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INTEGER PARTITIONS INTO ARITHMETIC PROGRESSIONS WITH AN ODD COMMON DIFFERENCE

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Abstract

Thomas E. Mason has shown that the number of ways in which a number n may be partitioned into consecutive parts, including the case of a single term, equals the number of odd divisors of n. This result is generalized by determining the number of partitions of n into arithmetic progressions with odd common difference, including the case of a single term.

1. Introduction

Let n be an integer. A partition of n is an integral solution of the system:

$$\begin{cases} n = n_1 + \cdots + n_k, \\ 1 \leq n_1 \leq \cdots \leq n_k. \end{cases}$$

The positive integers $n_1, \ldots, n_k$ are called *parts*, and k is the *length* of the partition. In partition identities, we are often interested in the number of partitions that satisfy some conditions. For example, the number of partitions into odd parts, the number of partitions into even parts, the number of partitions into distinct parts, and so on. For more on integer partitions, see for instance [1-5] and [7]. Thomas E. Mason [6] was interested in the number of ways in which a number n may be partitioned into consecutive parts, including the case of a single term, and he has shown that this number equals the number of odd divisors of n. Let $n = 2^\alpha \cdot p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots\ p_r^{\alpha_r}$, where the ps are distinct odd primes and the αs are the powers to which they occur. Then the number of ways in which a number n may be partitioned into consecutive parts, including the case of a single term, equals $(\alpha_1 + 1) \cdot (\alpha_2 + 1) \cdots (\alpha_r + 1)$, except in the case $n = 2^\alpha$ where the number of ways is 1. In this paper we treat the case of partitions of n into arithmetic progressions with an odd common difference. For $1 \leq d \leq n-2$, if $n_i - n_{i-1} = d$, $i = 2, \ldots, k$, we say that we have a partition into an arithmetic progression of common difference d. Our problem can be formulated in terms of partitions: For a given positive integer n, what is the number of partitions of n into arithmetic progressions with odd common difference d?

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Partitioning n into an arithmetic progression of common difference d means that there exist two positive integers $l \geq 1$ and $m \geq 0$, such that

$$n = l + (l + d) + (l + 2d) + \cdots + (l + md). \tag{1}$$

Let us denote such a partition briefly by $\pi(l, m)$. From (1), we have

$$n = (m + 1)l + \frac{m(m + 1)}{2}\, d.$$

Hence,

$$dm^2 + (2l + d)m + 2l - 2n = 0. \tag{2}$$

The discriminant Δ of the equation (2) in m equals $\Delta = (2l + d)^2 - 8d(l - n) = (2l - d)^2 + 8dn$. Then,

$$m = \frac{-(2l + d) + \sqrt{(2l - d)^2 + 8dn}}{2d}\cdot \tag{3}$$

Since m is an integer, $(2l - d)^2 + 8dn$ is a perfect square, i.e., $(2l - d)^2 + 8dn = u^2$, with u odd. Then,

$$2dn = \frac{(u - (2l - d))}{2} \times \frac{(u + (2l - d))}{2}\cdot$$

Denoting A and B by:

$$A = \frac{u - (2l - d)}{2} \quad \text{and} \quad B = \frac{u + (2l - d)}{2},$$

we have,

$$A + B = u \;\text{ and }\; B - A = 2l - d. \tag{4}$$

Since u and d are odd, A and B must have different parity.

From (3) and (4) we get

$$l = \frac{d + B - A}{2} \quad \text{and} \quad m = \frac{A}{d} - 1\cdot \tag{5}$$

Consequently, d divides A. Hence

$$2n = \frac{A}{d} \times B. \tag{6}$$

`Case 1.` If A is even then B is odd and $n = \frac{A}{2d} \times B$. In this case from (5)

$$l = \frac{d + a - \frac{2dn}{a}}{2} \;\text{ and }\; m = \frac{2n}{a} - 1, \tag{7}$$

where $a = B$, is an odd divisor of n. Since $l \geq 1$, it follows that a must satisfy $(2n - a)d \leq a(a - 2)$.

`Case 2.` If A is odd then $\frac{A}{d}$ is odd as well and $n = \frac{A}{d} \times \frac{B}{2}$. In this case from (5) we get

$$l = \frac{d + \frac{2n}{a} - ad}{2} \quad \text{and} \quad m = a - 1, \tag{8}$$

where $a = \frac{A}{d}$, is an odd divisor of n. Because $l \geq 1$, a must satisfy $a(a - 1)d \leq 2(n - a)$. For every such odd divisor of n there exists a partition $\pi(l, m)$ into an arithmetic progression with an odd common difference, and vice versa. It is easy to see that an odd divisor a of n can not simultaneously satisfy the inequalities $(2n-a)d \leq a(a-2)$, $a(a-1)d \leq 2(n-a)$, otherwise we would get $(a-1)d \leq (a-2)$, which is a contradiction. Now we are able to formulate our result as follows:

Theorem 1. *Let $n \geq 3$ be a positive integer and let $1 \leq d \leq n - 2$ be an odd integer. Then the number of partitions of n into an arithmetic progression with a common difference d, including the case of a single term, is equal to the number of odd divisors a of n, satisfying $\big(2n - a\big)d \leq a(a - 2)$ or $a(a - 1)d \leq 2(n - a)$ and for every such odd divisor, the partition $\pi(l, m)$ is given by:*

$$\begin{cases} l = \frac{d + \frac{2n}{a} - ad}{2} \quad \textit{and} \quad m = a - 1 & \textit{if} \;\; a(a-1)d \leq 2(n-a), \\[2ex] l = \frac{d + a - \frac{2dn}{a}}{2} \quad \textit{and} \quad m = \frac{2n}{a} - 1 & \textit{if} \;\; (2n-a)d \leq a(a-2). \end{cases}$$

The following example illustrates the above theorem.

Example. Let $n = 15$ and $d = 3$. The odd divisors of 15 satisfying $\big(2n - a\big)d \leq a(a-2)$ or $a(a-1)d \leq 2(n-a)$ are 1, 3 and 15, so 15 admits three partitions into an arithmetic progression with a common difference equal to 3; each one is associated with one of these divisors and this can be shown as follows:

► **a=1** satisfies $a(a-1)d \leq 2(n-a)$. We have $l = \frac{d + \frac{2n}{a} - ad}{2} = 15$ and $m = a - 1 = 0$. The partition $\pi(15, 0)$ is **15**.

► **a=3** satisfies $a(a-1)d \leq 2(n-a)$. We have $l = \frac{d + \frac{2n}{a} - ad}{2} = 2$ and $m = a - 1 = 2$. The partition $\pi(2, 2)$ is **2 + 5 + 8**.

► **a=15** satisfies $\big(2n-a\big)d \leq a(a-2)$. We have $l = \frac{d + a - \frac{2dn}{a}}{2} = 6$ and $m = \frac{2n}{a} - 1 = 1$. The partition $\pi(6, 1)$ is **6 + 9**.

2. Concluding Remarks

Theorem 1 is an extension of Mason's theorem [6]. Indeed, for $d = 1$ and for every odd divisor a of n, one and only one of the two inequalities $a(a-1) \leq 2(n-a)$, $(2n-a) \leq a(a-2)$ necessarily holds. In fact, if $a = 2p+1$ and $n = ak$, then

$$\begin{cases} a(a-1) \leq 2(n-a) \Leftrightarrow p \leq k-1, \\ \\ (2n-a) \leq a(a-2) \Leftrightarrow p \geq k. \end{cases}$$

Hence, if $p \leq k-1$, we get $l = \frac{1-a+2k}{2}$ and $m = a-1$; otherwise $l = \frac{1+a-2k}{2}$ and $m = 2k-1$. For example, $n = 15$ has four odd divisors: 1, 3, 5 and 15:

► **a=1** $\Rightarrow p = 0 < k = 15$. We have $l = \frac{1-a+2k}{2} = 15$ and $m = a-1 = 0$.
The partition $\pi(15, 0)$ is **15**.

► **a=3** $\Rightarrow p = 1 < k = 5$. We have $l = \frac{1-a+2k}{2} = 4$ and $m = a-1 = 2$.
The partition $\pi(4, 2)$ is **4 + 5 + 6**.

► **a=5** $\Rightarrow p = 2 < k = 3$. We have $l = \frac{1-a+2k}{2} = 1$ and $m = a-1 = 4$.
The partition $\pi(1, 4)$ is **1 + 2 + 3 + 4 + 5**.

► **a=15** $\Rightarrow p = 7 > k = 1$. We have $l = \frac{1+a-2k}{2} = 7$ and $m = 2k-1 = 1$.
The partition $\pi(7, 1)$ is **7 + 8**.

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