating function of the Fibonacci numbers with even indices:

$$F^e(x) := \sum_{n\geq 0} F_{2n}x^n = \frac{x}{1-3x+x^2}.$$

According to the partial fraction decomposition

$$\begin{aligned} \frac{F^e(\lambda x)}{1 - xF^e(\lambda x)} &= \frac{\lambda x}{1 - 3\lambda x - \lambda x^2 + \lambda^2 x^2} \\ &= \frac{\lambda}{\sqrt{4\lambda + 5\lambda^2}}\Big\{\frac{1}{1 - x(\lambda+\alpha)} - \frac{1}{1 - x(\lambda+\gamma)}\Big\} \end{aligned}$$

we can similarly evaluate the multiple convolution

$$\Omega_m(\lambda, F^e) = \frac{\lambda}{\sqrt{4\lambda + 5\lambda^2}}\Big\{(\lambda+\alpha)^m - (\lambda+\gamma)^m\Big\}$$

where the parameters λ and γ are defined in (3) as before. This reads explicitly as the following summation formula.

Proposition 3 (Multiple convolution formula)**.**

$$\sum_{n=0}^{m} \lambda^{m-n} \sum_{\mathbf{k}\in\sigma_n(m)} \prod_{i=0}^{n} F_{2k_i} = \frac{\lambda}{\sqrt{4\lambda+5\lambda^2}}\Big\{(\lambda+\alpha)^m - (\lambda+\gamma)^m\Big\}.$$

For $\lambda \in \mathbb{N}$, the first nine examples are tabulated as follows:

λ	$\Omega_m(\lambda, F^e)$
1	3^{m-1}
2	$\frac{1}{\sqrt{7}}\big\{(3+\sqrt{7})^m - (3-\sqrt{7})^m\big\}$
3	$\sqrt{\frac{3}{19}}\big\{(\frac{9+\sqrt{57}}{2})^m - (\frac{9-\sqrt{57}}{2})^m\big\}$
4	$\frac{1}{\sqrt{6}}\big\{(6+2\sqrt{6})^m - (6-2\sqrt{6})^m\big\}$
5	$\sqrt{\frac{5}{29}}\big\{(\frac{15+\sqrt{145}}{2})^m - (\frac{15-\sqrt{145}}{2})^m\big\}$
6	$\sqrt{\frac{3}{17}}\big\{(9+\sqrt{51})^m - (9-\sqrt{51})^m\big\}$
7	$\sqrt{\frac{7}{39}}\big\{(\frac{21+\sqrt{273}}{2})^m - (\frac{21-\sqrt{273}}{2})^m\big\}$
8	$\sqrt{\frac{2}{11}}\big\{(12+2\sqrt{22})^m - (12-2\sqrt{22})^m\big\}$
9	$\frac{3^{m+1}}{7}\{8^m - 1\}$

In addition, when $m = 4$, we can explicitly write down, from Proposition 3 specified with $\lambda = 1$ and $\lambda = 9$, the following two numerical equalities

$$F_8 + 2F_2F_4 = 27 \quad \text{and} \quad 9^4F_8 + 2\times 9^3F_2F_4 = 142155.$$

Instead, noticing that

$$F(x) - F(-x) = \frac{x}{1-x-x^2} + \frac{x}{1+x-x^2} = \frac{2x(1-x^2)}{1-3x^2+x^4}$$

we derive the generating function of the Fibonacci numbers with odd indices:

$$F^o(x) := \sum_{n\geq 0} F_{1+2n}x^n = \frac{1-x}{1-3x+x^2}.$$

Taking into account the partial fraction decomposition

$$\frac{F^o(\lambda x)}{1-xF^o(\lambda x)} = \frac{1-\lambda x}{1-x-3\lambda x+\lambda x^2+\lambda^2x^2}$$
$$= \frac{1}{\sqrt{1+2\lambda+5\lambda^2}}\Big\{\frac{\hat{\alpha}-\lambda}{1-x\hat{\alpha}} - \frac{\hat{\gamma}-\lambda}{1-x\hat{\gamma}}\Big\},$$

where $\hat{\alpha}$ and $\hat{\gamma}$ are given respectively by

$$\hat{\alpha} = \frac{1+3\lambda+\sqrt{1+2\lambda+5\lambda^2}}{2} \quad \text{and} \quad \hat{\gamma} = \frac{1+3\lambda-\sqrt{1+2\lambda+5\lambda^2}}{2}, \tag{4}$$

the corresponding multiple convolution has the following closed form:

$$\Omega_m(\lambda, F^o) = \frac{1}{\sqrt{1+2\lambda+5\lambda^2}}\Big\{(\hat{\alpha}-\lambda)\,\hat{\alpha}^m - (\hat{\gamma}-\lambda)\,\hat{\gamma}^m\Big\}.$$

We state it explicitly as the following proposition.

Proposition 4 (Multiple convolution formula)**.**

$$\sum_{n=0}^{m} \lambda^{m-n} \sum_{\mathbf{k}\in\sigma_n(m)} \prod_{i=0}^{n} F_{1+2k_i} = \frac{1}{\sqrt{1+2\lambda+5\lambda^2}}\Big\{(\hat{\alpha}-\lambda)\,\hat{\alpha}^m - (\hat{\gamma}-\lambda)\,\hat{\gamma}^m\Big\}.$$

For $\lambda \in \mathbb{N}$, the first nine examples are tabulated as follows:

λ	$\Omega_m(\lambda, F^o)$
1	$\frac{1}{2\sqrt{2}}\big\{(1+\sqrt{2})(2+\sqrt{2})^m - (1-\sqrt{2})(2-\sqrt{2})^m\big\}$
2	$\frac{1}{5}\big\{1+4\times 6^m\big\}$
3	$\frac{1}{2\sqrt{13}}\big\{(2+\sqrt{13})(5+\sqrt{13})^m - (2-\sqrt{13})(5-\sqrt{13})^m\big\}$
4	$\frac{1}{\sqrt{89}}\big\{(\frac{5+\sqrt{89}}{2})(\frac{13+\sqrt{89}}{2})^m - (\frac{5-\sqrt{89}}{2})(\frac{13-\sqrt{89}}{2})^m\big\}$
5	$\frac{1}{2\sqrt{34}}\big\{(3+\sqrt{34})(8+\sqrt{34})^m - (3-\sqrt{34})(8-\sqrt{34})^m\big\}$
6	$\frac{1}{\sqrt{193}}\big\{(\frac{7+\sqrt{193}}{2})(\frac{19+\sqrt{193}}{2})^m - (\frac{7-\sqrt{193}}{2})(\frac{19-\sqrt{193}}{2})^m\big\}$
7	$\frac{1}{2\sqrt{65}}\big\{(4+\sqrt{65})(11+\sqrt{65})^m - (4-\sqrt{65})(11-\sqrt{65})^m\big\}$
8	$\frac{1}{\sqrt{337}}\big\{(\frac{9+\sqrt{337}}{2})(\frac{25+\sqrt{337}}{2})^m - (\frac{9-\sqrt{337}}{2})(\frac{25-\sqrt{337}}{2})^m\big\}$
9	$\frac{1}{2\sqrt{106}}\big\{(5+\sqrt{106})(14+\sqrt{106})^m - (5-\sqrt{106})(14-\sqrt{106})^m\big\}$

We point out the identity corresponding to $\lambda = 2$ has been proposed by Tauraso [6]. When $m = 3$, we can also explicitly write down, from Proposition 4 specified with $\lambda = 2$ and $\lambda = 15$, the following two numerical equalities:

$$\begin{aligned} 2^3F_7 \;+\; 2^3F_1F_5 \;+\; 2^2F_3^2 \;+\; 2\times 3F_1^2F_3 \;+\; F_1^4 \quad &= \quad 173, \\ 15^3F_7 + 2\times 15^2F_1F_5 + 15^2F_3^2 + 3\times 15F_1^2F_3 + F_1^4 \quad &= \quad 47116. \end{aligned}$$

2. Lucas Numbers

Lucas numbers (cf. [4, P 312]) are defined through the recurrence relation

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad \text{for} \quad n \geq 2 \tag{5a}$$

and the ordinary generating function

$$L(x) := \sum_{n\geq 0} L_n x^n = \frac{2-x}{1-x-x^2}. \tag{5b}$$

Then we have the decomposition into partial fractions

$$\frac{L(\lambda x)}{1-xL(\lambda x)} = \frac{2-\lambda x}{1-2x-\lambda x+\lambda x^2-\lambda^2x^2} = \frac{1}{\sqrt{4+5\lambda^2}}\Big\{\frac{2\mu-\lambda}{1-x\mu}-\frac{2\nu-\lambda}{1-x\nu}\Big\}$$

where μ and ν are given respectively by

$$\mu = \frac{2+\lambda+\sqrt{4+5\lambda^2}}{2} \quad \text{and} \quad \nu = \frac{2+\lambda-\sqrt{4+5\lambda^2}}{2}. \tag{6}$$

By means of Theorem 1, we obtain the following closed formula

$$\Omega_m(\lambda, L) = \frac{1}{\sqrt{4+5\lambda^2}}\Big\{(2\mu-\lambda)\mu^m-(2\nu-\lambda)\nu^m\Big\}$$

which leads us consequently to the following proposition.

Proposition 5 (Multiple convolution formula on Lucas numbers)**.**

$$\sum_{n=0}^{m}\lambda^{m-n}\sum_{\mathbf{k}\in\sigma_n(m)}\prod_{i=0}^{n}L_{k_i} = \frac{1}{\sqrt{4+5\lambda^2}}\Big\{(2\mu-\lambda)\mu^m-(2\nu-\lambda)\nu^m\Big\}.$$

For $\lambda \in \mathbb{N}$, the first nine examples are tabulated as follows:

λ	$\Omega_m(\lambda, L)$
1	$5\times 3^{m-1}$
2	$\frac{1}{\sqrt{6}}\big\{(\sqrt{6}+1)(2+\sqrt{6})^m+(\sqrt{6}-1)(2-\sqrt{6})^m\big\}$
3	$\frac{1}{7}\{9\times 6^m+5(-1)^m\}$
4	$\frac{1}{\sqrt{21}}\big\{(\sqrt{21}+1)(3+\sqrt{21})^m+(\sqrt{21}-1)(3-\sqrt{21})^m\big\}$
5	$\frac{1}{\sqrt{129}}\big\{(2+\sqrt{129})(\frac{7+\sqrt{129}}{2})^m-(2-\sqrt{129})(\frac{7-\sqrt{129}}{2})^m\big\}$
6	$\frac{1}{\sqrt{46}}\big\{(\sqrt{46}+1)(4+\sqrt{46})^m+(\sqrt{46}-1)(4-\sqrt{46})^m\big\}$
7	$\frac{1}{\sqrt{249}}\big\{(2+\sqrt{249})(\frac{9+\sqrt{249}}{2})^m-(2-\sqrt{249})(\frac{9-\sqrt{249}}{2})^m\big\}$
8	$\frac{2}{9}\{5\times 14^m-(-4)^{m+1}\}$
9	$\frac{1}{\sqrt{409}}\big\{(2+\sqrt{409})(\frac{11+\sqrt{409}}{2})^m-(2-\sqrt{409})(\frac{11-\sqrt{409}}{2})^m\big\}$

Noting that

$$L(x)+L(-x) = \frac{2-x}{1-x-x^2}+\frac{2+x}{1+x-x^2} = \frac{4-6x^2}{1-3x^2+x^4}$$

we get the generating function of the Lucas numbers with even indices:

$$L^e(x) := \sum_{n\geq 0} L_{2n}x^n = \frac{2-3x}{1-3x+x^2}.$$

From the decomposition into partial fractions

$$\frac{L^e(\lambda x)}{1-xL^e(\lambda x)} = \frac{2-3\lambda x}{1-2x-3\lambda x+3\lambda x^2+\lambda^2x^2}$$
$$= \frac{1}{\sqrt{4+5\lambda^2}}\Big\{\frac{2\mu-\lambda}{1-x(\lambda+\mu)} - \frac{2\nu-\lambda}{1-x(\lambda+\nu)}\Big\}$$

we can similarly evaluate the multiple convolution

$$\Omega_m(\lambda, L^e) = \frac{1}{\sqrt{4+5\lambda^2}}\Big\{(2\mu-\lambda)(\lambda+\mu)^m - (2\nu-\lambda)(\lambda+\nu)^m\Big\}$$

where the parameters μ and ν are defined as before. This reads explicitly as the following summation formula.

Proposition 6 (Multiple convolution formula)**.**

$$\sum_{n=0}^{m}\lambda^{m-n}\sum_{\mathbf{k}\in\sigma_n(m)}\prod_{i=0}^{n}L_{2k_i} = \frac{1}{\sqrt{4+5\lambda^2}}\Big\{(2\mu-\lambda)(\lambda+\mu)^m - (2\nu-\lambda)(\lambda+\nu)^m\Big\}.$$

For $\lambda \in \mathbb{N}$, the first nine examples are tabulated as follows:

λ	$\Omega_m(\lambda, L^e)$
1	$\frac{1}{3}\{1+5\times 4^m\}$
2	$\frac{1}{\sqrt{6}}\{(\sqrt{6}+1)(4+\sqrt{6})^m+(\sqrt{6}-1)(4-\sqrt{6})^m\}$
3	$\frac{1}{7}\{9^{m+1}+5\times 2^m\}$
4	$\frac{1}{\sqrt{21}}\{(\sqrt{21}+1)(7+\sqrt{21})^m+(\sqrt{21}-1)(7-\sqrt{21})^m\}$
5	$\frac{1}{\sqrt{129}}\{(2+\sqrt{129})(\frac{17+\sqrt{129}}{2})^m-(2-\sqrt{129})(\frac{17-\sqrt{129}}{2})^m\}$
6	$\frac{1}{\sqrt{46}}\{(\sqrt{46}+1)(10+\sqrt{46})^m+(\sqrt{46}-1)(10-\sqrt{46})^m\}$
7	$\frac{1}{\sqrt{249}}\{(2+\sqrt{249})(\frac{23+\sqrt{249}}{2})^m-(2-\sqrt{249})(\frac{23-\sqrt{249}}{2})^m\}$
8	$\frac{2}{9}\{5\times 22^m+4^{m+1}\}$
9	$\frac{1}{\sqrt{409}}\{(2+\sqrt{409})(\frac{29+\sqrt{409}}{2})^m-(2-\sqrt{409})(\frac{29-\sqrt{409}}{2})^m\}$

Moreover, observing that

$$L(x)-L(-x) = \frac{2-x}{1-x-x^2} - \frac{2+x}{1+x-x^2} = \frac{2x(1+x^2)}{1-3x^2+x^4}$$

we derive the generating function of the Lucas numbers with odd indices:

$$L^o(x) := \sum_{n\geq 0}L_{1+2n}x^n = \frac{1+x}{1-3x+x^2}.$$

In view of the partial fraction decomposition

$$\frac{L^o(\lambda x)}{1-xL^o(\lambda x)} = \frac{1+\lambda x}{1-x-3\lambda x-\lambda x^2+\lambda^2x^2}$$
$$= \frac{1}{\sqrt{1+10\lambda+5\lambda^2}}\Big\{\frac{\hat\mu+\lambda}{1-x\hat\mu} - \frac{\hat\nu+\lambda}{1-x\hat\nu}\Big\}$$

where $\hat{\mu}$ and $\hat{\nu}$ are given respectively by

$$\hat{\mu} = \frac{1+3\lambda+\sqrt{1+10\lambda+5\lambda^2}}{2} \quad \text{and} \quad \hat{\nu} = \frac{1+3\lambda-\sqrt{1+10\lambda+5\lambda^2}}{2} \tag{7}$$

we can similarly evaluate the corresponding multiple convolution

$$\Omega_m(\lambda, L^o) = \frac{1}{\sqrt{1+10\lambda+5\lambda^2}}\Big\{(\hat{\mu}+\lambda)\,\hat{\mu}^m - (\hat{\nu}+\lambda)\,\hat{\nu}^m\Big\}$$

which leads us to the following proposition.

Proposition 7 (Multiple convolution formula)**.**

$$\sum_{n=0}^{m} \lambda^{m-n} \sum_{\mathbf{k}\in\sigma_n(m)} \prod_{i=0}^{n} L_{1+2k_i} = \frac{1}{\sqrt{1+10\lambda+5\lambda^2}}\Big\{(\hat{\mu}+\lambda)\,\hat{\mu}^m - (\hat{\nu}+\lambda)\,\hat{\nu}^m\Big\}.$$

For $\lambda \in \mathbb{N}$, the first nine examples are tabulated as follows:

λ	$\Omega_m(\lambda, L^o)$
1	$5 \times 4^{m-1}$
2	$\frac{1}{\sqrt{41}}\big\{\frac{11+\sqrt{41}}{2}(\frac{7+\sqrt{41}}{2})^m - \frac{11-\sqrt{41}}{2}(\frac{7-\sqrt{41}}{2})^m\big\}$
3	$\frac{1}{2\sqrt{19}}\big\{(8+\sqrt{19})(5+\sqrt{19})^m - (8-\sqrt{19})(5-\sqrt{19})^m\big\}$
4	$\frac{1}{11}\big\{16 \times 12^m - 5\big\}$
5	$\frac{1}{4\sqrt{11}}\big\{(13+2\sqrt{11})(8+2\sqrt{11})^m - (13-2\sqrt{11})(8-2\sqrt{11})^m\big\}$
6	$\frac{1}{\sqrt{241}}\big\{\frac{31+\sqrt{241}}{2}(\frac{19+\sqrt{241}}{2})^m - \frac{31-\sqrt{241}}{2}(\frac{19-\sqrt{241}}{2})^m\big\}$
7	$\frac{1}{2\sqrt{79}}\big\{(18+\sqrt{79})(11+\sqrt{79})^m - (18-\sqrt{79})(11-\sqrt{79})^m\big\}$
8	$\frac{1}{\sqrt{401}}\big\{\frac{41+\sqrt{401}}{2}(\frac{25+\sqrt{401}}{2})^m - \frac{41-\sqrt{401}}{2}(\frac{25-\sqrt{401}}{2})^m\big\}$
9	$\frac{1}{4\sqrt{31}}\big\{(23+2\sqrt{31})(14+2\sqrt{31})^m - (23-2\sqrt{31})(14-2\sqrt{31})^m\big\}$

3. Pell Numbers

The Pell numbers (cf. Sloane [5, A000129]) are given by the recurrence relation

$$P_0 = 1, \quad P_1 = 2, \quad P_n = 2P_{n-1} + P_{n-2} \quad \text{for} \quad n \geq 2 \tag{8a}$$

with the following ordinary generating function

$$P(x) := \sum_{n\geq 0} P_n x^n = \frac{1}{1-2x-x^2}. \tag{8b}$$

By means of the partial fraction decomposition

$$\frac{P(\lambda x)}{1-xP(\lambda x)} = \frac{1}{1-x-2\lambda x-\lambda^2x^2} = \frac{1}{\sqrt{1+4\lambda+8\lambda^2}}\Big\{\frac{\xi}{1-x\xi} - \frac{\eta}{1-x\eta}\Big\}$$

where ξ and η are given respectively by

$$\xi = \frac{1+2\lambda+\sqrt{1+4\lambda+8\lambda^2}}{2} \quad \text{and} \quad \eta = \frac{1+2\lambda-\sqrt{1+4\lambda+8\lambda^2}}{2} \tag{9}$$

we get through Theorem 1 the following multiple convolution formula

$$\Omega_m(\lambda,P) = \frac{1}{\sqrt{1+4\lambda+8\lambda^2}}\big(\xi^{m+1}-\eta^{m+1}\big).$$

This reads explicitly as the following proposition.

Proposition 8 (Multiple convolution formula on Pell numbers)**.**

$$\sum_{n=0}^{m}\lambda^{m-n}\sum_{\mathbf{k}\in\sigma_n(m)}\prod_{i=0}^{n}P_{k_i} = \frac{1}{\sqrt{1+4\lambda+8\lambda^2}}\big(\xi^{m+1}-\eta^{m+1}\big).$$

For $\lambda \in \mathbb{N}$, the first nine examples are tabulated as follows:

λ	$\Omega_m(\lambda,P)$
1	$\frac{1}{\sqrt{13}}\big\{\big(\frac{3+\sqrt{13}}{2}\big)^{m+1}-\big(\frac{3-\sqrt{13}}{2}\big)^{m+1}\big\}$
2	$\frac{1}{\sqrt{41}}\big\{\big(\frac{5+\sqrt{41}}{2}\big)^{m+1}-\big(\frac{5-\sqrt{41}}{2}\big)^{m+1}\big\}$
3	$\frac{1}{\sqrt{85}}\big\{\big(\frac{7+\sqrt{85}}{2}\big)^{m+1}-\big(\frac{7-\sqrt{85}}{2}\big)^{m+1}\big\}$
4	$\frac{1}{\sqrt{145}}\big\{\big(\frac{9+\sqrt{145}}{2}\big)^{m+1}-\big(\frac{9-\sqrt{145}}{2}\big)^{m+1}\big\}$
5	$\frac{1}{\sqrt{221}}\big\{\big(\frac{11+\sqrt{221}}{2}\big)^{m+1}-\big(\frac{11-\sqrt{221}}{2}\big)^{m+1}\big\}$
6	$\frac{1}{\sqrt{313}}\big\{\big(\frac{13+\sqrt{313}}{2}\big)^{m+1}-\big(\frac{13-\sqrt{313}}{2}\big)^{m+1}\big\}$
7	$\frac{1}{\sqrt{421}}\big\{\big(\frac{15+\sqrt{421}}{2}\big)^{m+1}-\big(\frac{15-\sqrt{421}}{2}\big)^{m+1}\big\}$
8	$\frac{1}{\sqrt{545}}\big\{\big(\frac{17+\sqrt{545}}{2}\big)^{m+1}-\big(\frac{17-\sqrt{545}}{2}\big)^{m+1}\big\}$
9	$\frac{1}{\sqrt{685}}\big\{\big(\frac{19+\sqrt{685}}{2}\big)^{m+1}-\big(\frac{19-\sqrt{685}}{2}\big)^{m+1}\big\}$

Observing that

$$P(x)+P(-x) = \frac{1}{1-2x-x^2}+\frac{1}{1+2x-x^2} = \frac{2-2x^2}{1-6x^2+x^4}$$

we derive the generating function of the Pell numbers with even indices:

$$P^e(x) := \sum_{n\geq 0}P_{2n}x^n = \frac{1-x}{1-6x+x^2}.$$

Taking into account the partial fraction decomposition

$$\frac{P^e(\lambda x)}{1-xP^e(\lambda x)} = \frac{1-\lambda x}{1-x-6\lambda x+\lambda x^2+\lambda^2x^2}$$
$$= \frac{1}{\sqrt{1+8\lambda+32\lambda^2}}\Big\{\frac{\tilde\xi-\lambda}{1-x\tilde\xi}-\frac{\tilde\eta-\lambda}{1-x\tilde\eta}\Big\}$$

where $\tilde{\xi}$ and $\tilde{\eta}$ are given respectively by

$$\tilde{\xi} = \frac{1+6\lambda+\sqrt{1+8\lambda+32\lambda^2}}{2} \quad \text{and} \quad \tilde{\eta} = \frac{1+6\lambda-\sqrt{1+8\lambda+32\lambda^2}}{2} \tag{10}$$

we can similarly evaluate the multiple convolution

$$\Omega_m(\lambda, P^e) = \frac{1}{\sqrt{1+8\lambda+32\lambda^2}}\Big\{(\tilde{\xi}-\lambda)\tilde{\xi}^m - (\tilde{\eta}-\lambda)\tilde{\eta}^m\Big\}$$

which may explicitly be restated as the following summation formula.

Proposition 9 (Multiple convolution formula)**.**

$$\sum_{n=0}^{m} \lambda^{m-n} \sum_{\mathbf{k}\in\sigma_n(m)} \prod_{i=0}^{n} P_{2k_i} = \frac{1}{\sqrt{1+8\lambda+32\lambda^2}}\Big\{(\tilde{\xi}-\lambda)\tilde{\xi}^m - (\tilde{\eta}-\lambda)\tilde{\eta}^m\Big\}.$$

For $\lambda \in \mathbb{N}$, the first nine examples are tabulated as follows:

λ	$\Omega_m(\lambda, P^e)$
1	$\frac{1}{\sqrt{41}}\Big\{\frac{5+\sqrt{41}}{2}\big(\frac{7+\sqrt{41}}{2}\big)^m - \frac{5-\sqrt{41}}{2}\big(\frac{7-\sqrt{41}}{2}\big)^m\Big\}$
2	$\frac{1}{\sqrt{145}}\Big\{\frac{9+\sqrt{145}}{2}\big(\frac{13+\sqrt{145}}{2}\big)^m - \frac{9-\sqrt{145}}{2}\big(\frac{13-\sqrt{145}}{2}\big)^m\Big\}$
3	$\frac{1}{\sqrt{313}}\Big\{\frac{13+\sqrt{313}}{2}\big(\frac{19+\sqrt{313}}{2}\big)^m - \frac{13-\sqrt{313}}{2}\big(\frac{19-\sqrt{313}}{2}\big)^m\Big\}$
4	$\frac{1}{\sqrt{545}}\Big\{\frac{17+\sqrt{545}}{2}\big(\frac{25+\sqrt{545}}{2}\big)^m - \frac{17-\sqrt{545}}{2}\big(\frac{25-\sqrt{545}}{2}\big)^m\Big\}$
5	$\frac{1}{29}\Big\{4 + 25 \times 30^m\Big\}$
6	$\frac{1}{\sqrt{1201}}\Big\{\frac{25+\sqrt{1201}}{2}\big(\frac{37+\sqrt{1201}}{2}\big)^m - \frac{25-\sqrt{1201}}{2}\big(\frac{37-\sqrt{1201}}{2}\big)^m\Big\}$
7	$\frac{1}{5\sqrt{65}}\Big\{\frac{29+5\sqrt{65}}{2}\big(\frac{43+5\sqrt{65}}{2}\big)^m - \frac{29-5\sqrt{65}}{2}\big(\frac{43-5\sqrt{65}}{2}\big)^m\Big\}$
8	$\frac{1}{\sqrt{2113}}\Big\{\frac{33+\sqrt{2113}}{2}\big(\frac{49+\sqrt{2113}}{2}\big)^m - \frac{33-\sqrt{2113}}{2}\big(\frac{49-\sqrt{2113}}{2}\big)^m\Big\}$
9	$\frac{1}{\sqrt{2665}}\Big\{\frac{37+\sqrt{2665}}{2}\big(\frac{55+\sqrt{2665}}{2}\big)^m - \frac{37-\sqrt{2665}}{2}\big(\frac{55-\sqrt{2665}}{2}\big)^m\Big\}$

In addition, noticing that

$$P(x) - P(-x) = \frac{1}{1-2x-x^2} - \frac{1}{1+2x-x^2} = \frac{4x}{1-6x^2+x^4}$$

we get the generating function of the Pell numbers with odd indices:

$$P^o(x) := \sum_{n\geq 0} P_{1+2n}x^n = \frac{2}{1-6x+x^2}.$$

Then we can similarly have the following partial fractions

$$\begin{aligned}\frac{P^o(\lambda x)}{1-xP^o(\lambda x)} &= \frac{2}{1-2x-6\lambda x+\lambda^2x^2}\\ &= \frac{1}{\sqrt{1+6\lambda+8\lambda^2}}\Big\{\frac{\hat{\xi}}{1-x\hat{\xi}} - \frac{\hat{\eta}}{1-x\hat{\eta}}\Big\}\end{aligned}$$

where $\hat{\xi}$ and $\hat{\eta}$ are given respectively by

$$\hat{\xi} = 1+3\lambda+\sqrt{1+6\lambda+8\lambda^2} \quad \text{and} \quad \hat{\eta} = 1+3\lambda-\sqrt{1+6\lambda+8\lambda^2}. \tag{11}$$

Hence, we have further the multiple convolution formula

$$\Omega_m(\lambda, P^o) = \frac{1}{\sqrt{1+6\lambda+8\lambda^2}}\left(\hat{\xi}^{m+1} - \hat{\eta}^{m+1}\right)$$

which leads us to the following proposition.

Proposition 10 (Multiple convolution formula)**.**

$$\sum_{n=0}^{m} \lambda^{m-n} \sum_{\mathbf{k}\in\sigma_n(m)} \prod_{i=0}^{n} P_{1+2k_i} = \frac{1}{\sqrt{1+6\lambda+8\lambda^2}}\left(\hat{\xi}^{m+1} - \hat{\eta}^{m+1}\right).$$

For $\lambda \in \mathbb{N}$, the first nine examples are tabulated as follows:

λ	$\Omega_m(\lambda, P^o)$
1	$\frac{1}{\sqrt{15}}\left\{(4+\sqrt{15})^{m+1} - (4-\sqrt{15})^{m+1}\right\}$
2	$\frac{1}{3\sqrt{5}}\left\{7+3\sqrt{5})^{m+1} - (7-3\sqrt{5})^{m+1}\right\}$
3	$\frac{1}{\sqrt{91}}\left\{(10+\sqrt{91})^{m+1} - (10-\sqrt{91})^{m+1}\right\}$
4	$\frac{1}{3\sqrt{17}}\left\{(13+3\sqrt{17})^{m+1} - (13-3\sqrt{17})^{m+1}\right\}$
5	$\frac{1}{\sqrt{231}}\left\{(16+\sqrt{231})^{m+1} - (16-\sqrt{231})^{m+1}\right\}$
6	$\frac{1}{5\sqrt{13}}\left\{(19+5\sqrt{13})^{m+1} - (19-5\sqrt{13})^{m+1}\right\}$
7	$\frac{1}{\sqrt{435}}\left\{(22+\sqrt{435})^{m+1} - (22-\sqrt{435})^{m+1}\right\}$
8	$\frac{1}{\sqrt{561}}\left\{(25+\sqrt{561})^{m+1} - (25-\sqrt{561})^{m+1}\right\}$
9	$\frac{1}{\sqrt{703}}\left\{(28+\sqrt{703})^{m+1} - (28-\sqrt{703})^{m+1}\right\}$

From the results displayed in this paper, we see that Theorem 1 contains many multiple convolution identities as special cases. There exist numerous other sequences having rational functions as their generating functions, or equivalently, satisfying linear recurrence relations of constant coefficients, which may serve to produce multiple convolution formulae, for example, the Fibonacci polynomials appeared in [1, 2, 8, 9]. In particular, it is worthwhile mentioning that the two trigonometric sequences $\{\sin n\theta\}_{n\in\mathbb{N}_0}$ and $\{\cos n\theta\}_{n\in\mathbb{N}_0}$ satisfy the same recurrence relation

$$W_n = 2\cos\theta\, W_{n-1} - W_{n-2}$$

with the following generating functions

$$\begin{aligned}\sum_{n\geq 0} x^n \sin n\theta &= \frac{x\sin\theta}{1-2x\cos\theta+x^2}, \\ \sum_{n\geq 0} x^n \cos n\theta &= \frac{1-x\cos\theta}{1-2x\cos\theta+x^2}.\end{aligned}$$

The multiple convolution formula corresponding to $\{\sin n\theta\}_{n\in\mathbb{N}_0}$ reads as

$$\sum_{n=0}^{m}\lambda^{m-n}\sum_{\mathbf{k}\in\sigma_n(m)}\prod_{i=0}^{n}\sin(k_i\theta)=\frac{\rho^m-\varrho^m}{2}\sqrt{\frac{\lambda\sin\theta}{1-\lambda\sin\theta}}$$

where ρ and ϱ are defined respectively by

$$\rho=\lambda\cos\theta+\sqrt{\lambda\sin\theta(1-\lambda\sin\theta)}\quad\text{and}\quad\varrho=\lambda\cos\theta-\sqrt{\lambda\sin\theta(1-\lambda\sin\theta)}.$$

Similarly, we have the convolution formula corresponding to $\{\cos n\theta\}_{n\in\mathbb{N}_0}$

$$\sum_{n=0}^{m}\lambda^{m-n}\sum_{\mathbf{k}\in\sigma_n(m)}\prod_{i=0}^{n}\cos(k_i\theta)=\frac{(\hat{\rho}-\lambda\cos\theta)\hat{\rho}^m-(\hat{\varrho}-\lambda\cos\theta)\hat{\varrho}^m}{\sqrt{1-4\lambda^2\sin^2\theta}}$$

where $\hat{\rho}$ and $\hat{\varrho}$ are defined respectively by

$$\hat{\rho}=\frac{1+2\lambda\cos\theta+\sqrt{1-4\lambda^2\sin^2\theta}}{2}\quad\text{and}\quad\hat{\varrho}=\frac{1+2\lambda\cos\theta-\sqrt{1-4\lambda^2\sin^2\theta}}{2}.$$

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