SOME ARITHMETIC PROPERTIES OF OVERPARTITION K-TUPLES

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Abstract

Recently, Lovejoy introduced the construct of overpartition pairs which are a natural generalization of overpartitions. Here we generalize that idea to overpartition k-tuples and prove several congruences related to them. We denote the number of overpartition k-tuples of a positive integer n by $\overline{p}_k(n)$ and prove, for example, that for all $n \geq 0$, $\overline{p}_{t-1}(tn+r) \equiv 0 \pmod{t}$ where t is prime and r is a quadratic nonresidue mod t.

1. Introduction

As defined by Corteel and Lovejoy [5], an *overpartition* of a positive integer n is a non-increasing sequence of natural numbers whose sum is n in which the first occurrence of a part may be overlined. For example, the overpartitions of the integer 3 are

$$3,\ \overline{3},\ 2+1,\ \overline{2}+\overline{1},\ \overline{2}+1,\ 2+\overline{1},\ 1+1+1,\ \overline{1}+1+1.$$

The number of overpartitions of a positive integer n is denoted by $\overline{p}(n)$, with $\overline{p}(0) = 1$ by definition. Thus $\overline{p}(3) = 8$ from the above example. As noted in Corteel and Lovejoy [5], the generating function for overpartitions is

$$\sum_{n=0}^{\infty} \overline{p}(n) q^n \;=\; \prod_{n=1}^{\infty} \frac{1+q^n}{1-q^n}.$$

As the topic of overpartitions has already been examined rather thoroughly [3, 4, 5, 6, 7, 8, 10, 11], we look to new constructions. One such construction is that of an *overpartition pair* of a positive integer n, defined by Lovejoy [9] as a pair of overpartitions wherein the sum of all listed parts is n. For example, the overpartition pairs of 2 are

$$(2\ ;\ \emptyset),\ (\overline{2}\ ;\ \emptyset),\ (\emptyset\ ;\ \overline{2}),\ (\emptyset\ ;\ 2),\ (1+1\ ;\ \emptyset),\ (\overline{1}+1\ ;\ \emptyset),\ (\emptyset\ ;\ 1+1),\ (\emptyset\ ;\ \overline{1}+1),$$
$$(1\ ;\ \overline{1}),\ (1\ ;\ 1),\ (\overline{1}\ ;\ \overline{1}),\ (\overline{1}\ ;\ 1).$$

Lovejoy denoted the number of overpartition pairs of a positive integer n by $\overline{pp}(n)$, with $\overline{pp}(0) = 1$ by definition. Thus $\overline{pp}(2) = 12$ from the above example. Following lines similar to that for overpartitions, the generating function for overpartition pairs is

$$\sum_{n=0}^{\infty} \overline{pp}(n) q^n = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^2 .$$

Several arithmetic properties of both overpartitions and their pairs have appeared in the literature. Since our interest here is primarily on congruence properties, there are a few theorems that are especially noteworthy. The first one is straightforward and proven intuitively.

Theorem 1. *For all* $n > 0$, $\overline{p}(n) \equiv 0 \pmod{2}$.

Next we have a theorem easily proven using results of Mahlburg [10].

Theorem 2. *For all* $n > 0$,

$$\overline{p}(n) \equiv \begin{cases} 2 \pmod{4} & \text{if } n \text{ is a square,} \\ 0 \pmod{4} & \text{otherwise.} \end{cases}$$

Several other congruences in arithmetic progressions were proven by Hirschhorn and Sellers. For example, the following were proven in [7].

Theorem 3. *For all* $n \geq 0$,

$$\begin{aligned} \overline{p}(5n+2) &\equiv 0 \pmod{4}, \\ \overline{p}(5n+3) &\equiv 0 \pmod{4}, \\ \overline{p}(4n+3) &\equiv 0 \pmod{8}, \\ \text{and} \qquad \overline{p}(8n+7) &\equiv 0 \pmod{64}. \end{aligned}$$

Also, Hirschhorn and Sellers [6] proved that $\overline{p}(n)$ satisfies congruences modulo non-powers of 2 by proving the following:

Theorem 4. *For all* $n \geq 0$ *and all* $\alpha \geq 0$, $\overline{p}\left(9^{\alpha}(27n+18)\right) \equiv 0 \pmod{12}$.

Finally, we note a theorem proven by Bringmann and Lovejoy [2]. This result provides much inspiration for the main result in the next section.

Theorem 5. *For all* $n \geq 0$, $\overline{pp}(3n+2) \equiv 0 \pmod{3}$.

We now introduce a generalization of overpartition pairs. An *overpartition k-tuple* of a positive integer n is a k-tuple of overpartitions wherein all listed parts sum to n. We denote the number of overpartition k-tuples of n by $\overline{p}_k(n)$, with $\overline{p}_k(0) = 1$ by definition. Consequently, the number of overpartition pairs of n is denoted as $\overline{p}_2(n)$. The generating function for $\overline{p}_k(n)$ is easily seen to be

$$\sum_{n\geq 0} \overline{p}_k(n)q^n = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n}\right)^k.$$

The aim of this note is to prove several congruence properties for families of overpartition k-tuples. In the process, we will prove several natural generalizations of results quoted above.

2. Results for Overpartition k-Tuples

Our first theorem of this section provides a natural generalization of Bringmann and Lovejoy's Theorem 5 above. Moreover, the proof technique is extremely elementary, making this a very satisfying result.

Theorem 6. *For all $n \geq 0$, $\overline{p}_{t-1}(tn+r) \equiv 0 \pmod{t}$, where t is an odd prime and r is a quadratic nonresidue mod t.*

Remarks. First, note that the $t = 3$ case of this theorem is exactly Theorem 5. Secondly, note that, for each odd prime t, this theorem provides $\frac{t-1}{2}$ congruence properties for $\overline{p}_{t-1}(n)$.

Proof. Consider the following generating function manipulations:

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{p}_{t-1}(n)q^n &= \prod_{i=1}^{\infty}\left(\frac{1+q^i}{1-q^i}\right)^{t-1} \\
&= \left[\prod_{i=1}^{\infty}\frac{1+q^i}{1-q^i}\right]^{t-1} \\
&= \left[\prod_{i=1}^{\infty}\frac{1+q^i}{1-q^i}\right]^{t}\left[\prod_{i=1}^{\infty}\frac{1-q^i}{1+q^i}\right] \\
&\equiv \left[\prod_{i=1}^{\infty}\frac{1+q^{ti}}{1-q^{ti}}\right]\left[\prod_{i=1}^{\infty}\frac{1-q^i}{1+q^i}\right] \quad \pmod{t} \text{ since } t \text{ is prime} \\
&= \sum_{m=0}^{\infty}\overline{p}(m)q^{tm}\left[\prod_{i=1}^{\infty}\frac{1-q^i}{1+q^i}\right] \\
&= \sum_{m=0}^{\infty}\overline{p}(m)q^{tm}\sum_{s=-\infty}^{\infty}(-1)^s q^{s^2} \text{ thanks to Gauss [1, Cor. 2.10].}
\end{aligned}$$

But note that $tn+r$ can never be represented as $tm+s^2$ for some integers m and s if r is a quadratic nonresidue mod t. This implies that $\overline{p}_{t-1}(tn+r) \equiv 0 \pmod{t}$ for all $n \geq 0$. □

The next theorem is a broad generalization of Theorem 1. It is found with proof in [12], but is included here for the sake of completeness. We require a brief technical lemma.

Lemma 7. *Let m be a nonnegative integer. For all $1 \leq n \leq 2^m$,*

$$\binom{2^m}{n} 2^n \equiv 0 \pmod{2^{m+1}}.$$

Proof. Let $\mathrm{ord}_2(N)$ be the exponent of the highest power of 2 dividing N. Thus, for example, $\mathrm{ord}_2(8) = 3$ while $\mathrm{ord}_2(80) = 4$. To prove Lemma 7, we need to prove that

$$\mathrm{ord}_2\left(\binom{2^m}{n} 2^n\right) \geq m+1. \tag{1}$$

Note that

$$\begin{aligned}
\mathrm{ord}_2\left(\binom{2^m}{n} 2^n\right) &= \mathrm{ord}_2\left(\frac{2^m(2^m-1)(2^m-2)\cdots(2^m-(n-1))}{n!} \cdot 2^n\right) \\
&\geq \mathrm{ord}_2\left(\frac{2^{m+n}}{n!}\right) \\
&= m+n-\mathrm{ord}_2(n!) \\
&= m+n-\left(\left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{n}{4}\right\rfloor + \left\lfloor\frac{n}{8}\right\rfloor + \cdots\right)
\end{aligned}$$

where $\lfloor x \rfloor$ is the floor function of x.

Now assume $n = c_0 2^0 + c_1 2^1 + \cdots + c_t 2^t$ where each $c_i \in \{0, 1\}$. Then

$$\begin{aligned}
\left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{n}{4}\right\rfloor + \left\lfloor\frac{n}{8}\right\rfloor + \cdots &= c_1 2^0 + c_2 2^1 + \cdots c_t 2^{t-1} \\
&\quad + c_2 2^0 + c_3 2^1 + \cdots c_t 2^{t-2} \\
&\quad + c_3 2^0 + c_4 2^1 + \cdots c_t 2^{t-3} \\
&\quad \vdots \\
&\quad + c_t 2^0 \\
&= (2-1)c_1 + (2^2-1)c_2 + (2^3-1)c_3 + \cdots + (2^t-1)c_t \\
&= n - (c_0 + c_1 + c_2 + \cdots + c_t) \\
&\leq n-1
\end{aligned}$$

since at least one of the c_i must equal 1. Therefore,

$$\begin{aligned} ord_2\left(\binom{2^m}{n}2^n\right) &\geq m+n-\left(\left\lfloor\frac{n}{2}\right\rfloor+\left\lfloor\frac{n}{4}\right\rfloor+\left\lfloor\frac{n}{8}\right\rfloor+\cdots\right) \\ &\geq m+n-(n-1) \\ &= m+1. \end{aligned}$$

This is the desired result as noted in (1) above. □

We are now in a position to prove the following theorem:

Theorem 8. *Let $k=(2^m)\,r$, where m is a nonnegative integer and r is odd. Then, for all positive integers n, we have $\overline{p}_k(n) \equiv 0 \pmod{2^{m+1}}$.*

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty}\overline{p}_k(n)q^n &= \prod_{i=1}^{\infty}\left[\frac{1+q^i}{1-q^i}\right]^k \\ &= \prod_{i=1}^{\infty}\left[\frac{1+q^i}{1-q^i}\right]^{(2^m)r} \\ &= \left(\prod_{i=1}^{\infty}\left[\frac{1+q^i}{1-q^i}\right]^{2^m}\right)^r \\ &= \left(\prod_{i=1}^{\infty}\left[1+\frac{2q^i}{1-q^i}\right]^{2^m}\right)^r \\ &= \left(\prod_{i=1}^{\infty}\left[1+\sum_{n=1}^{2^m}\binom{2^m}{n}2^n\left(\frac{q^i}{1-q^i}\right)^n\right]\right)^r \\ &\equiv 1 \pmod{2^{m+1}} \text{ by Lemma 7.} \end{aligned}$$

□

The following theorem is inspired by Theorem 2. As with Theorem 8, it primarily hinges upon the use of the binomial theorem.

Theorem 9. *Let $k=(2^m)\,r$, $m>0$ and r is odd. Then, for all $n \geq 1$,*

$$\overline{p}_k(n) \equiv \begin{cases} 2^{m+1} \pmod{2^{m+2}} & \textit{if } n \textit{ is a square or twice a square,} \\ 0 \pmod{2^{m+2}} & \textit{otherwise.} \end{cases}$$

Proof. We prove this result by induction on m.
`Basis Step.` Let $m = 1$. We must show that

$$\overline{p}_{2r}(n) \equiv \begin{cases} 4 \pmod 8 & \text{if } n \text{ is a square or twice a square,} \\ 0 \pmod 8 & \text{otherwise.} \end{cases}$$

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{p}_{2r}(n) q^n &= \prod_{i=1}^{\infty} \left(\frac{1+q^i}{1-q^i} \right)^{2r} \\
&= \left(\left[\sum_{n=0}^{\infty} \overline{p}(n) q^n \right]^2 \right)^r \\
&= \left(\left[1 + \sum_{\substack{n>0 \\ \text{square}}} \overline{p}(n) q^n + \sum_{\substack{n>0 \\ \text{not square}}} \overline{p}(n) q^n \right]^2 \right)^r \\
&= \left[1 + 2 \left(\sum_{\substack{n>0 \\ \text{square}}} \overline{p}(n) q^n \right) + \left(\sum_{\substack{n>0 \\ \text{square}}} \overline{p}(n) q^n \right)^2 \right. \\
&\quad + 2 \left(\sum_{\substack{n>0 \\ \text{square}}} \overline{p}(n) q^n \right) \left(\sum_{\substack{n>0 \\ \text{not square}}} \overline{p}(n) q^n \right) \\
&\quad \left. + 2 \left(\sum_{\substack{n>0 \\ \text{not square}}} \overline{p}(n) q^n \right) + \left(\sum_{\substack{n>0 \\ \text{not square}}} \overline{p}(n) q^n \right)^2 \right]^r
\end{aligned}$$

From Theorem 2, we know that $\overline{p}(n) \equiv 2$ or $6 \pmod 8$ when n is a square and $\overline{p}(n) \equiv 0$ or $4 \pmod 8$ otherwise. Since 2×0, 2×4, 6×0, 6×4, 0×0, 0×4, and 4×4 are all congruent to $0 \pmod 8$,

$$\begin{aligned}
2 \left(\sum_{\substack{n>0 \\ \text{square}}} \overline{p}(n) q^n \right) \left(\sum_{\substack{n>0 \\ \text{not square}}} \overline{p}(n) q^n \right) &\equiv 0 \pmod 8, \\
2 \left(\sum_{\substack{n>0 \\ \text{not square}}} \overline{p}(n) q^n \right) &\equiv 0 \pmod 8, \\
\text{and } \left(\sum_{\substack{n>0 \\ \text{not square}}} \overline{p}(n) q^n \right)^2 &\equiv 0 \pmod 8.
\end{aligned}$$

This gives

$$\sum_{n=0}^{\infty} \overline{p}_{2r}(n)q^n \equiv \left[1 + 2\left(\sum_{n=1}^{\infty}\overline{p}(n^2)q^{n^2}\right) + \left(\sum_{n=1}^{\infty}\overline{p}(n^2)q^{n^2}\right)^2\right]^r \pmod 8$$
$$\equiv \left[1 + 4\left(\sum_{n=1}^{\infty}q^{n^2}\right) + 4\left(\sum_{n=1}^{\infty}q^{n^2}\right)^2\right]^r \pmod 8$$

again thanks to Theorem 2.

Given that $(q^{n_1} + q^{n_2} + \cdots)^2 = \left(q^{2n_1} + q^{2n_2} + \cdots\right) + 2\left(q^{n_1+n_2} + \cdots\right)$, we then have

$$\sum_{n=0}^{\infty} \overline{p}_{2r}(n)q^n \equiv \left[1 + 4\left(\sum_{n=1}^{\infty}q^{n^2}\right) + 4\left(\sum_{n=1}^{\infty}q^{2n^2} + 2\sum_{\substack{n_1,n_2>0\\ n_1\neq n_2}} q^{n_1^2+n_2^2}\right)\right]^r \pmod 8$$
$$\equiv \left[1 + 4\left(\sum_{m=1}^{\infty}q^{m^2} + \sum_{n=1}^{\infty}q^{2n^2}\right)\right]^r \pmod 8$$
$$= \sum_{j=0}^{\infty}\binom{r}{j}4^j\left(\sum_{m=1}^{\infty}q^{m^2} + \sum_{n=1}^{\infty}q^{2n^2}\right)^j$$
$$\equiv 1 + 4\left(\sum_{m=1}^{\infty}q^{m^2} + \sum_{n=1}^{\infty}q^{2n^2}\right) \pmod 8 \text{ since } r \text{ is odd.}$$

This proves the result needed for the basis step.

`Induction Step.` Assume that

$$\overline{p}_{(2^m)r}(n) \equiv \begin{cases} 2^{m+1} \pmod{2^{m+2}} & \text{if n is a square or twice a square,} \\ 0 \pmod{2^{m+2}} & \text{otherwise.} \end{cases}$$

We must show that

$$\overline{p}_{(2^{m+1})r}(n) \equiv \begin{cases} 2^{m+2} \pmod{2^{m+3}} & \text{if n is a square or twice a square,} \\ 0 \pmod{2^{m+3}} & \text{otherwise.} \end{cases}$$

Consider the generating function for $\overline{p}_{2^{m+1}}(n)$:

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{p}_{(2^{m+1})r}(n)q^n &= \prod_{i=1}^{\infty}\left(\frac{1+q^i}{1-q^i}\right)^{(2^{m+1})r} \\
&= \left(\prod_{i=1}^{\infty}\left(\frac{1+q^i}{1-q^i}\right)^{2^m r}\right)^2 \\
&= \left(\sum_{n=0}^{\infty} \overline{p}_{(2^m)r}(n)q^n\right)^2 \\
&= \left(1 + \sum_{\substack{n>0 \\ \text{not sq.} \\ \text{and not} \\ \text{twice sq.}}} \overline{p}_{(2^m)r}(n)q^n + \sum_{\substack{n>0 \\ \text{square or} \\ \text{twice sq.}}} \overline{p}_{(2^m)r}(n)q^n\right)^2 \\
&= 1 + 2\left(\sum_{\substack{n>0 \\ \text{square or} \\ \text{twice sq.}}} \overline{p}_{(2^m)r}(n)q^n\right) + \left(\sum_{\substack{n>0 \\ \text{square or} \\ \text{twice sq.}}} \overline{p}_{(2^m)r}(n)q^n\right)^2 \\
&\quad + 2\left(\sum_{\substack{n>0 \\ \text{square or} \\ \text{twice sq.}}} \overline{p}_{(2^m)r}(n)q^n\right)\left(\sum_{\substack{n>0 \\ \text{not sq.} \\ \text{and not} \\ \text{twice sq.}}} \overline{p}_{(2^m)r}(n)q^n\right) \\
&\quad + 2\left(\sum_{\substack{n>0 \\ \text{not sq.} \\ \text{and not} \\ \text{twice sq.}}} \overline{p}_{(2^m)r}(n)q^n\right) + \left(\sum_{\substack{n>0 \\ \text{not sq.} \\ \text{and not} \\ \text{twice sq.}}} \overline{p}_{(2^m)r}(n)q^n\right)^2.
\end{aligned}$$

Using a very similar argument about the coefficients to that of the basis step, we use the induction hypothesis to conclude that

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{p}_{(2^{m+1})r}(n)q^n &\equiv 1 + 2\sum_{n=1}^{\infty} \overline{p}_{(2^m)r}(n^2)q^{n^2} + \sum_{s=1}^{\infty} \overline{p}_{(2^m)r}(2s^2)q^{2s^2} \\
&\quad + \left(\sum_{n=1}^{\infty} \overline{p}_{(2^m)r}(n^2)q^{n^2} + \sum_{s=1}^{\infty} \overline{p}_{(2^m)r}(2s^2)q^{2s^2}\right)^2 \pmod{2^{m+3}} \\
&\equiv 1 + 2\left(\sum_{n=1}^{\infty} \overline{p}_{(2^m)r}(n^2)q^{n^2} + \sum_{s=1}^{\infty} \overline{p}_{(2^m)r}(2s^2)q^{2s^2}\right) \pmod{2^{m+3}}.
\end{aligned}$$

We know that all coefficients of the last term are congruent to 2^{m+1} or $2^{m+1} + 2^{m+2} \pmod{2^{m+3}}$ from the induction hypothesis. But the last term is multiplied by 2. So then all coefficients are congruent to $2^{m+2} \pmod{2^{m+3}}$ or $2^{m+2} + 2^{m+3} \equiv 2^{m+2} \pmod{2^{m+3}}$, which implies

$$\sum_{n=0}^{\infty} \overline{p}_{(2^{m+1})r}(n) q^n \equiv 1 + 2^{m+2} \left(\sum_{n=1}^{\infty} q^{n^2} + \sum_{s=1}^{\infty} q^{2s^2} \right) \pmod{2^{m+3}}.$$

This completes the induction and proves the theorem. □

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