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A MULTIPLICITY PROBLEM RELATED TO SCHUR NUMBERS

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Abstract

For each natural number n, let C_n represent the set of all 2-colorings of the set $\{1, 2, \ldots, n\}$. Given a natural number n and a coloring $\Delta \in C_n$, let $S(\Delta)$ represent the set

$$S(\Delta) = \{x_3 \mid \exists\ x_1, x_2 \ s.t.\ x_1 + x_2 = x_3 \ \ and\ \Delta(x_1) = \Delta(x_2) = \Delta(x_3)\}.$$

Given a natural number n, let

$$f(n) = \min_{\Delta \in C_n} S(\Delta).$$

For all natural numbers n and r where $\frac{n}{2} \le r \le n$, let $C_{n,r}$ represent the set of all 2-colorings of the set $\{1, 2, \ldots, n\}$ where $\max\{|\Delta^{-1}(0)|, |\Delta^{-1}(1)|\} = r$. Given natural numbers n and r where $\frac{n}{2} \le r \le n$, let

$$f(n, r) = \min_{\Delta \in C_{n,r}} S(\Delta).$$

In this paper it is determined that for all natural numbers n,

$$f(n) = \begin{cases} 0 & 1 \le n \le 4 \\ \lfloor \frac{n-3}{2} \rfloor & n \ge 5 \end{cases}$$

and for all natural numbers n and r where $n \ge 5$ and $\frac{n}{2} \le r \le n$,

$$f(n, r) = \begin{cases} r - 2 & r < n \\ r - 1 & r = n. \end{cases}$$

1. Introduction

Let $\mathbb{N}$ represent the set of natural numbers and let $[a, b]$ denote the set $\{n \in \mathbb{N} |\, a \le n \le b\}$. A function $\Delta : [1, n] \to [0, t-1]$ is referred to as a *t-coloring of the set* $[1, n]$ or as a *t-coloring* *of length* n. For every natural number n, let C_n represent the set of all 2-colorings of length n. For a given natural number n' where $n' < n$, a coloring Δ restricted to the set $[1, n']$ will be denoted by $\Delta|_{n'}$. Given a t-coloring Δ and a linear equation L in m variables, a solution $(x_1, x_2, \dots, x_m)$ to L is *monochromatic* if and only if $\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m)$.

In 1916, I. Schur [21] proved that for every $t \geq 2$, there exists a least integer $n = Schur(t)$ such that for every t-coloring of length n, there exists a monochromatic solution to

$$x_1 + x_2 = x_3. \tag{1}$$

Note that the integers x_1 and x_2 need not be distinct. The integers $Schur(t)$ are called *Schur numbers*. It is known that $Schur(2) = 5$, $Schur(3) = 14$ and $Schur(4) = 45$, but no other Schur numbers are known [22]. A monochromatic solution to equation (1) is called a *monochromatic Schur triple*. In 1933, R. Rado found necessary and sufficient conditions to determine if an arbitrary system of linear equations admits a monochromatic solution for every coloring of the natural numbers with a finite number of colors [5, 15, 16, 17].

Recently, several other problems related to Schur numbers have been considered [1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14]. In 1996, R. Graham, V. Rödl and A. Rucinski proposed the following problem [6]. Find (asymptotically) the least number of monochromatic Schur triples that must occur in an arbitrary 2-coloring of length n. A problem of this nature where the number of monochromatic solutions is to be determined is referred to as a *multiplicity* problem. This problem was solved independently by A. Robertson and D. Zeilberger [19] and by T. Schoen [20] and was found to be $\frac{1}{22}n^2 + O(n)$.

In this paper we modify the problem of Graham, Rödl and Rucinski by asking the following question. In an arbitrary 2-coloring of length n, how many integers must there be that are the third integer (i.e. x_3) in at least one monochromatic Schur triple? For convenience in this paper, we shall refer to the third integer in any monochromatic Schur triple as *special*, formally defined below.

Definition 1. *For every* 2*-coloring* $\Delta : [1, n] \to [0, 1]$, *an integer* $s \in [1, n]$ *is* special *if and only if there exist integers* $x_1, x_2 \in [1, n]$, *such that* $x_1 + x_2 = s$ *and* $\Delta(x_1) = \Delta(x_2) = \Delta(s)$. *The set of all special integers for a* 2*-coloring* Δ *is denoted by* $S(\Delta)$.

Definition 2. *For every natural number* n, *let*

$$f(n) = \min_{\Delta \in C_n} S(\Delta).$$

In this paper it is determined that

$$f(n) = \begin{cases} 0 & 1 \leq n \leq 4 \\ \left\lfloor \frac{n-3}{2} \right\rfloor & n \geq 5. \end{cases}$$

More importantly, and perhaps more surprisingly, it is found that the number of special integers in a 2-coloring Δ is more directly related to the number of monochromatic integers in Δ, rather than the length of Δ. For instance, the number of special integers that must occur in an arbitrary 2-coloring of length seventy with fourty integers colored 0 is the same as the number that must occur in an arbitrary 2-coloring of length sixty with fourty integers colored 0 or an arbitrary 2-coloring of length fifty with fourty integers colored 0. We say that a coloring Δ has r monochromatic integers if $\max\{|\Delta^{-1}(0)|, |\Delta^{-1}(1)|\} = r$. For all natural numbers n and r where $\frac{n}{2} \leq r \leq n$, let $C_{n,r}$ represent the set of all 2-colorings of length n with r monochromatic integers.

Definition 3. *For all integers n and r where $n \geq 5$ and $\frac{n}{2} \leq r \leq n$, let*

$$f(n,r) = \min_{\Delta \in C_{n,r}} S(\Delta).$$

In this paper it is determined that

$$f(n,r) = \begin{cases} r-2 & r < n \\ r-1 & r = n. \end{cases}$$

The above formula for $f(n)$ follows immediately from this result.

2. Main Results

Theorem 1. *If n and r are natural numbers such that $n \geq 5$ and $\frac{n}{2} \leq r \leq n$, then*

$$f(n,r) = \begin{cases} r-2 & \text{if } r < n \\ r-1 & \text{if } r = n. \end{cases}$$

Proof. First we shall prove the case where $r = n$. Let a natural number $n \geq 5$ be given, let $r = n$ and let $\Delta : [1,n] \to [0,1]$ be a coloring with r monochromatic integers. Clearly $S(\Delta) = [2,n]$, so $f(n,r) = r-1$.

We shall now consider the case where $r < n$. First we shall show that $f(n,r) \leq r-2$. Let integers n and r be given such that $n \geq 5$ and $\frac{n}{2} \leq r < n$. We shall exhibit a coloring $\Delta : [1,n] \to [0,1]$ where $|\Delta^{-1}(0)| = r$ and $|S(\Delta)| = r-2$. If $r = n-1$ let Δ be defined by

$$\Delta(x) = \begin{cases} 1 & x = 1 \\ 0 & 2 \leq x \leq n. \end{cases}$$

It is clear that $S(\Delta) = [4,n]$ and that $|S(\Delta)| = n-3 = r-2$. If $r \leq n-2$, let Δ be defined by

$$\Delta(x) = \begin{cases} 1 & 1 \leq x \leq n-r-1 \\ 0 & n-r \leq x \leq n-1 \\ 1 & x = n. \end{cases}$$

If $r = n-2$, then $\Delta(S) = [4, n-1]$; if $r = \frac{n}{2}$, then $\Delta(S) = [2, n-r-1]$; and if $\frac{n}{2} < r < n-2$, then $\Delta(S) = [2, n-r-1] \cup [2n-2r, n-1]$. Since in each case $|S(\Delta)| = r-2$, it follows that $f(n,r) \leq r-2$.

Next we shall show that $f(n,r) \geq r-2$ for all integers n and r where $n \geq 5$ and $\frac{n}{2} \leq r < n$. We will use induction on the integer n. When $n = 5$ we must consider the two cases of $r = 3$ and $r = 4$. Since the 2-color Schur number is 5, every 2-coloring of the set $[1,5]$ with 3 monochromatic integers has at least 1 special integer, so $f(5,3) \geq 1$. It is easy to check that every 2-coloring of the set $[1,5]$ with 4 monochromatic integers has as least 2 special integers, so $f(5,4) \geq 2$. In both cases $f(5,r) \geq r-2$, so the basis step is complete. Now, let an integer $n_0 \geq 6$ be given. We may assume that

$$f(n,r) \geq r-2 \text{ for every } n \in [5, n_0-1] \text{ and for every } r \text{ such that } \frac{n}{2} \leq r < n. \tag{2}$$

We must show that $f(n_0, r) \geq r-2$ for all integers r such that $\frac{n_0}{2} \leq r < n_0$. Let an integer r such that $\frac{n_0}{2} \leq r < n_0$ be given and let a coloring $\Delta : [1, n_0] \to [0,1]$ with r monochromatic integers be given. We must show that $|S(\Delta)| \geq r-2$.

Without loss of generality, we may assume that Δ has r integers colored 0. If $\Delta(n_0) = 1$, then the coloring $\Delta|_{n_0-1}$ has r integers colored 0. If $r \leq n_0 - 2$, then by (2) it follows that $\Delta|_{n_0-1}$ has at least $r-2$ special integers, so Δ does as well. If $r = n_0 - 1$ it has been previously shown that $\Delta|_{n_0-1}$ has $r-1$ special integers. In each case $|S(\Delta)| \geq r-2$, so we may assume that $\Delta(n_0) = 0$.

We shall consider three cases.

Case 1: Assume that $r > \frac{n_0+1}{2}$.

Since Δ has r integers colored 0 and $\Delta(n_0) = 0$, it follows that $\Delta|_{n_0-1}$ has $r-1$ integers colored 0. Hence, from (2) we have that

$$|S(\Delta|_{n_0-1})| \geq r-3,$$

so it suffices to show that n_0 is special. We shall consider the two subcases where n_0 is even and where n_0 is odd.

(i) Assume that n_0 is even. It follows that $r > \frac{n_0}{2}$.

If $\Delta\left(\frac{n_0}{2}\right) = 0$, then since $\Delta(n_0) = 0$ and $\frac{n_0}{2} + \frac{n_0}{2} = n_0$, it follows that n_0 is special. In this case we are done, so we may assume that $\Delta\left(\frac{n_0}{2}\right) = 1$.

Let $A = \left[1, \frac{n_0}{2} - 1\right] \cup \left[\frac{n_0}{2} + 1, n_0 - 1\right]$. Since there are $r-1$ integers colored 0 in the set A and $r > \frac{n_0}{2}$, there are at least $\frac{n_0}{2}$ integers in the set A colored 0. For every $i \in \left[1, \frac{n_0}{2} - 1\right]$ let $A_i = \{i, n_0 - i\}$. Hence the set $\left\{A_1, A_2, \ldots, A_{\frac{n_0}{2}-1}\right\}$ is a partition of the set A into $\frac{n_0}{2} - 1$ subsets and there exists an $i \in \left[1, \frac{n_0}{2} - 1\right]$ such that $\Delta(i) = \Delta(n_0 - i) = 0$. Then $(i, n_0 - 1, n_0)$ is a monochromatic Schur triple, so n_0 is special.

(ii) Assume that n_0 is odd.

This subcase is very similar to the n_0 even subcase. Let $A = [1, n_0 - 1]$ and for every $i \in [1, \frac{n_0-1}{2}]$, let $A_i = \{i, n_0 - i\}$. Since the set A has $r-1$ integers colored 0 and $r-1 \geq \frac{n_0+1}{2}$, there exists an $i \in [1, \frac{n_0-1}{2}]$ such that $(i, n_0 - i, n_0)$ is a monochromatic Schur triple, so n_0 is special.

Since in both subcases n_0 is special and it was previously shown that $|S(\Delta|_{n_0-1})| \geq r-3$, we have that $|S(\Delta)| \geq r-2$.

Case 2: Assume that n_0 is even and $r = \frac{n_0}{2}$.

It follows that $\Delta|_{n_0-1}$ has $r-1$ integers colored 0 and r integers colored 1. From (2) it follows that $\Delta|_{n_0-1}$ has at least $r-2$ special integers, so Δ does as well and $|S(\Delta)| \geq r-2$.

Case 3: Assume that n_0 is odd and $r = \frac{n_0+1}{2}$.

Since $n_0 - 2$ is odd, it follows that $\Delta|_{n_0-2}$ has $\frac{n_0-1}{2} = r-1$ monochromatic integers. From (2) it follows that $\Delta|_{n_0-2}$ has at least $r-3$ special integers, so it will be sufficient to show that $S(\Delta) \cap \{n_0 - 1, n_0\} \neq \emptyset$.

Recall that Δ has $r = \frac{n_0+1}{2}$ integers colored 0 and $\Delta(n_0) = 0$. Hence, exactly $\frac{n_0-1}{2}$ integers in the set $[1, n_0 - 1]$ are colored 0. Let $A = [1, n_0 - 1]$ and for every $i \in \left[1, \frac{n_0-1}{2}\right]$ let $A_i = \{i, n_0 - i\}$. Hence the set $\left\{A_1, A_2, \ldots, A_{\frac{n_0-1}{2}}\right\}$ is a partition of the set A into $\frac{n_0-1}{2}$ subsets. If, for any $i \in \left[1, \frac{n_0-1}{2}\right]$, the set A_i is monochromatic in 0, then $(i, n_0 - i, n_0)$ is a monochromatic Schur triple and n_0 is special. In this case we are done, so we may assume that, for every $i \in \left[1, \frac{n_0-1}{2}\right]$, A_i contains one integer colored 0 and one integer colored 1.

Now we will prove the following claim.

Claim. *Let $\Delta(n_0 - 1) = a$ and let $b = 1 - a$. If $S(\Delta) \cap \{n_0 - 1, n_0\} = \emptyset$, then $\Delta(x) = b$ for every $x \in \left[1, \frac{n_0-1}{2}\right]$.*

Proof. Assume that the hypothesis of the claim is satisfied and note that $\{a, b\} = \{0, 1\}$. We shall show that $\Delta\left(\frac{n_0+1}{2} - i\right) = b$ for every $i \in \left[1, \frac{n_0-1}{2}\right]$ by using induction on i. First we shall show that

$$\Delta\left(\frac{n_0+1}{2} - 1\right) = b.$$

If $\Delta\left(\frac{n_0+1}{2} - 1\right) = a$, then $\left(\frac{n_0+1}{2} - 1, \frac{n_0+1}{2} - 1, n_0 - 1\right)$ is a monochromatic Schur triple and $n_0 - 1 \in S(\Delta)$, which is a contradiction. Hence, we may assume that $\Delta\left(\frac{n_0+1}{2} - 1\right) = b$.

Now, let $i_0 \in \left[2, \frac{n_0-1}{2}\right]$ be given and assume that $\Delta\left(\frac{n_0+1}{2} - (i_0 - 1)\right) = b$. We will show that

$$\Delta\left(\frac{n_0+1}{2} - i_0\right) = b.$$

Now, $A_{\frac{n_0+3}{2}-i_0} = \left\{\frac{n_0+1}{2} - (i_0 - 1), \frac{n_0-3}{2} + i_0\right\}$. Since $A_{\frac{n_0+3}{2}-i_0}$ contains an integer colored a, it follows that

$$\Delta\left(\frac{n_0-3}{2} + i_0\right) = a.$$

If $\Delta\left(\frac{n_0+1}{2} - i_0\right) = a$, then $\left(\frac{n_0-3}{2} + i_0, \frac{n_0+1}{2} - i_0, n_0 - 1\right)$ is a monochromatic Schur triple and $n_0 - 1 \in S_3(\Delta)$, which is a contradiction. Hence, we may assume that $\Delta\left(\frac{n_0+1}{2} - i_0\right) = b$ and the proof of the claim is complete. □

Now, as noted above, if $S(\Delta) \cap \{n_0 - 1, n_0\} \neq \emptyset$, then $|S(\Delta)| \geq r - 2$. If $S(\Delta) \cap \{n_0 - 1, n_0\} = \emptyset$, then from the claim we have that $\Delta(x) = b$ for every $x \in \left[1, \frac{n_0-1}{2}\right]$. This implies that $\left[2, \frac{n_0-1}{2}\right] \subseteq S(\Delta)$ and that $|S(\Delta)| \geq \frac{n_0-3}{2} = r - 2$. In either case we have that $|S(\Delta)| \geq r - 2$, so we are done with Case 3.

Since in all three cases we showed that $|S(\Delta)| \geq r - 2$, the proof of Theorem 1 is complete. □

We are now ready to state and prove Theorem 2. The theorem follows directly from previous results.

Theorem 2. *For every natural number n,*

$$f(n) = \begin{cases} 0 & 1 \leq n \leq 4 \\ \left\lfloor \frac{n-3}{2} \right\rfloor & n \geq 5. \end{cases}$$

Proof. The case where $1 \leq n \leq 4$ follows directly from the fact that the 2-color Schur number is 5. If $n \geq 5$, then every coloring of the set $[1, n]$ contains at least $\left\lfloor \frac{n+1}{2} \right\rfloor$ monochromatic integers. From Theorem 1 it follows that every coloring of the set $[1, n]$ contains at least $\left\lfloor \frac{n+1}{2} \right\rfloor - 2 = \left\lfloor \frac{n-3}{2} \right\rfloor$ special integers. □

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