THE (EXPONENTIAL) BIPARTITIONAL POLYNOMIALS AND POLYNOMIAL SEQUENCES OF TRINOMIAL TYPE: PART II

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Abstract

In a previous paper we investigated the (exponential) bipartitional polynomials involving polynomial sequences of trinomial type. Our aim is to give properties of bipartitional polynomials related to the derivatives of polynomial sequences of trinomial type. Furthermore, we deduce identities involving Bell polynomials.

1. Introduction

For $(m,n)\in\mathbb{N}^2$, the complete bipartional polynomial $A_{m,n}$ with variables $x_{0,1},x_{1,0}$, $\ldots,x_{m,n}$ is defined by the sum

$$A_{m,n}:=\sum\frac{m!n!}{k_{0,1}!k_{1,0}!\cdots k_{m,n}!}\left(\frac{x_{0,1}}{0!1!}\right)^{k_{0,1}}\left(\frac{x_{1,0}}{1!0!}\right)^{k_{1,0}}\cdots\left(\frac{x_{m,n}}{m!n!}\right)^{k_{m,n}},\qquad(1)$$

where the summation is carried over all partitions of the bipartite number (m,n), that is, over all nonnegative integers $k_{0,1},k_{1,0},\ldots,k_{m,n}$ which are solutions of the equations

$$\sum_{i=1}^{m}i\sum_{j=0}^{n}k_{i,j}=m,\quad \sum_{j=1}^{n}j\sum_{i=0}^{m}k_{i,j}=n.\qquad(2)$$

The partial bipartitional polynomial $A_{m,n,k}$ with variables $x_{0,1},x_{1,0},\ldots,x_{m,n}$, of degree $k\in\mathbb{N}$, is defined by the sum

$$A_{m,n,k}:=\sum\frac{m!n!}{k_{0,1}!k_{1,0}!\cdots k_{m,n}!}\left(\frac{x_{0,1}}{0!1!}\right)^{k_{0,1}}\left(\frac{x_{1,0}}{1!0!}\right)^{k_{1,0}}\cdots\left(\frac{x_{m,n}}{m!n!}\right)^{k_{m,n}},\qquad(3)$$

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where the summation is carried over all partitions of the bipartite number (m, n) into k parts, that is, over all nonnegative integers $k_{0,1}, k_{1,0}, \ldots, k_{m,n}$ which are solutions of the equations

$$\sum_{i=1}^{m} i \sum_{j=0}^{n} k_{i,j} = m, \quad \sum_{j=1}^{n} j \sum_{i=0}^{m} k_{i,j} = n, \quad \sum_{i=0}^{m}\sum_{j=0}^{n} k_{i,j} = k, \quad \text{with the convention } k_{0,0} = 0. \tag{4}$$

These polynomials were introduced in [2, pp. 454], with properties such as those given in (5), (6), (7), (8) below. Indeed, for all real numbers α, β, γ we have

$$A_{m,n,k}\left(\beta\gamma x_{0,1}, \alpha\gamma x_{1,0}, \ldots, \alpha^m \beta^n \gamma x_{m,n}\right) = \alpha^m \beta^n \gamma^k A_{m,n,k}\left(x_{0,1}, x_{1,0}, \ldots, x_{m,n}\right) \tag{5}$$

and

$$A_{m,n}\left(\alpha x_{0,1}, \beta x_{1,0}, \ldots, \alpha^m \beta^n x_{m,n}\right) = \alpha^m \beta^n A_{m,n}\left(x_{0,1}, x_{1,0}, \ldots, x_{m,n}\right). \tag{6}$$

Moreover, we can check that the exponential generating functions for $A_{m,n}$ and $A_{m,n,k}$ are respectively provided by

$$1 + \sum_{m+n \geq 1} A_{m,n}\left(x_{0,1}, x_{1,0}, \ldots, x_{m,n}\right) \frac{t^m}{m!} \frac{u^n}{n!} = \exp\left(\sum_{i+j \geq 1} x_{i,j} \frac{t^i}{i!} \frac{u^j}{j!}\right), \tag{7}$$

and

$$\sum_{m+n \geq k} A_{m,n,k}\left(x_{0,1}, x_{1,0}, \ldots, x_{m,n}\right) \frac{t^m}{m!} \frac{u^n}{n!} = \frac{1}{k!}\left(\sum_{i+j \geq 1} x_{i,j} \frac{t^i}{i!} \frac{u^j}{j!}\right)^k. \tag{8}$$

We present some recursive formulae related to the bipartitional polynomials and Bell polynomials. We also describe a connection with the successive derivatives of polynomials $(f_{m,n}(x))$ of trinomial type, defined by

$$\left(\sum_{i,j \geq 0} f_{i,j}(1) \frac{t^i}{i!} \frac{u^j}{j!}\right)^x = \sum_{m,n \geq 0} f_{m,n}(x) \frac{t^m}{m!} \frac{u^n}{n!}, \quad \text{with } f_{0,0}(x) := 1. \tag{9}$$

We use the notations $A_{m,n,k}(x_{i,j})$ for $A_{m,n,k}(x_{0,1}, \ldots, x_{m,n})$, and $A_{m,n}(x_{i,j})$ for $A_{m,n}(x_{0,1}, \ldots, x_{m,n})$. Moreover, we represent by $B_{n,k}(x_j)$ and $B_n(x_j)$ the partial and complete Bell polynomials $B_{n,k}(x_1, x_2, \ldots)$ and $B_n(x_1, x_2, \ldots)$, respectively. For $m < 0$ or $n < 0$, we set $f_{m,n}(x) = 0$ and $A_{m,n}(x_{i,j}) = 0$, and for $m < 0$ or $n < 0$ or $k < 0$, we set $A_{m,n,k}(x_{i,j}) = 0$. With the above notation, in [6] we proved the following results.

Theorem 1. *We have*

$$A_{0,n,k}\left(x_{I,J}\right)=B_{n,k}\left(x_{0,j}\right),\quad A_{m,0,k}\left(x_{I,J}\right)=B_{m,k}\left(x_{j,0}\right), \tag{10}$$

$$A_{0,n}\left(x_{I,J}\right)=B_{n}\left(x_{0,j}\right),\quad A_{m,0}\left(x_{I,J}\right)=B_{m}\left(x_{j,0}\right), \tag{11}$$

$$A_{m+k,n,k}\left(ix_{i+j}\right)=\frac{\binom{m+k}{k}}{\binom{m+n+k}{k}}B_{m+n+k,k}\left(jx_{j}\right), \tag{12}$$

$$A_{m,n+k,k}\left(jx_{i+j}\right)=\frac{\binom{n+k}{k}}{\binom{m+n+k}{k}}B_{m+n+k,k}\left(jx_{j}\right). \tag{13}$$

Theorem 2. *Let $(f_{m,n}(x))$ be a sequence of trinomial type and let a,b be real numbers. We have*

$$\begin{aligned}A_{m+k,n+k,k}&\left(xij\frac{f_{i-1,j-1}\left(a\left(i-1\right)+b\left(j-1\right)+x\right)}{a\left(i-1\right)+b\left(j-1\right)+x}\right)\\&=k!kx\binom{m+k}{k}\binom{n+k}{k}\frac{f_{m,n}\left(am+bn+kx\right)}{am+bn+kx},\end{aligned} \tag{14}$$

$$A_{m+k,n,k}\left(xi\frac{f_{i-1,j}\left(a\left(i-1\right)+bj+x\right)}{a\left(i-1\right)+bj+x}\right)=kx\binom{m+k}{k}\frac{f_{m,n}\left(am+bn+kx\right)}{am+bn+kx}, \tag{15}$$

$$A_{m,n}\left(\frac{x}{ai+bj}f_{i,j}\left(ai+bj\right)\right)=\frac{x}{am+bn+x}f_{m,n}\left(am+bn+x\right). \tag{16}$$

Theorem 3. *Let $(f_{m,n}(x))$ be a sequence of trinomial type and let a,b be real numbers. Then the sequence $(h_{m,n}(x))$ defined by*

$$h_{m,n}\left(x\right):=\frac{x}{am+bn+x}f_{m,n}\left(am+bn+x\right) \tag{17}$$

is of trinomial type.

2. Some Formulae Related to Bipartitional and Bell Polynomials

We present several identities related to bipartitional polynomials and Bell polynomials via recursive relations.

Theorem 4. *Let $(x_{m,n})$ be a sequence of real numbers with $x_{1,1}=1$ and let p, q and d be nonnegative integers, $d\geq\max(p,q)+1$. Then*

$$\begin{aligned}A_{m+k,n+k,k}&\left(d\frac{A_{i-1+r,\ j-1+r,\ r}\left(ghx_{g,h}\right)}{(r-1)!\binom{i-1+r}{i}\binom{j-1+r}{j}}\right)\\&=d\frac{k!k\binom{m+k}{k}\binom{n+k}{k}}{R!R\binom{m+R}{m}\binom{n+R}{n}}A_{m+R,n+R,R}\left(ghx_{g,h}\right),\end{aligned} \tag{18}$$

where $r=p\left(i-1\right)+q\left(j-1\right)+d$ and $R=pm+qn+dk$.

Proof. Let $(f_{m,n}(x))$ be a trinomial type sequence of polynomials with

$$f_{m,n}(1) := x_{m+1,n+1}.$$

Taking $a = b = 0$ and $x = 1$ in (14), we get

$$f_{m,n}(k) = \frac{1}{k!}\binom{m+k}{k}^{-1}\binom{n+k}{k}^{-1} A_{m+k,n+k,k}(ijx_{i,j}). \tag{19}$$

To obtain (18), in (14) set $a = p$, $b = q$ and $x = d$ and use (19) to express $f_{m,n}(pn + qn + dk)$ and $f_{i-1,j-1}(p(i-1) + q(j-1) + d)$ in terms of partial bipartitional polynomials. □

Theorem 5. *Let p, q and d be nonnegative integers. Then, for a given sequence $(x_{m,n})$ of real numbers with $x_{1,0} = 1$, we have*

$$\begin{aligned} &A_{m+k,n,k}\left(d\frac{A_{(p+1)(i-1)+qj+d,\ j,\ p(i-1)+qj+d}(gx_{g,h})}{\binom{(p+1)(i-1)+qj+d}{i}}\right) \\ &= dk\binom{m+k}{k}\frac{A_{(p+1)m+qn+kd,\ n,\ pm+qn+kd}(gx_{g,h})}{(pm+qn+kd)\binom{(p+1)m+qn+kd}{m}}, \quad d \geq \max(p-q,0)+1, \end{aligned} \tag{20}$$

and for a given sequence of real numbers $(x_{m,n})$ with $x_{0,1} = 1$, we have

$$\begin{aligned} &A_{m,n+k,k}\left(d\frac{A_{i,\ pi+(q+1)(j-1)+d,\ pi+q(j-1)+d}(hx_{g,h})}{\binom{pi+(q+1)(j-1)+d}{j}}\right) \\ &= dk\binom{n+k}{k}\frac{A_{m,\ pm+(q+1)n+dk,\ pm+qn+dk}(hx_{g,h})}{(pm+qn+dk)\binom{pm+(q+1)n+dk}{n}}, \quad d \geq \max(q-p,0)+1. \end{aligned} \tag{21}$$

Proof. Let $(f_{m,n}(x))$ be a trinomial type sequence of polynomials with

$$f_{m,n}(1) := x_{m+1,n}.$$

Setting $a = b = 0$ and $x = 1$ in (15), we get

$$f_{m,n}(k) = \binom{m+k}{k}^{-1} A_{m+k,n,k}(ix_{i,j}), \quad m, n \geq 0. \tag{22}$$

To obtain (20), in (15) set $a = p$, $b = q$ and $x = d$ and use (22) to express $f_{m,n}(pn + qn + dk)$ and $f_{i-1,j}(p(i-1) + qj + d)$ in terms of partial bipartitional polynomials. Using the symmetry of m and n, the same goes for relation (21). □

Theorem 5 can be reduced using some particular cases related to Bell polynomials as follows.

Corollary 6. *Let p, q and d be nonnegative integers. Then, for a given sequence (x_n) of real numbers, we have*

$$A_{m+k,n,k}\left(di\frac{B_{(p+1)(i-1)+(q+1)j+d,\ p(i-1)+qj+d}\left(gx_g\right)}{\left(p\left(i-1\right)+qj+d\right)\binom{(p+1)(i-1)+(q+1)j+d}{i+j-1}}\right) \tag{23}$$
$$= dk\binom{m+k}{k}\frac{B_{(p+1)m+(q+1)n+kd,\ pm+qn+kd}\left(gx_g\right)}{\left(pm+qn+kd\right)\binom{(p+1)m+(q+1)n+kd}{m+n}},\quad d\geq\max\left(p-q,0\right)+1,$$

and

$$A_{m,n+k,k}\left(dj\frac{B_{(p+1)i+(q+1)(j-1)+d,\ pi+q(j-1)+d}\left(gx_g\right)}{\left(pi+q\left(j-1\right)+d\right)\binom{(p+1)i+(q+1)(j-1)+d}{i+j-1}}\right) \tag{24}$$
$$= dk\binom{n+k}{k}\frac{B_{(p+1)m+(q+1)n+kd,\ pm+qn+kd}\left(gx_g\right)}{\left(pm+qn+kd\right)\binom{(p+1)m+(q+1)n+kd}{m+n}},\quad d\geq\max\left(q-p,0\right)+1.$$

Proof. In (20) set $x_{m,n} := x_{m+n}$ and use relation (12). By symmetry, we obtain (24). $\square$

Example 7.

1. Let $x_n = j\frac{x}{a(j-1)+x}f_{j-1}\left(a\left(j-1\right)+x\right)$ for some binomial type sequence of polynomials $\left(f_n\left(x\right)\right)$. From the identities (23), (24) and using

$$B_{n,k}\left(j\frac{xf_{j-1}\left(a\left(j-1\right)+x\right)}{a\left(j-1\right)+x}\right)=\binom{n}{k}\frac{kxf_{n-k}\left(a\left(n-k\right)+kx\right)}{a\left(n-k\right)+kx}, \tag{25}$$

(see [4, Proposition 1]), we obtain

$$A_{m+k,n,k}\left(Gi\frac{f_{i+j-1}\left(E\left(i-1\right)+Fj+G\right)}{A\left(i-1\right)+Bj+G}\right)$$
$$=Gk\binom{m+k}{k}\frac{f_{m+n}\left(Em+Fn+kG\right)}{Em+Fn+kG},\ d\geq\max\left(p-q,0\right)+1,$$

$$A_{m,n+k,k}\left(Gj\frac{f_{i+j-1}\left(Ei+F\left(j-1\right)+G\right)}{Ei+F\left(j-1\right)+G}\right)$$
$$=Gk\binom{n+k}{k}\frac{f_{m+n}\left(Em+Fn+kG\right)}{Em+Fn+kG},\ d\geq\max\left(q-p,0\right)+1,$$

where $E=a+px,\ F=a+qx$ and $G=dx$.

2. Let $x_n = \frac{1}{n}$. From (23) and the known identity $B_{n,k}\left(1,1,1,\ldots\right)=S\left(n,k\right)$, the numbers $S\left(n,k\right)$ being Stirling numbers of the second kind, we obtain for

$d \geq \max(p-q,0)+1$:

$$A_{m+k,n,k}\left(di\frac{S\left((p+1)(i-1)+(q+1)j+d,\ p(i-1)+qj+d\right)}{(p(i-1)+qj+d)\binom{(p+1)(i-1)+(q+1)j+d}{i+j-1}}\right)$$
$$=dk\binom{m+k}{k}\frac{S\left((p+1)m+(q+1)n+kd,\ pm+qn+kd\right)}{(pm+qn+kd)\binom{(p+1)m+(q+1)n+kd}{m+n}}.$$

Theorem 8. *Let* $(x_{m,n})$ *be a sequence of real numbers with* $x_{1,1}=1$, d *be an integer and* $p,\ q,\ m,\ n$ *be nonnegative integers with* $(p+q)(pm+qn+d)\geq 1$. *We have*

$$A_{m,n}\left(d\frac{A_{(p+1)i+qj,\ pi+(q+1)j,\ pi+qj}(ghx_{g,h})}{(pi+qj)\binom{(p+1)i+qj}{i}\binom{pi+(q+1)j}{j}(pi+qj)!}\right) \tag{26}$$
$$=d\frac{A_{(p+1)m+qn+d,\ pm+(q+1)n+d,\ pm+qn+d}(ghx_{g,h})}{(pm+qn+d)\binom{(p+1)m+qn+d}{m}\binom{pm+(q+1)n+d}{n}(pm+qn+d)!}.$$

Proof. To obtain (26), in (16) set $a=p$, $b=q$, $x=d$ and use (19) to express $f_{i,j}(pi+qj)$ in terms of partial bipartitional polynomials. □

Theorem 9. *Let* d *be an integer and* $p,\ q,\ m,\ n$ *be nonnegative integers such that* $(p+q)(pm+qn+d)\geq 1$. *Then, for any sequence* $(x_{m,n})$ *of real numbers with* $x_{1,0}=1$, *we have*

$$A_{m,n}\left(d\frac{A_{(p+1)i+qj,\ j,\ pi+qj}(gx_{g,h})}{(pi+qj)\binom{(p+1)i+qj}{i}}\right)=d\frac{A_{(p+1)m+qn+d,\ n,\ pm+qn+d}(gx_{g,h})}{(pm+qn+d)\binom{(p+1)m+qn+d}{m}}, \tag{27}$$

and for any sequence $(x_{m,n})$ *of real numbers with* $x_{0,1}=1$, *we have*

$$A_{m,n}\left(d\frac{A_{i,\ pi+(q+1)j,\ pi+qj}(hx_{g,h})}{(pi+qj)\binom{pi+(q+1)j}{j}}\right)=d\frac{A_{m,\ pm+(q+1)n+d,\ pm+qn+d}(hx_{g,h})}{(pm+qn+d)\binom{pm+(q+1)n+d}{m}}. \tag{28}$$

Proof. To obtain (27), in (16) set $a=p$, $b=q$, $x=d$ and use (22) to express $f_{i,j}(pi+qj)$ in terms of partial bipartitional polynomials. By symmetry, we obtain (28). □

Theorem 9 can be restricted to some particular cases related to Bell polynomials as follows.

Corollary 10. *Let* d *be an integer and* $p,\ q,\ m,\ n$ *be nonnegative integers such that* $(p+q)(pm+qn+d)\geq 1$. *Then, for any sequence* (x_n) *of real numbers with* $x_1=1$, *we have*

$$A_{m,n}\left(d\frac{B_{(p+1)i+(q+1)j,\ pi+qj}(gx_g)}{(pi+qj)\binom{(p+1)i+(q+1)j}{i+j}}\right)=d\frac{B_{(p+1)m+(q+1)n+d,\ pm+qn+d}(gx_g)}{(pm+qn+d)\binom{(p+1)m+(q+1)n+d}{m+n}}. \tag{29}$$

Proof. Set $x_{m,n} := x_{m+n}$ in (27) or in (28), and use relation (12) or (13). $\square$

Example 11.

1. Let $x_n = j\frac{x}{a(j-1)+x}f_{j-1}(a(j-1)+x)$ for some binomial type sequence of polynomials $(f_n(x))$. From (25) and (29), we obtain

$$A_{m,n}\left(G\frac{f_{i+j}(Ei+Fj)}{Ei+Fj}\right) = G\frac{f_{m+n}(Em+Fn+G)}{Em+Fn+G},$$

where $E = a + px,\ F = a + qx$ and $G = dx$.

2. Let $x_n = \frac{1}{n}$. From (29) and the known identity $B_{n,k}(1,1,1,\ldots) = S(n,k)$, we obtain

$$\begin{aligned} &A_{m,n}\left(d\frac{S((p+1)i+(q+1)j,\ pi+qj)}{(pi+qj)\binom{(p+1)i+(q+1)j}{i+j}}\right) \\ &= d\frac{S((p+1)m+(q+1)n+d,\ pm+qn+d)}{(pm+qn+d)\binom{(p+1)m+(q+1)n+d}{m+n}}. \end{aligned}$$

Remark 12. When we set $n = 0$ in (23) or $m = 0$ in (24) and use relations (10), we get Proposition 4 given in [4]. When we set $n = 0$ or $m = 0$ in (29) and use relations (11), we obtain Proposition 8 given in [4].

3. Bipartitional Polynomials and Derivatives of Polynomials of Trinomial Type

Now, we consider connections between bipartitional polynomials and derivatives of polynomial sequences of trinomial type.

Lemma 13. *Let $(f_{m,n}(x))$ be a sequence of trinomial type. Then for all real numbers α, we have*

$$\begin{aligned} &A_{m,n,k}\left(ijD_{z=0}\left(e^{\alpha z}f_{i-1,j-1}(x+z)\right)\right) \\ &= k!\binom{m}{k}\binom{n}{k}D_{z=0}^{k}\left(e^{\alpha z}f_{m-k,n-k}(kx+z)\right), \quad m,n \geq k, \end{aligned} \tag{30}$$

$$\begin{aligned} &A_{m,n,k}\left(iD_{z=0}\left(e^{\alpha z}f_{i-1,j}(x+z)\right)\right) \\ &= \binom{m}{k}D_{z=0}^{k}\left(e^{\alpha z}f_{m-k,n}(kx+z)\right), \quad m \geq k, \end{aligned} \tag{31}$$

$$A_{m,n,k}\left(jD_{z=0}\left(e^{\alpha z}f_{i,j-1}\left(x+z\right)\right)\right) \tag{32}$$
$$=\binom{n}{k}D_{z=0}^{k}\left(e^{\alpha z}f_{m,n-k}\left(kx+z\right)\right),\quad n\geq k,$$

where $D_{z=0}f:=\frac{df}{dz}(0)$ *and* $D_{z=0}^{k}f:=\frac{d^{k}f}{dz^{k}}(0)$.

Proof. Let $F(t,u)$ be the exponential generating function of the sequence $(f_{m,n}(x))$ as in (9) and let $x_{m,n}=mnD_{z=0}\left(e^{\alpha z}f_{m-1,n-1}\left(x+z\right)\right).$ Then

$$\begin{aligned}
\frac{1}{k!}\left(\sum_{i+j\geq 1}x_{i,j}\frac{t^{i}}{i!}\frac{u^{j}}{j!}\right)^{k} &= \frac{1}{k!}\left(\sum_{i,j\geq 0}ijD_{z=0}\left(e^{\alpha z}f_{i-1,j-1}\left(x+z\right)\right)\frac{t^{i}}{i!}\frac{u^{j}}{j!}\right)^{k}\\
&= \frac{(tu)^{k}}{k!}\left(\sum_{i,j\geq 0}D_{z=0}\left(e^{\alpha z}f_{i,j}\left(x+z\right)\right)\frac{t^{i}}{i!}\frac{u^{j}}{j!}\right)^{k}\\
&= \frac{(tu)^{k}}{k!}\left(D_{z=0}e^{\alpha z}\sum_{i,j\geq 0}f_{i,j}\left(x+z\right)\frac{t^{i}}{i!}\frac{u^{j}}{j!}\right)^{k}\\
&= \frac{(tu)^{k}}{k!}\left(D_{z=0}e^{\alpha z}F\left(t,u\right)^{x+z}\right)^{k}\\
&= \frac{(tu)^{k}}{k!}\left(F\left(t,u\right)^{x}D_{z=0}e^{(\alpha+\ln F(t,u))z}\right)^{k}\\
&= \frac{(tu)^{k}}{k!}F\left(t,u\right)^{kx}\left(\alpha+\ln F\left(t,u\right)\right)^{k}\\
&= \frac{(tu)^{k}}{k!}F\left(t,u\right)^{kx}D_{z=0}^{k}e^{(\alpha+\ln F(t,u))z}\\
&= \frac{(tu)^{k}}{k!}D_{z=0}^{k}\left(e^{\alpha z}F\left(t,u\right)^{kx+z}\right)\\
&= \frac{(tu)^{k}}{k!}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}D_{z=0}^{k}\left(e^{\alpha z}f_{i,j}\left(kx+z\right)\right)\frac{t^{i}}{i!}\frac{u^{j}}{j!}\\
&= k!\sum_{m,n\geq k}\binom{m}{k}\binom{n}{k}D_{z=0}^{k}\left(e^{\alpha z}f_{m-k,n-k}\left(kx+z\right)\right)\frac{t^{m}}{m!}\frac{u^{n}}{n!},
\end{aligned}$$

which gives, by identification with (8), the identity (30). The same goes for identities (31) and (32). □

Theorem 14. *Let* $p,\ q,\ d$ *be nonnegative integers and* $a,\ b,\ \alpha,\ x$ *be real numbers.*

Then, we have

$$A_{m+k,n+k,k}\left(dij\left(xD_{z=0}^{r}+D_{z=0}^{r-1}\right)\left(\frac{f_{i-1,j-1}\left(\overline{r}+z\right)}{\overline{r}+z}e^{\alpha z}\right)\right) \tag{33}$$
$$=dk!k\binom{m+k}{k}\binom{n+k}{k}\left(xD_{z=0}^{R}+D_{z=0}^{R-1}\right)\left(\frac{f_{m,n}\left(\overline{R}+z\right)}{\overline{R}+z}e^{\alpha z}\right),$$

where $r=p\left(i-1\right)+q\left(j-1\right)+d,\ \overline{r}=a\left(i-1\right)+b\left(j-1\right)+dx,\ R=pm+qn+dk,\ \overline{R}=am+bn+dkx.$

Proof. Setting $x_{m,n}=D_{z=0}\left(e^{\alpha z}f_{m-1,n-1}\left(x+z\right)\right)$ in (18) and using identity (30), we obtain

$$A_{m+k,n+k,k}\left(dijD_{z=0}^{r}\left(\frac{f_{i-1,j-1}\left(rx+z\right)}{r}e^{\alpha z}\right)\right)$$
$$=\quad dk!k\binom{m+k}{k}\binom{n+k}{k}D_{z=0}^{R}\left(\frac{f_{m,n}\left(Rx+z\right)}{R}e^{\alpha z}\right).$$

Use the trinomial sequence $\left(h_{m,n}\left(x\right)\right)$ given by (17) :

$$h_{m,n}\left(x\right)=\frac{x}{\left(a-px\right)m+\left(b-px\right)n+x}f_{m,n}\left(\left(a-px\right)m+\left(b-px\right)n+x\right)$$

instead of $\left(f_{m,n}\left(x\right)\right)$. □

For the next theorem, we set by convention $D_{z=0}^{-s}g\left(z\right)=0,\ \ s=1,2,\ldots.$

Theorem 15. *Let* $p,\ q,\ d$ *be nonnegative integers and* $a,\ b,\ \alpha,\ x$ *be real numbers. Then*

$$A_{m+k,n,k}\left(di\left(xD_{z=0}^{r}+D_{z=0}^{r-1}\right)\left(\frac{f_{i-1,j}\left(\overline{r}+z\right)}{\overline{r}+z}e^{\alpha z}\right)\right) \tag{34}$$
$$=dk\binom{m+k}{k}\left(xD_{z=0}^{R}+D_{z=0}^{R-1}\right)\left(\frac{f_{m,n}\left(\overline{R}+z\right)}{\overline{R}+z}e^{\alpha z}\right),$$

where $r=p\left(i-1\right)+qj+d,\ \overline{r}=a\left(i-1\right)+bj+dx,\ R=pm+qn+dk,\ \overline{R}=am+bn+dkx,$

and

$$A_{m,n+k,k}\left(dj\left(xD_{z=0}^{r}+D_{z=0}^{r-1}\right)\left(\frac{f_{i,j-1}\left(\overline{r}+z\right)}{\overline{r}+z}e^{\alpha z}\right)\right) \tag{35}$$
$$=dk\binom{n+k}{k}\left(xD_{z=0}^{R}+D_{z=0}^{R-1}\right)\left(\frac{f_{m,n}\left(\overline{R}+z\right)}{\overline{R}+z}e^{\alpha z}\right),$$

where $r=pi+q\left(j-1\right)+d,\ \overline{r}=ai+b\left(j-1\right)+dx,\ R=pm+qn+dk,\ \overline{R}=am+bn+dkx.$

Proof. For (34), when we set $x_{m,n} = D_{z=0}\left(e^{\alpha z} f_{m-1,n}(x+z)\right)$ in (23) and use the identity of Lemma (31), we get

$$\begin{aligned} & A_{m,n,k}\left(diD_{z=0}^{p(i-1)+qj+d}\left(\frac{f_{m-k,n}\left(\left(p\left(i-1\right)+qj+d\right)x+z\right)}{p\left(i-1\right)+qj+d}e^{\alpha z}\right)\right) \\ = \quad & dk\binom{m+k}{k}D_{z=0}^{pm+qn+kd}\left(\frac{f_{m,n}\left(\left(pm+qn+kd\right)x+z\right)}{pm+qn+kd}e^{\alpha z}\right). \end{aligned}$$

Taking the trinomial sequence $(h_{m,n}(x))$ given by (17)

$$h_{m,n}(x) = \frac{x}{(a-px)m+bn+x} f_{m,n}\left((a-px)m+bn+x\right)$$

instead of $(f_{m,n}(x))$ and using (31) and (24) instead of (23), we obtain (35). $\square$

Theorem 16. *Let d be an integer, p, q, m, n be nonnegative integers such that $(pm+qn+d)\,pq \geq 1$ and a, b, x be real numbers. Then we have*

$$\begin{aligned} & A_{m,n}\left(d\left(xD_{z=0}^{pi+qj}+D_{z=0}^{pi+qj-1}\right)\left(\frac{f_{i,j}\left(ai+bj+z\right)}{ai+bj+z}e^{\alpha z}\right)\right) \\ & = d\left(xD_{z=0}^{pm+qn+d}+D_{z=0}^{pm+qn+d-1}\right)\left(\frac{f_{m,n}\left(am+bn+dx+z\right)}{am+bn+dx+z}e^{\alpha z}\right). \end{aligned} \tag{36}$$

Proof. When we set $x_{m,n} = D_{z=0}\left(e^{\alpha z} f_{m-1,n-1}(x+z)\right)$ in (26) and use the first identity of Lemma 13, we obtain

$$\begin{aligned} & A_{m,n}\left(dD_{z=0}^{pi+qj}\left(\frac{f_{i,j}\left(\left(pi+qj\right)x+z\right)}{pi+qj}e^{\alpha z}\right)\right) \\ = \quad & dD_{z=0}^{pm+qn+d}\left(\frac{f_{m,n}\left(\left(pm+qn+d\right)x+z\right)}{pm+qn+d}e^{\alpha z}\right). \end{aligned}$$

Use the trinomial sequence $(h_{m,n}(x))$ given by (17) :

$$h_{m,n}(x) = \frac{x}{(a-px)m+(b-qx)n+x} f_{m,n}\left((a-px)m+(b-qx)n+x\right)$$

instead of $(f_{m,n}(x))$. $\square$

4. Some Consequences for Bell Polynomials

The next corollaries present some consequences of the above Theorems for Bell polynomials (see [1]). The identities given below can be viewed as complementary identities to those given in [4] and [5].

Corollary 17. *Let q, d be nonnegative integers, $d \geq 1$, b, x be real numbers and let $(f_n(x))$ be a sequence of binomial type. For $n \geq k \geq 1$, we have*

$$B_{n,k}\left(dj\left(xD_{z=0}^{q(j-1)+d}+D_{z=0}^{q(j-1)+d-1}\right)\left(\frac{f_{j-1}\left(b\left(j-1\right)+dx+z\right)}{b\left(j-1\right)+dx+z}e^{\alpha z}\right)\right) \tag{37}$$
$$=dk\binom{n}{k}\left(xD_{z=0}^{q(n-k)+dk}+D_{z=0}^{q(n-k)+dk-1}\right)\left(\frac{f_{n-k}\left(b\left(n-k\right)+dkx+z\right)}{b\left(n-k\right)+dkx+z}e^{\alpha z}\right).$$

Proof. From the definition in (9), it is easy to verify that the sequence $(f_{0,n}(x))$ is of binomial type. Conversely, for any sequence of binomial type $(f_n(x))$, one can construct a sequence of trinomial type $(f_{m,n}(x))$ such that $f_n(x)=f_{0,n}(x)$. Then, we consider a sequence of trinomial type $(f_{m,n}(x))$ for which $f_n(x)=f_{0,n}(x)$. Now, by setting $m=0$ in (35), we get

$$A_{0,n+k,k}\left(dj\left(xD_{z=0}^{r}+D_{z=0}^{r-1}\right)\left(\frac{f_{i,j-1}\left(\overline{r}+z\right)}{\overline{r}+z}e^{\alpha z}\right)\right)$$
$$=\quad dk\binom{n+k}{k}\left(xD_{z=0}^{R}+D_{z=0}^{R-1}\right)\left(\frac{f_{0,n}\left(\overline{R}+z\right)}{\overline{R}+z}e^{\alpha z}\right),$$

and by using the first relation of (10), the last identity becomes

$$B_{n,k}\left(dj\left(xD_{z=0}^{q(j-1)+d}+D_{z=0}^{q(j-1)+d-1}\right)\left(\frac{f_{0,j-1}\left(b\left(j-1\right)+dx+z\right)}{b\left(j-1\right)+dx+z}e^{\alpha z}\right)\right)$$
$$=\quad dk\binom{n}{k}\left(xD_{z=0}^{q(n-k)+dk}+D_{z=0}^{q(n-k)+dk-1}\right)\left(\frac{f_{0,n-k}\left(b\left(n-k\right)+dkx+z\right)}{b\left(n-k\right)+dkx+z}e^{\alpha z}\right).$$

To obtain the identity (38), it suffices to replace $f_{0,n}(x)$ by $f_n(x)$ in the last identity. □

Corollary 18. *Let d be an integer, q, n be nonnegative integers with $(qn+d)\,q \geq 1$, b, x be real numbers and let $(f_n(x))$ be a sequence of binomial type. We have*

$$B_n\left(d\left(xD_{z=0}^{qj}+D_{z=0}^{qj-1}\right)\left(\frac{f_j\left(bj+z\right)}{bj+z}e^{\alpha z}\right)\right) \tag{38}$$
$$=d\left(xD_{z=0}^{qn+d}+D_{z=0}^{qn+d-1}\right)\left(\frac{f_n\left(bn+dx+z\right)}{bn+dx+z}e^{\alpha z}\right).$$

Proof. As above, let $(f_{m,n}(x))$ be a sequence of trinomial type such that $f_n(x)=f_{0,n}(x)$. Then, (38) follows by setting $m=0$ or $n=0$ in (36) and using the relations in (11). □

The above identities on Bell polynomials can be simplified when $x=0$ as follows:

$$B_{n,k}\left(djD_{z=0}^{q(j-1)+d-1}\left(\frac{f_{j-1}\left(b\left(j-1\right)+z\right)}{b\left(j-1\right)+z}e^{\alpha z}\right)\right)$$
$$=dk\binom{n}{k}D_{z=0}^{q(n-k)+dk-1}\left(\frac{f_{n-k}\left(b\left(n-k\right)+z\right)}{b\left(n-k\right)+z}e^{\alpha z}\right),$$

$$B_n\left(dD_{z=0}^{qj-1}\left(\frac{f_j\left(bj+z\right)}{bj+z}e^{\alpha z}\right)\right)=dD_{z=0}^{qn+d-1}\left(\frac{f_n\left(bn+z\right)}{bn+z}e^{\alpha z}\right).$$

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