#A32 INTEGERS 11 (2011)

THE K-PERIODIC FIBONACCI SEQUENCE AND AN EXTENDED BINET'S FORMULA

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Received: 10/31/10, Revised: 3/9/11, Accepted: 3/19/11, Published: 5/18/11

Abstract

It is well-known that a continued fraction is periodic if and only if it is the representation of a quadratic irrational α. In this paper, we consider the family of sequences obtained from the recurrence relation generated by the numerators of the convergents of these numbers α. These sequences are generalizations of most of the Fibonacci-like sequences, such as the Fibonacci sequence itself, r-Fibonacci sequences, and the Pell sequence, to name a few. We show that these sequences satisfy a linear recurrence relation when considered modulo k, even though the sequences themselves do not. We then employ this recurrence relation to determine the generating functions and Binet-like formulas. We end by discussing the convergence of the ratios of the terms of these sequences.

1. Introduction

Generalizations of the Fibonacci numbers have been extensively studied. From Lucas and Catalan numbers to Gibonacci and k-Bonacci, all are evidence of the interest Fibonacci-like sequences still hold. To generalize the Fibonacci sequence, some authors [3, 4, 6, 13, 17] have altered the starting values, while others [2, 8, 9, 10, 12, 14] have preserved the first two terms of the sequence but changed the recurrence relation. In a previous paper [2], we gave a generalization of the latter type, called the generalized Fibonacci sequence. It is defined using a non-linear recurrence relation depending on two real parameters (a, b) as follows. For any two nonzero real numbers a and b, the generalized Fibonacci sequence, say $\left\{F_n^{(a,b)}\right\}_{n=0}^{\infty}$, is defined recursively by

$$F_0^{(a,b)} = 0, \quad F_1^{(a,b)} = 1, \quad F_n^{(a,b)} = \begin{cases} aF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)}, & \text{if n is even} \\ bF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)}, & \text{if n is odd} \end{cases} \qquad (n \geq 2).$$

This generalization has its own Binet-like formula and satisfies identities that are analogous to the identities satisfied by the classical Fibonacci sequence.

We now introduce a further generalization of the Fibonacci sequence; we shall call it the *k-periodic Fibonacci sequence.* This new generalization is defined using a non-linear recurrence relation that depends on k real parameters, and is an extension of the generalized Fibonacci sequence. The non-linear recurrence relation we consider in this article can be viewed as a linear recurrence relation with nonconstant coefficients.

Definition 1. For any k-tuple $(x_1, x_2, \ldots, x_k) \in \mathbb{Z}^k$, we recursively define the *k-periodic Fibonacci sequence,* denoted $\left\{F_n^{(x_1,x_2,\ldots,x_k)}\right\}_{n=0}^{\infty}$, by

$$F_0^{(x_1,x_2,\ldots,x_k)} = 0, \quad F_1^{(x_1,x_2,\ldots,x_k)} = 1, \quad F_{n+1}^{(x_1,x_2,\ldots,x_k)} = x_n F_n^{(x_1,x_2,\ldots,x_k)} + F_{n-1}^{(x_1,x_2,\ldots,x_k)}$$

for all $n \geq 1$, where $x_n = x_i$, for $1 \leq i \leq k$, if $n \equiv i \pmod{k}$.

To avoid cumbersome notation, let us denote $F_n^{(x_1,x_2,\ldots,x_k)}$ by q_n. Thus, the sequence $\{q_n\}$ satisfies

$$q_0 = 0, \quad q_1 = 1, \quad q_n = \begin{cases} x_1 q_{n-1} + q_{n-2}, & \text{if } n \equiv 2 \pmod{k} \\ x_2 q_{n-1} + q_{n-2}, & \text{if } n \equiv 3 \pmod{k} \\ \quad\vdots & \\ x_{k-1} q_{n-1} + q_{n-2}, & \text{if } n \equiv 0 \pmod{k} \\ x_k q_{n-1} + q_{n-2}, & \text{if } n \equiv 1 \pmod{k} \end{cases} \qquad (n \geq 2).$$

We now note that this new generalization is in fact a family of sequences where each new combination of $x_1, x_2, \ldots, x_k$ produces a distinct sequence. When $x_1 = x_2 = \ldots = x_k = 1$, we have the classical Fibonacci sequence and when $x_1 = x_2 = \ldots = x_k = 2$, we get the Pell numbers. Even further, if we set $x_1 = x_2 = \ldots = x_k = r$, for some positive integer r, we get the r-Fibonacci numbers, and if k is even, we can obtain the generalized Fibonacci sequence by assigning a to the odd-numbered subscripts and b to the even-numbered subscripts.

Example 2. The sequence descriptions that follow give reference numbers found in Sloane's On-Line Encyclopedia of Integer Sequences, [16]. When $k = 3$ and $(x_1, x_2, x_3) = (1, 0, 1)$, we obtain the sequence [A092550], beginning

$$0, 1, 1, 1, 2, 3, 2, 5, 7, 5, 12, 17, 12, 29, \ldots.$$

This sequence is described in [16] as a "two-steps-forward-and-one-step-back Fibonacci-based switched sequence inspired by Per Bak's sand piles." When $(x_1, x_2, x_3) = (2, 1, 1)$, we obtain the sequence [A179238] and when $(x_1, x_2, x_3) = (1, -1, 2)$, we obtain the sequence [A011655]. When $k = 4$ and $(x_1, x_2, x_3, x_4) = (2,1,2,1)$, we get the sequence [A048788] and when $(x_1, x_2, x_3, x_4) = (1,2,1,2)$, we get the sequence [A002530].

We now consider the connection between this family of sequences (for positive x_i) and the set of quadratic irrational numbers. If an irrational number α satisfies a quadratic equation with integer coefficients, α is said to be a quadratic irrational. In addition, we say a continued fraction expansion of a number x is periodic if it can be written in the form

$$x = [x_0; x_1, x_2, \ldots, x_m, \overline{x_{m+1}, \ldots, x_{m+k}}].$$

It is well known that a number α is a quadratic irrational if and only if it has a periodic continued fraction expansion. If we restrict our quadratic irrational α to the interval $[0, 1]$, we get a continued fraction expansion of the form $\alpha = [0; \overline{x_1, x_2, \ldots, x_k}]$. Therefore, given a quadratic irrational $\widehat{\alpha} = [x_0; x_1, x_2, \ldots, x_m, \overline{x_{m+1}, \ldots, x_{m+k}}]$, if we associate it with the quadratic irrational $\alpha = [0; \overline{x_{m+1}, x_{m+2}, \ldots, x_{m+k}}]$, which is purely periodic with period k, we have that for each quadratic irrational there is a corresponding k-periodic Fibonacci sequence with associated k-tuple $(x_1, x_2, \ldots, x_k)$. Furthermore, we can derive this k-periodic Fibonacci sequence directly from the convergents of α.

Consider the sequence of convergents of α,

$$\frac{p_0}{q_0}, \; \frac{p_0}{q_0}, \; \frac{p_1}{q_1}, \; \ldots, \frac{p_n}{q_n}, \; \ldots .$$

It can be shown that the numerators and denominators of these convergents satisfy the following recurrence relations. If $p_{-2} = 0, p_{-1} = 1$, then the sequence of numerators satisfies the relation

$$p_n = x_n p_{n-1} + p_{n-2},$$

and if $q_{-2} = 1, q_{-1} = 0$, then the sequence of denominators satisfies the relation

$$q_n = x_n q_{n-1} + q_{n-2}.$$

It is not difficult to see that we obtain the k-periodic Fibonacci sequence of α from the sequence of numerators of the convergents of α, as the two sequences have the same initial values and satisfy the same recurrence relation. For further reading on continued fractions, the books [5, 11] are excellent sources.

Example 3. From the continued fraction expansion of $\phi = [1; 1, 1, 1, \ldots]$, the golden ratio, we have $x_1 = x_2 = \ldots = x_k = 1$ and obtain the Fibonacci sequence. From the continued fraction expansion of $\sqrt{2} = [1; 2, 2, 2, \ldots]$, we consider the periodic portion and derive the Pell sequence from $-1 + \sqrt{2} = [0; 2, 2, 2, \ldots]$, by setting $x_1 = x_2 = \ldots = x_k = 2$.

We will describe the terms of the sequence $\{q_n\}$ explicitly by using a generalization of Binet's formula. In order to do this, we must first show that for some constant A, the q_n satisfy the recurrence relation

$$q_{mk+j} = Aq_{(m-1)k+j} + (-1)^{k-1} q_{(m-2)k+j} \quad \text{for } m \geq 2k, \quad 0 \leq j \leq k-1.$$

Therefore, we begin by establishing that the $\{q_n\}$ satisfy a linear recurrence modulo k, and we follow by deriving a generalization of Binet's formula (via generating functions). Finally, we consider the convergence of the ratios of successive terms of the sequence. It is well-known that the ratios of successive Fibonacci numbers approach the golden mean, Φ, so it is natural to ask if analogous results exist for the variations and extensions of the Fibonacci sequence. We show in [2] that successive terms of the generalized Fibonacci sequence do not converge, though we show convergence of ratios of terms when increasing by two's or ratios of even or odd terms. We end with a discussion of the convergence of the ratios of subsequent terms modulo k.

2. The Recurrence Relation

In order to obtain the generating function, we first show that our sequences satisfy a linear recurrence relation modulo k. So, fix k and the k-tuple $(x_1, x_2, \ldots, x_k)$. We consider the sequence $\left\{F_n^{(x_1, x_2, \ldots, x_k)}\right\}$, and use the short-hand notation $\{q_n\}$, as defined in the introduction. We show that for some constant A,

$$q_{mk+j} = Aq_{(m-1)k+j} + (-1)^{k-1} q_{(m-2)k+j} \quad \text{for } m \geq 2k, \quad 0 \leq j \leq k-1.$$

To achieve this, we introduce a family of sequences related to $\{q_n\}$.

Definition 4. For each j, where $0 \leq j \leq k-1$, we define a sequence $\left\{q_n^j\right\}$ as follows. Let $q_0^j = 0$ and $q_1^j = 1$. For $n = mk + r \geq 2$, with $0 \leq r \leq k-1$, we define

$$q_n^j \;=\; q_{mk+r}^j \;=\; x_{j-r+1} q_{mk+r-1}^j + q_{mk+r-2}^j,$$

where $x_{j-r+1} = x_i$, for $1 \leq i \leq k$, if $(j - r + 1) \equiv i \pmod{k}$.

For example, the sequence q_n^0 begins,

$$0,\ 1,\ x_{k-1},\ x_{k-1}x_{k-2} + 1,\ x_{k-1}x_{k-2}x_{k-3} + x_{k-1} + x_{k-3}, \ldots.$$

Note that there are k-many sequences $\left\{q_n^j\right\}$ associated with $\{q_n\}$.

We set the constant $A = q^0_{k+1} + q_{k-1}$. Now through a series of lemmas, we will arrive at the linear recurrence

$$q_{mk+j} = Aq_{(m-1)k+j} + (-1)^{k-1}q_{(m-2)k+j} \qquad \text{for } m \geq 2k, \quad 0 \leq j \leq k-1.$$

Throughout the remainder of this paper, we assume k to be a fixed positive integer.

Lemma 5. *For $0 \leq j \leq k-1$, $q_{k+j} = q^j_{k+j}$.*

Proof. To begin, we see that

$$\begin{aligned} q_{k+j} &= x_{j-1}q_{k+j-1} + q_{k+j-2} \\ &= q^j_2 q_{k+j-1} + q^j_1 q_{k+j-2} \\ &= q^j_2(x_{j-2}q_{k+j-2} + q_{k+j-3}) + q^j_1 q_{k+j-2} \\ &= (x_{j-2}q^j_2 + q^j_1)q_{k+j-2} + q^j_2 q_{k+j-3} \\ &= q^j_3 q_{k+j-2} + q^j_2 q_{k+j-3}. \end{aligned}$$

Following the same process, one can easily show that

$$q_{k+j} = q^j_{i+1}q_{k+j-i} + q^j_i q_{k+j-(i+1)},$$

where $0 \leq i \leq k+j-1$.
Hence,

$$\begin{aligned} q_{k+j} &= q^j_{k+j}q_{k+j-(k+j-1)} + q^j_{k+j-1}q_{k+j-(k+j)} \\ &= q^j_{k+j}q_1 + q^j_{k+j-1}q_0 \\ &= q^j_{k+j}q_1 \\ &= q^j_{k+j}. \end{aligned}$$

□

Similarly, we obtain the identity $q_j = q^j_j$.

Lemma 6. *For all $n \geq 2$ and $0 \leq j \leq k-1$,*

$$q^j_n = x_{j-1}q^{j-1}_{n-1} + q^{j-2}_{n-2} = q^j_2 q^{j-1}_{n-1} + q^{j-2}_{n-2}.$$

Proof. We proceed by strong induction on n. For $n = 2$, we have

$$\begin{aligned} q^j_2 &= x_{j-1} \\ &= x_{j-1}q^j_1 + q^j_0 \\ &= q^{j-1}_1 q^j_2 + q^j_0 \\ &= q^j_2 q^{j-1}_1 + q^{j-2}_0 \end{aligned}$$

Now, if we assume our claim is true for all $n \le t$, we shall show that $q^j_{t+1} = q^j_2 q^{j-1}_t + q^{j-2}_{t-1}$. We write $t = mk + r$ for some $0 \le r \le k-1$. Then

$$\begin{aligned}
q^j_{t+1} &= q^j_{mk+r+1} \\
&= x_{j-r} q^j_{mk+r} + q^j_{mk+r-1} \\
&= x_{j-r}\left(q^j_2 q^{j-1}_{mk+r-1} + q^{j-2}_{mk+r-2}\right) + \left(q^j_2 q^{j-1}_{mk+r-2} + q^{j-2}_{mk+r-3}\right) \\
&= q^j_2 q^{j-1}_{mk+r} + q^{j-2}_{mk+r-1} \\
&= q^j_2 q^{j-1}_t + q^{j-2}_{t-1}.
\end{aligned}$$

□

Lemma 7. *For all integers $0 \le j \le k-1$ we have $q^0_{k+1} = q^j_{k+1} - q_{k-1} + q^{j-1}_{k-1}$.*

Proof. Using Lemma 2, we have that

$$\begin{aligned}
q^0_{k+1} + q_{k-1} &= q^0_{k+1} + q^{k-1}_{k-1} \\
&= x_{k-1} q^{k-1}_k + q^{k-2}_{k-1} + q^{k-1}_{k-1} \\
&= q^{k-1}_{k+1} + q^{k-2}_{k-1} \\
&= x_{k-2} q^{k-2}_k + q^{k-3}_{k-1} + q^{k-2}_{k-1} \\
&= q^{k-2}_{k+1} + q^{k-3}_{k-1}.
\end{aligned}$$

By continuing this process for $k - j$ steps, we have $q^0_{k+1} + q_{k-1} = q^j_{k+1} + q^{j-1}_{k-1}$. □

Lemma 8. *For all integers $0 \le j \le k-1, m \ge 2,$ and $2 \le t \le km + j$,*

$$q_{km+j} = q^j_t q_{km+j-t+1} + q^j_{t-1} q_{km+j-t}.$$

Proof. Using Definitions 1 and 2, we have

$$\begin{aligned}
q_{km+j} &= x_{j-1} q_{km+j-1} + q_{km+j-2} \\
&= q^j_2 q_{km+j-1} + q^j_1 q_{km+j-2}.
\end{aligned}$$

We then achieve $q_{km+j} = q^j_t q_{km+j-t+1} + q^j_{t-1} q_{km+j-t}$ in $t-1$ steps by repeated applications of Definitions 1 and 2. □

Lemma 9. *For $0 \le j \le k-1$, and $k, m \ge 2$, we have that if $0 \le i \le k-1$,*

$$\begin{aligned}
q_{km+j} = A q_{k(m-1)+j} &+ (q^j_k q^{j-1}_i - q^j_{i+1} q^{j-1}_{k-1}) q_{k(m-1)+j-i} \\
&+ (q^j_k q^{j-1}_{i-1} - q^j_i q^{j-1}_{k-1}) q_{k(m-1)+j-(i+1)}.
\end{aligned}$$

Proof. By Lemmas 3 and 4, when $t = k+1$, we have that

$$\begin{aligned} q_{km+j} &= q^j_{k+1} q_{km+j-k} + q^j_k q_{km+j-k-1} \\ &= \left(q^0_{k+1} + q_{k-1} - q^{j-1}_{k-1}\right) q_{k(m-1)+j} + q^j_k q_{k(m-1)+j-1}. \end{aligned}$$

It remains to see that

$$\begin{aligned} &\left(-q^{j-1}_{k-1}\right) q_{k(m-1)+j} + q^j_k q_{k(m-1)+j-1} \\ &\quad = \left(q^j_k q^{j-1}_i - q^j_{i+1} q^{j-1}_{k-1}\right) q_{k(m-1)+j-i} + \left(q^j_k q^{j-1}_{i-1} - q^j_i q^{j-1}_{k-1}\right) q_{k(m-1)+j-(i+1)}. \end{aligned}$$

For this, we employ a similar method as in Lemma 4, always replacing the largest term of the sequence using the definition and gathering like terms.

□

Lemma 10. *For all integers $0 \le j \le k-1$, and for all $k \ge 2$,*

$$q^j_k q^{j-1}_{k-2} - q^j_{k-1} q^{j-1}_{k-1} = (-1)^{k-1}.$$

Proof. Employing the method used in Lemmas 4 and 5 (applying the definition and gathering like terms), we get

$$\begin{aligned} q^j_k q^{j-1}_{k-2} - q^{j-1}_{k-1} q^j_{k-1} &= \left(x_{j+1} q^j_{k-1} + q^j_{k-2}\right) q^{j-1}_{k-2} - q^{j-1}_{k-1} q^j_{k-1} \\ &= -q^{j-1}_{k-3} q^j_{k-1} + q^j_{k-2} q^{j-1}_{k-2} \\ &= q^j_{k-2} q^{j-1}_{k-4} - q^j_{k-3} q^{j-1}_{k-3}. \end{aligned}$$

Continuing in the same manner, we have that at step $k-2$,

$$q^j_k q^{j-1}_{k-2} - q^j_{k-1} q^{j-1}_{k-1} \;=\; (-1)^{k-2} q^j_2 q^{j-1}_0 - (-1)^{k-2} q^j_1 q^{j-1}_1 \;=\; (-1)^{k-1}.$$

□

Theorem 11. *For $0 \le j \le k-1$ and $m \ge 2$,*

$$q_{km+j} = A q_{k(m-1)+j} + (-1)^{k-1} q_{k(m-2)+j}.$$

Proof. This follows directly from Lemmas 5 and 6. □

Remark 12. The recurrence relation discussed in Section 2 is also discussed in a very recent paper by C. Cooper [1]. We note that in [1], there is no explicitly stated formula for the coefficient A. Instead, a very interesting combinatorial description is given, based on the number of ways to create a bracelet of length k using beads of length one or two.

3. Generating Function for the k-Periodic Fibonacci Sequence

Generating functions provide a powerful technique for solving linear homogeneous recurrence relations. Even though generating functions are typically used in conjunction with linear recurrence relations with constant coefficients, we will systematically make use of them for linear recurrence relations with nonconstant coefficients. In this section, we consider the generating functions for the k-periodic Fibonacci sequences.

Theorem 13. *The generating function for the k-periodic Fibonacci sequence given by $\{q_n\}$ is*

$$G(x) = \frac{\sum\limits_{r=0}^{k-1} q_r x^r + \sum\limits_{r=0}^{k-1} \left(q_{k+r} - Aq_r\right) x^{k+r}}{1 - Ax^k + (-1)^k x^{2k}}.$$

Proof. We begin with the formal power series representation of the generating function for $\{q_n\}$, $G(x) = q_0 + q_1 x + q_2 x^2 + \cdots + q_n x^n + \cdots = \sum_{m=0}^{\infty} q_m x^m$. We rewrite $G(x)$ as

$$G(x) = \sum_{r=0}^{k-1} \left(\sum_{j=0}^{\infty} q_{jk+r} x^{jk+r} \right).$$

Now denote the inner sum as $G_r(x) = \sum\limits_{j=0}^{\infty} q_{jk+r} x^{jk+r}$. Note that $G(x) = G_0(x) + G_1(x) + \cdots + G_{k-1}(x)$. To get the desired result, we consider each of the summands separately. For $0 \le r < k$, we have

$$\begin{aligned}
G_r(x) &= q_r x^r + q_{k+r} x^{k+r} + q_{2k+r} x^{2k+r} + \cdots + q_{jk+r} x^{jk+r} + \cdots \\
&= q_r x^r + q_{k+r} x^{k+r} + \sum_{j=2}^{\infty} q_{jk+r} x^{jk+r} \\
&= q_r x^r + q_{k+r} x^{k+r} + \sum_{j=2}^{\infty} \left[Aq_{(j-1)k+r} - (-1)^k q_{(j-2)k+r} \right] x^{jk+r} \\
&= q_r x^r + q_{k+r} x^{k+r} + Ax^k \sum_{j=2}^{\infty} q_{(j-1)k+r} x^{(j-1)k+r} \\
&\qquad\qquad - (-1)^k x^{2k} \sum_{j=2}^{\infty} q_{(j-2)k+r} x^{(j-2)k+r} \\
&= q_r x^r + q_{k+r} x^{k+r} + Ax^k \sum_{j=1}^{\infty} q_{jk+r} x^{jk+r} - (-1)^k x^{2k} \sum_{j=0}^{\infty} q_{jk+r} x^{jk+r} \\
&= q_r x^r + q_{k+r} x^{k+r} + Ax^k \left[G_r(x) - q_r x^r \right] - (-1)^k x^{2k} G_r(x) \\
&= q_r x^r + \left(q_{k+r} - Aq_r\right) x^{k+r} + \left[Ax^k - (-1)^k x^{2k} \right] G_r(x).
\end{aligned}$$

Therefore, $\left[1 - Ax^k + (-1)^k x^{2k}\right] G_r(x) = q_r x^r + (q_{k+r} - Aq_r)\, x^{k+r}$, resulting in

$$G_r(x) = \frac{q_r x^r + (q_{k+r} - Aq_r)\, x^{k+r}}{1 - Ax^k + (-1)^k x^{2k}}. \tag{1}$$

Thus,

$$\begin{aligned} G(x) &= G_0(x) + G_1(x) + \cdots + G_{k-1}(x) \\ &= \frac{q_0 + (q_k - Aq_0)\, x^k}{1 - Ax^k + (-1)^k x^{2k}} + \frac{q_1 x + (q_{k+1} - Aq_1)\, x^{k+1}}{1 - Ax^k + (-1)^k x^{2k}} \\ &\qquad + \cdots + \frac{q_{k-1} x^{k-1} + (q_{2k-1} - Aq_{k-1})\, x^{2k-1}}{1 - Ax^k + (-1)^k x^{2k}}. \end{aligned} \tag{2}$$

After simplifying the above expression, we get the desired result as claimed in the theorem;

$$G(x) = \frac{\sum_{r=0}^{k-1} q_r x^r + \sum_{r=0}^{k-1} (q_{k+r} - Aq_r)\, x^{k+r}}{1 - Ax^k + (-1)^k x^{2k}}. \qquad \square$$

4. Binet's Formula for the *k*-Periodic Fibonacci Sequence

In this section, we will state and prove an extension of Binet's formula for the k-periodic Fibonacci sequences.

Lemma 14. *If*

$$\alpha = \frac{(-1)^k A + \sqrt{A^2 - (-1)^k 4}}{2} \quad \textit{and} \quad \beta = \frac{(-1)^k A - \sqrt{A^2 - (-1)^k 4}}{2},$$

then α *and* β *are roots of* $z^2 - (-1)^k Az + (-1)^k = 0$.

Lemma 15. *If* α *and* β *are as in Lemma 14, then*

(a) $\alpha + \beta = (-1)^k A, \;\; \alpha - \beta = \sqrt{A^2 - (-1)^k 4}, \;\; \textit{and} \;\; \alpha\beta = (-1)^k$

(b) $\alpha^{m+1} + \beta\alpha^m = (-1)^k A\alpha^m, \;\; \beta^{m+1} + \alpha\beta^m = (-1)^k A\beta^m$

(c) $\dfrac{1}{1 - Ax^k + (-1)^k x^{2k}} = \dfrac{1}{\alpha - \beta}\left[\dfrac{\alpha}{1 - (-1)^k \alpha x^k} - \dfrac{\beta}{1 - (-1)^k \beta x^k}\right].$

Theorem 16. (Generalized Binet's Formula) *The terms of the k-periodic Fibonacci sequence* $\{q_n\}$ *are given by*

$$q_{km+r} = (-1)^{k(m+1)} \left[\left(\frac{\alpha^m - \beta^m}{\alpha - \beta}\right) q_{k+r} - \left(\frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta}\right) q_r\right],$$

where α *and* β *are as in Lemma 14.*

Proof. Suppose that $0 \leq r < k$. The generating function for the subsequence $\{q_{mk+r}\}$ is given by (see Equation (1))

$$
\begin{aligned}
G_r(x) &= \frac{q_r x^r + (q_{k+r} - Aq_r)\, x^{k+r}}{1 - Ax^k + (-1)^k x^{2k}} = x^r \frac{q_r + (q_{k+r} - Aq_r)\, x^k}{1 - Ax^k + (-1)^k x^{2k}} \\
&= \frac{x^r \left[q_r + (q_{k+r} - Aq_r)\, x^k\right]}{\alpha - \beta} \left[\frac{\alpha}{1 - (-1)^k \alpha x^k} - \frac{\beta}{1 - (-1)^k \beta x^k}\right] \\
&= \frac{x^r \left[q_r + (q_{k+r} - Aq_r)\, x^k\right]}{\alpha - \beta} \sum_{n=0}^{\infty} (-1)^{kn} \left(\alpha^{n+1} - \beta^{n+1}\right) x^{nk} \\
&= x^r \left[q_r + (q_{k+r} - Aq_r)\, x^k\right] \sum_{n=0}^{\infty} \frac{(-1)^{kn} \left(\alpha^{n+1} - \beta^{n+1}\right) x^{nk}}{\alpha - \beta} \\
&= x^r \left[\sum_{n=0}^{\infty} \frac{(-1)^{kn} \left(\alpha^{n+1} - \beta^{n+1}\right) q_r x^{nk}}{\alpha - \beta} \right. \\
&\qquad \left. + \sum_{n=0}^{\infty} \frac{(-1)^{kn} \left(\alpha^{n+1} - \beta^{n+1}\right) (q_{k+r} - Aq_r)\, x^{(n+1)k}}{\alpha - \beta}\right] \\
&= x^r \left[\sum_{n=0}^{\infty} \frac{(-1)^{kn} \left(\alpha^{n+1} - \beta^{n+1}\right) q_r x^{nk}}{\alpha - \beta} \right. \\
&\qquad \left. + (-1)^k \sum_{n=1}^{\infty} \frac{(-1)^{kn} (\alpha^n - \beta^n)(q_{k+r} - Aq_r)\, x^{nk}}{\alpha - \beta}\right] \\
&= x^r \left[q_r + \sum_{n=1}^{\infty} \frac{(-1)^{kn} \left(\alpha^{n+1} - \beta^{n+1}\right) q_r x^{nk}}{\alpha - \beta} \right. \\
&\qquad \left. + (-1)^k \sum_{n=1}^{\infty} \frac{(-1)^{kn} (\alpha^n - \beta^n)(q_{k+r} - Aq_r)\, x^{nk}}{\alpha - \beta}\right] \\
&= x^r \left[q_r + \sum_{n=1}^{\infty} (-1)^{kn} \frac{\left(\alpha^{n+1} - \beta^{n+1}\right) q_r + (-1)^k (\alpha^n - \beta^n)(q_{k+r} - Aq_r)}{\alpha - \beta} x^{nk}\right] \\
&= q_r x^r + \sum_{n=1}^{\infty} (-1)^{k(n+1)} \left[\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) q_{k+r} - \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) q_r\right] x^{nk+r}.
\end{aligned}
$$

Therefore,

$$G_r(x) = \sum_{n=0}^{\infty} (-1)^{k(n+1)} \left[\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) q_{k+r} - \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) q_r\right] x^{nk+r}.$$

Thus

$$q_{kn+r} = (-1)^{k(n+1)} \left[\left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) q_{k+r} - \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) q_r\right].$$

□

Theorem 17. *The ratios of succesive terms of the subsequence* $\{q_{mk+r}\}$ *converge to*

$$\eta = \frac{A + sgn(A)\sqrt{A^2 - 4(-1)^k}}{2}$$

if $|A| > 2$*, where* $sgn(A) = \dfrac{A}{|A|}$ *is the sign of* A*.*

Proof. We will show the case $A > 2$. First note that when k is odd, $|\alpha| < |\beta|$ and when k is even, $|\alpha| > |\beta|$. From Theorem 16, we have

$$\frac{q_{(m+1)k+r}}{q_{mk+r}} = (-1)^k \frac{\left(\alpha^{m+1} - \beta^{m+1}\right) q_{k+r} - (\alpha^m - \beta^m)\, q_r}{(\alpha^m - \beta^m)\, q_{k+r} - (\alpha^{m-1} - \beta^{m-1})\, q_r}.$$

$$= \begin{cases} -\beta \dfrac{\left[\left(\dfrac{\alpha}{\beta}\right)^{m+1} - 1\right] q_{k+r} - \dfrac{1}{\beta}\left[\left(\dfrac{\alpha}{\beta}\right)^{m} - 1\right] q_r}{\left[\left(\dfrac{\alpha}{\beta}\right)^{m} - 1\right] q_{k+r} - \dfrac{1}{\beta}\left[\left(\dfrac{\alpha}{\beta}\right)^{m-1} - 1\right] q_r} & \text{if k is odd} \\[4ex] \alpha \dfrac{\left[1 - \left(\dfrac{\beta}{\alpha}\right)^{m+1}\right] q_{k+r} - \dfrac{1}{\alpha}\left[1 - \left(\dfrac{\beta}{\alpha}\right)^{m}\right] q_r}{\left[1 - \left(\dfrac{\beta}{\alpha}\right)^{m}\right] q_{k+r} - \dfrac{1}{\alpha}\left[1 - \left(\dfrac{\beta}{\alpha}\right)^{m-1}\right] q_r} & \text{if k is even.} \end{cases}$$

Since $\left|\dfrac{\alpha}{\beta}\right| < 1$ when k is odd and $\left|\dfrac{\beta}{\alpha}\right| < 1$ when k is even, we get

$$\lim_{m\to\infty} \frac{q_{(m+1)k+r}}{q_{mk+r}} = \begin{cases} -\beta & ;\text{ if k is odd} \\ \alpha & ;\text{ if k is even} \end{cases} = \frac{A + \sqrt{A^2 - 4(-1)^k}}{2}.$$

The case $A < -2$ can be handled in the same fashion.

□

One can prove in a similar way that for each $r = 1, 2, \cdots, k-1$, the ratios $\dfrac{q_{mk+r}}{q_{mk+r-1}}$ converge to $\eta_r = \dfrac{q_{k+r} + (-1)^{k-1}\beta q_r}{q_{k+r-1} + (-1)^{k-1}\beta q_{r-1}}$ as $m \to \infty$.

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