# LUCAS NUMBERS AND DETERMINANTS

**A. R. Moghaddamfar**
*Department of Mathematics, K. N. Toosi University of Technology, Tehran, Iran*
and *School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran*
moghadam@kntu.ac.ir, moghadam@mail.ipm.ir

**H. Tajbakhsh**
*Department of Mathematics, K. N. Toosi University of Technology, Tehran, Iran*

*Received: 4/4/10, Revised: 5/9/11, Accepted: 5/27/11, Published: 7/15/11*

**Abstract**

In this article, we present two infinite-dimensional matrices whose entries are recursively defined, and show that the sequence of their principal minors form the Lucas sequence; that is, $(2, 1, 3, 4, 7, \ldots)$. It is worth mentioning that to construct these matrices we use nonhomogeneous recurrence relations.

## 1. Introduction

Given an arbitrary infinite matrix $A$, denote by $d_n$ the *nth principal minor* of $A$, which is defined as the determinant of the submatrix consisting of the entries in its first $n$ rows and columns. Actually, we shall be interested in the sequence of principal minors $(d_1, d_2, d_3, \ldots)$, especially, in the case that it forms the sequence including *Lucas numbers*. There are scattered results in the literature showing that there exist certain infinite families of matrices of this kind, for instance, see [2], [4] and [5].

Let $\alpha = (\alpha_i)_{i\geq 0}$ and $\beta = (\beta_i)_{i\geq 0}$ be two arbitrary sequences. The *convolution* of sequences $\alpha$ and $\beta$, is the sequence $\gamma = (\gamma_i)_{i\geq 0}$, where $\gamma_i = \sum_{k=0}^{i} \alpha_k \beta_{i-k}$. The *convolution matrix* of sequences $\alpha$ and $\beta$ is the infinite matrix $A = (A_{i,j})_{i,j\geq 0}$ whose first column, i.e., $\mathrm{C}_0(A)$, is $\alpha$ and whose $j$th column ($j = 1, 2, \ldots$) is the convolution of sequences $\mathrm{C}_{j-1}(A)$ (the $(j-1)$th column) and $\beta$. In [6], the authors show that any sequence can be represented in terms of principal minors associated with an infinite matrix. In fact, they proved the following theorem:

**Theorem 1.** *(see [6]) Let $A(\infty)$ be the following infinite matrix*

$$A(\infty) = \left( \left[ \begin{array}{l} \textit{Convolution sequence} \\ \text{of } ((-1)^i\alpha_i)_{i\geq 0} \text{ and } (\beta_i)_{i\geq 0} \end{array} \right] \middle| \left[ \begin{array}{l} \textit{Convolution matrix} \\ \text{of } (\beta_i)_{i\geq 0} \text{ and } (\xi_i)_{i\geq 0} \end{array} \right] \right),$$

*where $\beta_0 = 1$ and $(\xi_i)_{i\geq 0} = (\xi_0, 1, 0, 0, \ldots)$. Then, we have $\det A(n) = \alpha_{n-1}$, where $A(n)$ is the $n \times n$ upper left corner matrix of $A(\infty)$.*

Let $L_n$ be the $n$th Lucas number, defined by

$$L_0 = 2,\ L_1 = 1 \quad \text{with} \quad L_n = L_{n-1} + L_{n-2} \ \text{ if } \ n \geq 2.$$

As an immediate consequence of Theorem 1 we have the following corollary.

**Corollary 2.** *With the notation of Theorem 1, if $\alpha_i = L_i,\ i = 0, 1, 2, \ldots,$ then we have $\det A(n) = L_{n-1},\ (n = 1, 2, 3, \ldots)$.*

Let $\phi = (\phi_i)_{i\geq 0}$ and $\psi = (\psi_i)_{i\geq 0}$ be two sequences starting with a common first term $\phi_0 = \psi_0 (:= \gamma)$. Moreover, let $P_{\phi,\psi}^{[a,b,c]}(\infty) = [P_{i,j}]_{i,j\geq 0}$ be an infinite matrix with parameters $a, b, c$ associated with sequences $\phi$ and $\psi$, whose entries satisfy:

$$P_{0,0} = \gamma, \quad P_{i,0} = \phi_i \ \text{ and } \ P_{0,j} = \psi_i \quad \text{for } i, j \geq 1,$$

and

$$P_{i,j} = aP_{i,j-1} + bP_{i-1,j-1} + cP_{i-1,j} \quad \text{for } i, j \geq 1. \tag{1}$$

In special cases, when $[a, b, c] = [1, 0, 1]$ or $[a, b, c] = [0, 1, 1]$, the matrix $P_{\phi,\psi}^{[a,b,c]}(\infty)$ is called a *generalized Pascal triangle* or a 7*-matrix*, respectively (see [1], [3] and [7]).

In [5], we proved the following theorem:

**Theorem 3.** *Let $A_1$, $A_2$, $B_1$ and $B_2$ be integers. Let $\alpha = (\alpha_i)_{i\geq 0}$ and $\beta = (\beta_i)_{i\geq 0}$ be two integer sequences satisfy $\alpha_0 = \beta_0 (:= \gamma)$ and linear recurrences*

$$\alpha_i = A_1\alpha_{i-1} + A_2\alpha_{i-2} \quad \textit{and} \quad \beta_i = B_1\beta_{i-1} + B_2\beta_{i-2} \quad \textit{for all } \ i \geq 2.$$

*Then we have:*

*(a) for any nonnegative integer $n$, $\det P_{\alpha,\beta}^{[1,0,1]}(n) = L_n$ if and only if $\gamma$, $\alpha_1$, $\beta_1$, $A_1$, $A_2$, $B_1$ and $B_2$ satisfy one of the following conditions.*

| | $\gamma$ | $\alpha_1$ | $\beta_1$ | $A_1$ | $A_2$ | $B_1$ | $B_2$ | | $\gamma$ | $\alpha_1$ | $\beta_1$ | $A_1$ | $A_2$ | $B_1$ | $B_2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (L.1) | 2 | 5 | 3 | 2 | 1 | −3 | 8 | (L.17) | 2 | 3 | 5 | −3 | 8 | 2 | 1 |
| (L.2) | 2 | 5 | 3 | 6 | −9 | −7 | 14 | (L.18) | 2 | 3 | 5 | −7 | 14 | 6 | −9 |
| (L.3) | 2 | 5 | 3 | 1 | 4 | 12 | −17 | (L.19) | 2 | 3 | 5 | 12 | −17 | 1 | 4 |
| (L.4) | 2 | 5 | 3 | 5 | −6 | 8 | −11 | (L.20) | 2 | 3 | 5 | 8 | −11 | 5 | −6 |
| (L.5) | 2 | 5 | 3 | 8 | −15 | −1 | 4 | (L.21) | 2 | 3 | 5 | −1 | 4 | 8 | −15 |
| (L.6) | 2 | 5 | 3 | 4 | −5 | 3 | −2 | (L.22) | 2 | 3 | 5 | 3 | −2 | 4 | −5 |
| (L.7) | 2 | 5 | 3 | −1 | 10 | 6 | −7 | (L.23) | 2 | 3 | 5 | 6 | −7 | −1 | 10 |
| (L.8) | 2 | 5 | 3 | 3 | 0 | 2 | −1 | (L.24) | 2 | 3 | 5 | 2 | −1 | 3 | 0 |
| (L.9) | 2 | −1 | 1 | 0 | −1 | 1 | 0 | (L.25) | 2 | 1 | −1 | 1 | 0 | 0 | −1 |
| (L.10) | 2 | −1 | 1 | −4 | −3 | 5 | −2 | (L.26) | 2 | 1 | −1 | 5 | −2 | −4 | −3 |
| (L.11) | 2 | −1 | 1 | 2 | 1 | 7 | −2 | (L.27) | 2 | 1 | −1 | 7 | −2 | 2 | 1 |
| (L.12) | 2 | −1 | 1 | −2 | −1 | 11 | −4 | (L.28) | 2 | 1 | −1 | 11 | −4 | −2 | −1 |
| (L.13) | 2 | −1 | 1 | 3 | 2 | −8 | 3 | (L.29) | 2 | 1 | −1 | −8 | 3 | 3 | 2 |
| (L.14) | 2 | −1 | 1 | −1 | 0 | −4 | 1 | (L.30) | 2 | 1 | −1 | −4 | 1 | −1 | 0 |
| (L.15) | 2 | −1 | 1 | 5 | 4 | −2 | 1 | (L.31) | 2 | 1 | −1 | −2 | 1 | 5 | 4 |
| (L.16) | 2 | −1 | 1 | 1 | 2 | 2 | −1 | (L.32) | 2 | 1 | −1 | 2 | −1 | 1 | 2 |

(*b*) *For any nonnegative integer* $n$, $\det P^{[0,1,1]}_{\alpha,\beta}(n) = L_n$ *if and only if* $\gamma$, $\alpha_1$, $\beta_1$, $A_1$, $A_2$, $B_1$ *and* $B_2$ *satisfy one of the following conditions.*

| | $\gamma$ | $\alpha_1$ | $\beta_1$ | $A_1$ | $A_2$ | $B_1$ | $B_2$ | | $\gamma$ | $\alpha_1$ | $\beta_1$ | $A_1$ | $A_2$ | $B_1$ | $B_2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (L.1′) | 2 | 5 | 1 | 2 | 1 | −5 | 4 | (L.17′) | 2 | 3 | 3 | −3 | 8 | 0 | 2 |
| (L.2′) | 2 | 5 | 1 | 6 | −9 | −9 | 6 | (L.18′) | 2 | 3 | 3 | −7 | 14 | 4 | −4 |
| (L.3′) | 2 | 5 | 1 | 1 | 4 | 10 | −6 | (L.19′) | 2 | 3 | 3 | 12 | −17 | −1 | 4 |
| (L.4′) | 2 | 5 | 1 | 5 | −6 | 6 | −4 | (L.20′) | 2 | 3 | 3 | 8 | −11 | 3 | −2 |
| (L.5′) | 2 | 5 | 1 | 8 | −15 | −3 | 2 | (L.21′) | 2 | 3 | 3 | −1 | 4 | 6 | −8 |
| (L.6′) | 2 | 5 | 1 | 4 | −5 | 1 | 0 | (L.22′) | 2 | 3 | 3 | 3 | −2 | 2 | −2 |
| (L.7′) | 2 | 5 | 1 | −1 | 10 | 4 | −2 | (L.23′) | 2 | 3 | 3 | 6 | −7 | −3 | 8 |
| (L.8′) | 2 | 5 | 1 | 3 | 0 | 0 | 0 | (L.24′) | 2 | 3 | 3 | 2 | −1 | 1 | 2 |
| (L.9′) | 2 | −1 | −1 | 0 | −1 | −1 | 0 | (L.25′) | 2 | 1 | −3 | 1 | 0 | −2 | −2 |
| (L.10′) | 2 | −1 | −1 | −4 | −3 | 3 | 2 | (L.26′) | 2 | 1 | −3 | 5 | −2 | −6 | −8 |
| (L.11′) | 2 | −1 | −1 | 2 | 1 | 5 | 4 | (L.27′) | 2 | 1 | −3 | 7 | −2 | 0 | 2 |
| (L.12′) | 2 | −1 | −1 | −2 | −1 | 9 | 6 | (L.28′) | 2 | 1 | −3 | 11 | −4 | −4 | −4 |
| (L.13′) | 2 | −1 | −1 | 3 | 2 | −10 | −6 | (L.29′) | 2 | 1 | −3 | −8 | 3 | 1 | 4 |
| (L.14′) | 2 | −1 | −1 | −1 | 0 | −6 | −4 | (L.30′) | 2 | 1 | −3 | −4 | 1 | −3 | −2 |
| (L.15′) | 2 | −1 | −1 | 5 | 4 | −4 | −2 | (L.31′) | 2 | 1 | −3 | −2 | 1 | 3 | 8 |
| (L.16′) | 2 | −1 | −1 | 1 | 2 | 0 | 0 | (L.32′) | 2 | 1 | −3 | 2 | −1 | −1 | 2 |

Note that the recurrence relation (1) is homogeneous. Here, we will consider a more general case. In fact, we will allow the recursion entries to depend on nonhomogeneous recurrence relations:

$$P_{i,j} = aP_{i,j-1} + bP_{i-1,j-1} + cP_{i-1,j} + f(i,j), \quad i,j \geq 1,$$

where $f$ is a function on $\mathbb{N} \times \mathbb{N}$.

In this article, we will consider the following two infinite-dimensional matrices

$$A(\infty) = [A_{i,j}]_{i,j\geq 0} = \begin{pmatrix} 2 & 3 & 4 & 5 & \cdots \\ -3 & -4 & -6 & -9 & \cdots \\ -27 & -37 & -51 & -70 & \cdots \\ -125 & -170 & -231 & -313 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix},$$

and

$$B(\infty) = [B_{i,j}]_{i,j\geq 0} = \begin{pmatrix} 2 & 1 & 3 & -1 & \cdots \\ 1 & 1 & 2 & 0 & \cdots \\ -2 & -2 & -1 & -2 & \cdots \\ -1 & -10 & -9 & -9 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

As a matter of fact, these matrices are constructed as follows:

*The Matrix* $A(\infty)$. The first row and column is the sequence

$$(A_{0,j})_{j\geq 0} = (2, 3, 4, 5, , \ldots, A_{0,j} = j + 2, \ldots),$$

and

$$(A_{i,0})_{i\geq 0} = (2, -3, -27, \ldots, A_{i,0} = 4A_{i-1,0} + i^2 - 7i - 5, \ldots),$$

respectively. The remaining entries $A_{i,j}$ are obtained from the nonhomogeneous recurrence relation:

$$A_{i,j} = A_{i,j-1} + A_{i-1,j} - 2(i + j), \qquad i,j \geq 1.$$

*The Matrix* $B(\infty)$. The first row and column is the sequence

$$(B_{0,j})_{j\geq 0} = (2, 1, 3, -1, 7, -9, 23, -41, \ldots, B_{0,j} = -B_{0,j-1} + 2B_{0,j-2}, \ldots),$$

and

$$(B_{i,0})_{i\geq 0} = (2, 1, -2, -1, 42, \ldots, B_{i,0} = 3B_{i-1,0} + 5(3^{i-1} - 2i - 1)/2, \ldots),$$

respectively. The remaining entries $B_{i,j}$ are obtained from the nonhomogeneous recurrence relation:

$$B_{i,j} = B_{i-1,j-1} + B_{i-1,j} - 2i, \qquad i, j \geq 1.$$

We denote by $A(n)$ (resp. $B(n)$) the submatrix of $A(\infty)$ (resp. $B(\infty)$) consisting of the elements in its first $n$ rows and columns. The main result of the paper is essentially the following theorem:

**Theorem 4.** *With the notation defined above, we have* $\det A(n) = \det B(n) = L_{n-1}$.

Our notation and terminology are standard. Given a matrix $A$, we denote by $\mathrm{R}_i(A)$ and $\mathrm{C}_j(A)$ the row $i$ and the column $j$ of $A$, respectively. By $A^T$ we denote the transpose of $A$. We also denote the elementary row and column operations of type three by $O_{i,j}(\lambda)$ and $O'_{i,j}(\lambda)$, respectively, where $i \neq j$ and $\lambda$ a scalar. So that

$$\mathrm{R}_k(O_{ij}(\lambda)A) = \begin{cases} \mathrm{R}_i(A) + \lambda \mathrm{R}_j(A) & \text{if} \quad k = i; \\ \mathrm{R}_k(A) & \text{if} \quad k \neq i, \end{cases}$$

and

$$\mathrm{C}_i(AO'_{i,j}(\lambda)) = \begin{cases} \mathrm{C}_i(A) + \lambda \mathrm{C}_j(A) & \text{if} \quad k = i; \\ \mathrm{C}_k(A) & \text{if} \quad k \neq i. \end{cases}$$

We recall that a matrix $T(\infty) = [t_{i,j}]_{i,j\geq 0}$ is said to be Töeplitz if $t_{i,j} = t_{k,l}$ whenever $i - j = k - l$. In the case that $\phi = (\phi_i)_{i\geq 0}$ and $\psi = (\psi_i)_{i\geq 0}$ are two sequences with a common first term $\phi_0 = \psi_0$, we denote by $T_{\phi,\psi}(\infty) = [t_{i,j}]_{i,j\geq 0}$ the Töeplitz matrix with $C_0(T_{\phi,\psi}(\infty))^T = \phi$ and $R_0(T_{\phi,\psi}(\infty)) = \psi$. Moreover, we denote by $T_{\phi,\psi}(n)$ the submatrix of $T_{\phi,\psi}(\infty)$ consisting of the entries in its first $n$ rows and columns.

A lower Hessenberg matrix, $H(n) = [h_{i,j}]_{0\leq i,j\leq n-1}$, is an $n \times n$ matrix where $h_{i,j} = 0$ whenever $j > i + 1$ and $h_{i,i+1} \neq 0$ for some $i$, $0 \leq i \leq n - 2$, so we have

$$H(n) = \begin{pmatrix} h_{0,0} & h_{0,1} & 0 & \cdots & 0 \\ h_{1,0} & h_{1,1} & h_{1,2} & \ddots & 0 \\ h_{2,0} & h_{2,1} & h_{2,2} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & h_{n-2,n-1} \\ h_{n-1,0} & h_{n-1,1} & \cdots & h_{n-1,n-2} & h_{n-1,n-1} \end{pmatrix}.$$

## 2. Preliminaries

In order to prove the main results of this article, we need to state some technical lemmas. We start with the following simple observation.

**Lemma 5.** *For all integers $i \geq 3$, there holds*

(*a*) $\sum_{k=0}^{i}(-1)^k\binom{i}{k}(k^2-5k-11)=0.$

(*b*) $\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}(2k+3)=0.$

*Proof.* We only prove $(a)$, and $(b)$ goes similarly. Define the function $f$ as follows:

$$f(x)=(1-x)^i=\sum_{k=0}^{i}(-1)^k\binom{i}{k}x^k.$$

Then

$$f'(x)=-i(1-x)^{i-1}=\sum_{k=0}^{i}(-1)^k\binom{i}{k}kx^{k-1}.$$

Moreover, if we take $g(x):=xf(x)$, then an easy calculation shows that

$$g'(x)=(1-x)^{i-2}\cdot(*)=\sum_{k=0}^{i}(-1)^k\binom{i}{k}k^2x^{k-1}.$$

Putting $x=1$, we conclude that

$$\sum_{k=0}^{i}(-1)^k\binom{i}{k}=0,\quad \sum_{k=0}^{i}(-1)^k\binom{i}{k}k=0 \quad\text{and}\quad \sum_{k=0}^{i}(-1)^k\binom{i}{k}k^2=0,$$

and the result follows. □

For an arbitrary sequence $\sigma=(\sigma_i)_{i\geq 0}$, the binomial and inverse binomial transform of $\sigma$ are the sequences $\hat{\sigma}=(\hat{\sigma}_i)_{i\geq 0}$ and $\check{\sigma}=(\check{\sigma}_i)_{i\geq 0}$ defined by

$$\hat{\sigma}_i=\sum_{k=0}^{i}(-1)^{i+k}\binom{i}{k}\sigma_k \qquad\text{and}\qquad \check{\sigma}_i=\sum_{k=0}^{i}\binom{i}{k}\sigma_k.$$

Clearly, we have the following inverse relations

$$\sigma_i=\sum_{k=0}^{i}(-1)^{i+k}\binom{i}{k}\check{\sigma}_k=\sum_{k=0}^{i}\binom{i}{k}\hat{\sigma}_k. \tag{2}$$

Let $\alpha=(\alpha_i)_{i\geq 0}$, $\lambda=(\lambda_i)_{i\geq 0}$, $\mu=(\mu_i)_{i\geq 0}$ and $\nu=(\nu_i)_{i\geq 0}$ be three (non)homogeneous recurrence relations satisfying

$$\alpha_0=2,\ \alpha_1=-3,\quad\text{and}\quad \alpha_i=4\alpha_{i-1}+i^2-7i-5 \qquad (i\geq 2)$$
$$\lambda_0=2,\ \lambda_1=1,\quad\text{and}\quad \lambda_i=3\lambda_{i-1}+5(3^{i-1}-2i-1)/2 \qquad (i\geq 2)$$
$$\mu_0=2,\ \mu_1=1,\quad\text{and}\quad \mu_i=-\mu_{i-1}+2\mu_{i-2} \quad (i\geq 2)$$
$$\nu_0=-1,\ \nu_1=-17,\ \nu_2=-60,\quad\text{and}\quad \nu_i=2\nu_{i-1}-50\cdot 2^{i-3} \quad (i\geq 3).$$

Then, we have

$$\begin{aligned}
\alpha = (\alpha_i)_{i\geq 0} &= (2, -3, -27, -125, -517, -2083, \ldots) \\
\lambda = (\lambda_i)_{i\geq 0} &= (2, 1, -2, -1, 42, 301, 1478, 6219, \ldots), \\
\mu = (\mu_i)_{i\geq 0} &= (2, 1, 3, -1, 7, -9, 23, -41, 87, \ldots), \\
\nu = (\nu_i)_{i\geq 0} &= (-1, -17, -60, -170, -440, \ldots).
\end{aligned}$$

Solving these (non)homogeneous recurrence relations, we obtain

$$\begin{aligned}
\alpha_i &= (109 + 39i - 9i^2 - 55 \cdot 2^{2i})/27,\ i \geq 0, \\
\lambda_i &= 5i(3^{i-1} + 1)/2 - 3^{i+1} + 5,\ i \geq 0, \\
\mu_i &= (5 + (-2)^i)/3,\ i \geq 0, \quad \text{and} \\
\nu_i &= -5 \cdot 2^{i-2}(5i + 2),\ i \geq 2.
\end{aligned}$$

On the other hand, the binomial transform of the sequences $\alpha$, $\lambda$ and $\mu$ are:

$$\hat{\alpha} = (\hat{\alpha}_i)_{i\geq 0} = (2, -5, -19, -55, -165, \ldots, \sum_{k=0}^{i} (-1)^{i+k} \binom{i}{k} \alpha_k, \ldots).$$

$$\hat{\lambda} = (\hat{\lambda}_i)_{i\geq 0} = (2, -1, -2,\ 6,\ 32,\ 104, \ldots, \sum_{k=0}^{i} (-1)^{i+k} \binom{i}{k} \lambda_k, \ldots).$$

$$\hat{\mu} = (\hat{\mu}_i)_{i\geq 0} = (2, -1, 3,\ -9,\ 27,\ -81, \ldots, \sum_{k=0}^{i} (-1)^{i+k} \binom{i}{k} \mu_k, \ldots).$$

Throughout the rest of this article we fix the sequences $\alpha$, $\lambda$, $\mu$ and $\nu$.

**Lemma 6.** *With the notation defined above, we have*

*(a) for all positive integers $i \geq 4$, $\hat{\alpha}_i = 3\hat{\alpha}_{i-1}$.*

*(b) for all positive integers $i \geq 4$, $\hat{\alpha}_i - \sum_{k=0}^{i-3} \hat{\alpha}_{i-k-1} L_k + 7L_{i-2} + 2L_{i-1} + L_i = 0$.*

*(c) for all positive integer $i \geq 3$, $\hat{\lambda}_i - 2\hat{\lambda}_{i-1} - 5 \cdot 2^{i-2} = 0$,*

*(d) for all positive integer $i \geq 1$, $\hat{\mu}_i + (-3)^{i-1} = 0$.*

*Proof.* $(a)$ Easy computations show that

$$\begin{aligned}
\hat{\alpha}_i &= \textstyle\sum_{k=0}^{i}(-1)^{i+k}\binom{i}{k}\alpha_k \\
&= \textstyle\sum_{k=0}^{i}(-1)^{i+k}\left[\binom{i-1}{k-1}+\binom{i-1}{k}\right]\alpha_k \qquad \text{(by Pascal's rule)} \\
&= \textstyle\sum_{k=0}^{i}(-1)^{i+k}\binom{i-1}{k-1}\alpha_k + \sum_{k=0}^{i}(-1)^{i+k}\binom{i-1}{k}\alpha_k \\
&= \textstyle\sum_{k=0}^{i}(-1)^{i+k}\binom{i-1}{k-1}\alpha_k - \sum_{k=0}^{i-1}(-1)^{i+k-1}\binom{i-1}{k}\alpha_k \\
&= \textstyle\sum_{k=0}^{i}(-1)^{i+k}\binom{i-1}{k-1}\left(4\alpha_{k-1}+k^2-7k-5\right) - \hat{\alpha}_{i-1} \\
&= \textstyle 4\sum_{k=0}^{i}(-1)^{i+k}\binom{i-1}{k-1}\alpha_{k-1} + \sum_{k=0}^{i}(-1)^{i+k}\binom{i-1}{k-1}(k^2-7k-5) - \hat{\alpha}_{i-1} \\
&= \textstyle 4\sum_{k=0}^{i-1}(-1)^{i+k-1}\binom{i-1}{k}\alpha_k + \sum_{k=0}^{i-1}(-1)^{i+k+1}\binom{i-1}{k}(k^2-5k-11) - \hat{\alpha}_{i-1} \\
&= \textstyle 4\hat{\alpha}_{i-1} + (-1)^{i+1}\sum_{k=0}^{i-1}(-1)^{k}\binom{i-1}{k}(k^2-5k-11) - \hat{\alpha}_{i-1} \\
&= 3\hat{\alpha}_{i-1}, \qquad \text{(Since } i \geq 4\text{, we can apply Lemma 5}(a)\text{)}
\end{aligned}$$

as desired.

$(b)$ For the sake of brevity, throughout the proof we will set

$$\Phi_i := \hat{\alpha}_i - \sum_{k=0}^{i-3}\hat{\alpha}_{i-k-1}L_k + 7L_{i-2} + 2L_{i-1}$$

and we will show that $\Phi_i = -L_i$. First of all, it is easy to verify that

$$\begin{aligned}
\Phi_4 &= \textstyle\hat{\alpha}_4 - \sum_{k=0}^{1}\hat{\alpha}_{3-k}L_k + 7L_2 + 2L_3 \\
&= -165 + 55\cdot 2 + 19\cdot 1 + 7\cdot 3 + 2\cdot 4 \\
&= -7 \\
&= -L_4,
\end{aligned}$$

and

$$\begin{aligned}
\Phi_5 &= \textstyle\hat{\alpha}_5 - \sum_{k=0}^{2}\hat{\alpha}_{3-k}L_k + 7L_3 + 2L_4 \\
&= -495 + 165\cdot 2 + 55\cdot 1 + 19\cdot 3 + 7\cdot 4 + 2\cdot 7 \\
&= -11 \\
&= -L_5.
\end{aligned}$$

Next, to prove the result, it suffices to show that

$$\Phi_i = \Phi_{i-1} + \Phi_{i-2}, \qquad i \geq 6.$$

To do this, we see that

$$\begin{aligned}
\Phi_{i-1}+\Phi_{i-2} &= \hat{\alpha}_{i-1}-\textstyle\sum_{k=0}^{i-4}\hat{\alpha}_{i-k-2}L_k+7L_{i-3}+2L_{i-2}\\
&\quad +\hat{\alpha}_{i-2}-\textstyle\sum_{k=0}^{i-5}\hat{\alpha}_{i-k-3}L_k+7L_{i-4}+2L_{i-3}\\
&= \hat{\alpha}_{i-1}-\hat{\alpha}_{i-2}L_0-\textstyle\sum_{k=1}^{i-4}\hat{\alpha}_{i-k-2}L_k+7L_{i-3}+2L_{i-2}\\
&\quad +\hat{\alpha}_{i-2}-\textstyle\sum_{k=1}^{i-4}\hat{\alpha}_{i-k-2}L_{k-1}+7L_{i-4}+2L_{i-3}\\
&= \hat{\alpha}_{i-1}-\hat{\alpha}_{i-2}-\textstyle\sum_{k=1}^{i-4}\hat{\alpha}_{i-k-2}(L_k+L_{k-1})\\
&\quad +7(L_{i-3}+L_{i-4})+2(L_{i-3}+L_{i-2})\\
&= \hat{\alpha}_{i-1}-\hat{\alpha}_{i-2}-\textstyle\sum_{k=1}^{i-4}\hat{\alpha}_{i-k-2}L_{k+1}+7L_{i-2}+2L_{i-1}\\
&= \hat{\alpha}_{i-1}-\hat{\alpha}_{i-2}-\textstyle\sum_{k=2}^{i-3}\hat{\alpha}_{i-k-1}L_k+7L_{i-2}+2L_{i-1}\\
&= \hat{\alpha}_{i-1}-\textstyle\sum_{k=1}^{i-3}\hat{\alpha}_{i-k-1}L_k+7L_{i-2}+2L_{i-1}\\
&= 3\hat{\alpha}_{i-1}-\textstyle\sum_{k=0}^{i-3}\hat{\alpha}_{i-k-1}L_k+7L_{i-2}+2L_{i-1}\\
&= \hat{\alpha}_{i}-\textstyle\sum_{k=0}^{i-3}\hat{\alpha}_{i-k-1}L_k+7L_{i-2}+2L_{i-1} \qquad \text{(by part } (a))\\
&= \Phi_i,
\end{aligned}$$

as required.

$(c)$ Again, by easy computations we observe that

$$\begin{aligned}
\hat{\lambda}_i &= \textstyle\sum_{k=0}^{i}(-1)^{i+k}\binom{i}{k}\lambda_k\\
&= (-1)^i\lambda_0+\sum_{k=1}^{i}(-1)^{i+k}\left[\binom{i-1}{k-1}+\binom{i-1}{k}\right]\lambda_k \quad \text{(by Pascal's rule)}\\
&= (-1)^i\lambda_0+\sum_{k=1}^{i}(-1)^{i+k}\binom{i-1}{k-1}\lambda_k+\sum_{k=1}^{i}(-1)^{i+k}\binom{i-1}{k}\lambda_k\\
&= (-1)^{i+1}\lambda_1+\sum_{k=2}^{i}(-1)^{i+k}\binom{i-1}{k-1}\left(3\lambda_{k-1}+5(3^{k-1}-2k-1)/2\right)\\
&\quad -\sum_{k=0}^{i}(-1)^{i+k-1}\binom{i-1}{k}\lambda_k\\
&= (-1)^{i+1}\lambda_1+3\sum_{k=2}^{i}(-1)^{i+k}\binom{i-1}{k-1}\lambda_{k-1}+\frac{5}{2}\sum_{k=2}^{i}(-1)^{i+k}\binom{i-1}{k-1}3^{k-1}\\
&\quad -\frac{5}{2}\sum_{k=2}^{i}(-1)^{i+k}\binom{i-1}{k-1}(2k+1)-\hat{\lambda}_{i-1}
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{i+1}\lambda_1 + 3\sum_{k=1}^{i-1}(-1)^{i+k+1}\tbinom{i-1}{k}\lambda_k + \tfrac{5}{2}\sum_{k=1}^{i-1}(-1)^{i+k+1}\tbinom{i-1}{k}3^k \\
&\quad -\tfrac{5}{2}\sum_{k=1}^{i-1}(-1)^{i+k+1}\tbinom{i-1}{k}(2k+3) - \hat{\lambda}_{i-1} \\
&= (-1)^{i+1}\lambda_1 - 3(-1)^{i+1}\lambda_0 + 3\sum_{k=0}^{i-1}(-1)^{i+k+1}\tbinom{i-1}{k}\lambda_k \\
&\quad -\tfrac{5}{2}(-1)^{i+1} + \tfrac{5}{2}\sum_{k=0}^{i-1}(-1)^{i+k+1}\tbinom{i-1}{k}3^k \\
&\quad +\tfrac{15}{2}(-1)^{i+1} - \tfrac{5}{2}(-1)^{i+1}\sum_{k=0}^{i-1}(-1)^{k}\tbinom{i-1}{k}(2k+3) - \hat{\lambda}_{i-1} \\
&= 3\hat{\lambda}_{i-1} + \tfrac{5}{2}(-1)^{i+1}\sum_{k=0}^{i-1}\tbinom{i-1}{k}(-3)^k - \hat{\lambda}_{i-1} \quad \big(\text{since } i \geq 3 \text{ we can apply Lemma 5(b)}\big) \\
&= 2\hat{\lambda}_{i-1} + \tfrac{5}{2}(-1)^{i+1}(-2)^{i-1} = 2\hat{\lambda}_{i-1} + 5\cdot 2^{i-2},
\end{aligned}$$

which proves part $(c)$.

$(d)$ Finally, simple computations show that

$$\begin{aligned}
\hat{\mu}_i &= \textstyle\sum_{k=0}^{i}(-1)^{i+k}\binom{i}{k}\mu_k \\
&= \textstyle\sum_{k=0}^{i}(-1)^{i+k}\binom{i}{k}(5+(-2)^k)/3 \\
&= \tfrac{5}{3}\textstyle\sum_{k=0}^{i}(-1)^{i+k}\binom{i}{k} + \tfrac{1}{3}\sum_{k=0}^{i}(-1)^{i+k}\binom{i}{k}(-2)^k \\
&= \tfrac{1}{3}(-1)^i\textstyle\sum_{k=0}^{i}\binom{i}{k}2^k \\
&= (-1)^i 3^{i-1},
\end{aligned}$$

as desired. □

Next, we prove the following identity

**Lemma 7.** *For all positive integer $i \geq 5$, the following identity holds:*

$$L_i - 17\cdot L_{i-3} + 5\sum_{k=0}^{i-5}(-2)^k(5k+12)L_{i-(k+4)} - (-2)^{i-3}\cdot 5 = 0.$$

*Proof.* For convenience, we put

$$\Psi_i := 17\cdot L_{i-3} - 5\sum_{k=0}^{i-5}(-2)^k(5k+12)L_{i-(k+4)} + (-2)^{i-3}\cdot 5.$$

First, we assume that $i = 5$ or $6$. In these cases, using easy computations we obtain

$$\Psi_5 = 17\cdot L_2 - 5(12\cdot L_1) + (-2)^2\cdot 5 = 51 - 60 + 20 = 11 = L_5,$$

and

$$\Psi_6 = 17 \cdot L_3 - 5(12 \cdot L_2 - 2 \cdot 17 \cdot L_1) + (-2)^3 \cdot 5 = 68 - 10 - 40 = 18 = L_6.$$

Now, to prove the $\Psi_i = L_i$ $(i \geq 7)$, it is enough to show that:

$$\Psi_i = \Psi_{i-1} + \Psi_{i-2}, \quad (i \geq 7).$$

To do this, we see that:

$$\begin{aligned}
\Psi_{i-1} + \Psi_{i-2} &= 17 \cdot L_{i-4} - 5\sum_{k=0}^{i-6}(-2)^k(5k+12)L_{i-(k+5)} + (-2)^{i-4} \cdot 5 \\
&\quad +17 \cdot L_{i-5} - 5\sum_{k=0}^{i-7}(-2)^k(5k+12)L_{i-(k+6)} + (-2)^{i-5} \cdot 5 \\
&= 17(L_{i-4} + L_{i-5}) - 5\sum_{k=0}^{i-6}(-2)^k(5k+12)\big(L_{i-(k+5)} + L_{i-(k+6)}\big) \\
&\quad +5 \cdot (-2)^{i-6}\big(5(i-6)+12\big)L_0 + (-2)^{i-5} \cdot (-5) \\
&= 17 \cdot L_{i-3} - 5\sum_{k=0}^{i-6}(-2)^k(5k+12)L_{i-(k+4)} \\
&\quad -5 \cdot (-2)^{i-5}(5i-18) + (-2)^{i-5} \cdot (-5) \\
&= 17 \cdot L_{i-3} - 5\sum_{k=0}^{i-5}(-2)^k(5k+12)L_{i-(k+4)} \\
&\quad +5(-2)^{i-5}(5i-13) - 5(-2)^{i-5}(5i-18) + (-2)^{i-5} \cdot (-5) \\
&= 17 \cdot L_{i-3} - 5\sum_{k=0}^{i-5}(-2)^k(5k+12)L_{i-(k+4)} + (-2)^{i-3} \cdot 5 = \Psi_i,
\end{aligned}$$

as required. □

## 3. Principal Minors of the Matrix $A(\infty)$

In this section, we first introduce the following lower Hessenberg matrix

$$H(n) = [h_{i,j}]_{0 \leq i,j \leq n-1} = \left[\begin{array}{c|ccccc}
\hat{\alpha}_0 & 1 & 0 & 0 & \cdots & 0 \\
\hline
\hat{\alpha}_1 & & & & & \\
\hat{\alpha}_2 & & & & & \\
\vdots & & & T(n-1) & & \\
\hat{\alpha}_{n-1} & & & & &
\end{array}\right]_{n \times n},$$

where

$$T(n-1) = T_{(-2,-7,\hat{\alpha}_2,\hat{\alpha}_3,\hat{\alpha}_4,\ldots),(-2,-1,0,0,0,\ldots)}(n-1).$$

The matrix $H(6)$, for instance, is hence given by

$$H(6) = [h_{i,j}]_{0\le i,j\le 5} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ -5 & -2 & -1 & 0 & 0 & 0 \\ -19 & -7 & -2 & -1 & 0 & 0 \\ -55 & -19 & -7 & -2 & -1 & 0 \\ -165 & -55 & -19 & -7 & -2 & -1 \\ -495 & -165 & -55 & -19 & -7 & -2 \end{bmatrix}_{6\times 6}.$$

**Lemma 8.** *With the notation defined above, we have* $\det H(n) = L_{n-1}$.

*Proof.* First, we apply the following row operations:

$$\begin{aligned} H_1(n) &= \Big(\prod_{i=1}^{n-1} O_{i,0}(-h_{i,1})\Big) H(n), \\ H_2(n) &= \Big(\prod_{i=2}^{n-1} O_{i,1}(h_{i,2})\Big) H_1(n), \\ H_3(n) &= \Big(\prod_{i=3}^{n-1} O_{i,2}(h_{i,3})\Big) H_2(n), \\ &\vdots \\ H_{n-1}(n) &= \Big(\prod_{i=n-1}^{n-1} O_{i,n-2}(h_{i,n-1})\Big) H_{n-2}(n). \end{aligned}$$

As a matter of fact, step by step, the columns are "emptied" until finally the following matrix is obtained:

$$H_{n-1}(n) = \begin{bmatrix} \tilde{h}_{0,0} & 1 & 0 & 0 & 0 & \cdots & 0 \\ \tilde{h}_{1,0} & 0 & -1 & 0 & 0 & \cdots & 0 \\ \tilde{h}_{2,0} & 0 & 0 & -1 & 0 & \cdots & 0 \\ \tilde{h}_{3,0} & 0 & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{h}_{n-2,0} & 0 & 0 & 0 & 0 & \cdots & -1 \\ \tilde{h}_{n-1,0} & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n}.$$

Evidently $\tilde{h}_{0,0} = \hat{\alpha}_0 = 2 = L_0$. Now, we claim that

$$\tilde{h}_{i,0} = -L_i, \quad 1 \le i \le n-1. \tag{3}$$

We proceed by induction on $i$. The cases $i = 1, 2$, and 3 are trivial:

$$\begin{aligned}
\tilde{h}_{1,0} &= h_{1,0} - h_{1,1} \cdot 2 \\
&= -5 - (-2) \cdot 2 = -1 = -L_1, \\
\tilde{h}_{2,0} &= h_{2,0} - h_{2,1} \cdot 2 + h_{2,2} \cdot (-1) \\
&= -19 - (-7) \cdot 2 + (-2) \cdot (-1) = -3 = -L_2, \\
\tilde{h}_{3,0} &= h_{3,0} - h_{3,1} \cdot 2 + h_{3,2} \cdot (-1) + h_{3,3} \cdot (-3) \\
&= -55 - (-19) \cdot 2 + (-7) \cdot (-1) + (-2) \cdot (-3) = -4 = -L_3.
\end{aligned}$$

Assume now that $i \geq 4$ and Eq. (3) is true for $1, 2, \ldots, i-1$, that is, $\tilde{h}_{k,0} = -L_k,\ 1 \leq k \leq i-1$. We now prove that Eq. (3) is also true for $i$. In fact, by the applied row operations and induction hypothesis, we have

$$\begin{aligned}
\tilde{h}_{i,0} &= h_{i,0} - h_{i,1} \cdot 2 + \textstyle\sum_{k=1}^{i-3} \left\{ h_{i,i-k-1} \cdot (-L_k) \right\} \\
&\quad + (-7) \cdot (-L_{i-2}) + (-2) \cdot (-L_{i-1}) \\
&= h_{i,0} - \textstyle\sum_{k=0}^{i-3} \{ h_{i,i-k-1} L_k \} + 7L_{i-2} + 2L_{i-1} \\
&= \hat{\alpha}_i - \textstyle\sum_{k=0}^{i-3} \{ \hat{\alpha}_{i-k-1} L_k \} + 7L_{i-2} + 2L_{i-1}.
\end{aligned}$$

Now, using Lemma 6($b$), we conclude that $\tilde{h}_{i,0} = -L_i$, as claimed.

Evidently, $\det H(n) = \det H_{n-1}(n)$. The lemma follows now immediately, by expanding the determinant along the last row of $H_{n-1}(n)$. □

We are now in a position to prove the following proposition which is the first main result of this article.

**Proposition 9.** *Let* $[a_{i,j}]_{i,j \geq 0}$ *be the sequence given by the recurrence*

$$a_{i,j} = a_{i,j-1} + a_{i-1,j} - 2(i+j), \qquad i, j \geq 1 \tag{4}$$

*and the initial conditions* $a_{i,0} = \alpha_i$ *and* $a_{0,i} = i + 2$, $i \geq 0$. *Then, we have*

$$\det_{0 \leq i,j \leq n-1} [a_{i,j}] = L_{n-1}. \tag{5}$$

*Proof.* Let $A(n)$ denote the matrix $[a_{i,j}]_{0 \leq i,j \leq n-1}$. First, we claim that

$$A(n) = L(n) \cdot H(n) \cdot U(n), \tag{6}$$

where $L(n) = [L_{i,j}]_{0 \leq i,j < n}$ with

$$L_{i,j} = \begin{cases} 0 & \text{if} \quad i < j \\ \binom{i}{j} & \text{if} \quad i \geq j, \end{cases}$$

(which is called the unipotent lower triangular matrix of order $n$), $U(n) = L(n)^T$ and

$$H(n) = \left[\begin{array}{c|ccccc} \hat{\alpha}_0 & 1 & 0 & 0 & \cdots & 0 \\ \hline \hat{\alpha}_1 & & & & & \\ \hat{\alpha}_2 & & & & & \\ \vdots & & & & T(n-1) & \\ \hat{\alpha}_{n-1} & & & & & \end{array}\right]_{n\times n},$$

where

$$T(n-1) = T_{(-2,-7,\hat{\alpha}_2,\hat{\alpha}_3,\hat{\alpha}_4,\cdots),(-2,-1,0,0,0,\cdots)}(n-1).$$

Evidently, the claimed factorization of $A(n)$ immediately implies that

$$\det A(n) = \det H(n),$$

and we obtain Eq. (5) from Lemma 8.

Note that the entries of $L(n)$ satisfying in the following recurrence

$$L_{i,j} = L_{i-1,j-1} + L_{i-1,j}, \quad 1 \le i, j < n. \tag{7}$$

Similarly, we have

$$U_{i,j} = U_{i-1,j-1} + U_{i,j-1}, \quad 1 \le i, j < n. \tag{8}$$

In what follows, for convenience, we will let $A = A(n)$, $L = L(n)$, $H = H(n)$ and $U = U(n)$. Now, for the proof of the claimed factorization we compute the $(i, j)$-entry of $L \cdot H \cdot U$, that is,

$$(L \cdot H \cdot U)_{i,j} = \sum_{r=0}^{n-1} \sum_{s=0}^{n-1} L_{i,r} H_{r,s} U_{s,j}. \tag{9}$$

In fact, so as to prove the theorem, we should establish

$$\mathrm{R}_0(L \cdot H \cdot U) = \mathrm{R}_0(A) = (2, 3, \ldots, n+1),$$

$$\mathrm{C}_0(L \cdot H \cdot U) = \mathrm{C}_0(A) = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}),$$

and

$$(L \cdot H \cdot U)_{i,j} = (L \cdot H \cdot U)_{i-1,j-1} + (L \cdot H \cdot U)_{i-1,j} - 2(i + j), \tag{10}$$

for $1 \le i, j \le n - 1$.

Let us do the required calculations. First, suppose that $i = 0$. Then, we have

$$\begin{aligned}(L \cdot H \cdot U)_{0,j} &= \sum_{r=0}^{n-1}\sum_{s=0}^{n-1} L_{0,r}H_{r,s}U_{s,j} \\ &= \sum_{s=0}^{n-1} H_{0,s}U_{s,j} \\ &= H_{0,0}U_{0,j} + H_{0,1}U_{1,j} \\ &= 2 \cdot 1 + 1 \cdot j \\ &= j + 2,\end{aligned}$$

and so $\mathrm{R}_0(L \cdot H \cdot U) = \mathrm{R}_0(A) = (2, 3, \ldots, n+1)$.

Next, assume that $j = 0$. In this case, we obtain

$$\begin{aligned}(L \cdot H \cdot U)_{i,0} &= \sum_{r=0}^{n-1}\sum_{s=0}^{n-1} L_{i,r}H_{r,s}U_{s,0} \\ &= \sum_{r=0}^{n-1} L_{i,r}H_{r,0} \\ &= \sum_{r=0}^{n-1} \tbinom{i}{r}\hat{\alpha}_r \\ &= \alpha_i, \qquad \text{(by Eq. (2))}\end{aligned}$$

and hence we have $\mathrm{C}_0(L \cdot H \cdot U) = \mathrm{C}_0(A) = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$.

Finally, we must establish Eq. (10). Now, let us assume that $1 \leq i, j \leq n-1$. In this case we have

$$\begin{aligned}(L \cdot H \cdot U)_{i,j} &= \sum_{r=0}^{n-1}\sum_{s=0}^{n-1} L_{i,r}H_{r,s}U_{s,j} \\ &= \sum_{r=0}^{n-1} L_{i,r}H_{r,0}U_{0,j} + \sum_{r=0}^{n-1}\sum_{s=1}^{n-1} L_{i,r}H_{r,s}U_{s,j}.\end{aligned} \tag{11}$$

Let $\Omega(i,j) := \sum_{r=0}^{n-1}\sum_{s=1}^{n-1} L_{i,r}H_{r,s}U_{s,j}$. Then, using Eq. (8), we obtain

$$\begin{aligned}\Omega(i,j) &= \sum_{r=0}^{n-1}\sum_{s=1}^{n-1} L_{i,r}H_{r,s}\big(U_{s-1,j-1} + U_{s,j-1}\big) \\ &= \sum_{r=0}^{n-1}\sum_{s=1}^{n-1} L_{i,r}H_{r,s}U_{s-1,j-1} + \sum_{r=0}^{n-1}\sum_{s=1}^{n-1} L_{i,r}H_{r,s}U_{s,j-1} \\ &= \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i,r}H_{r,s}U_{s-1,j-1} + (L \cdot H \cdot U)_{i,j-1} \\ &\quad + \sum_{s=1}^{n-1} L_{i,0}H_{0,s}U_{s-1,j-1} - \sum_{r=0}^{n-1} L_{i,r}H_{r,0}U_{0,j-1}.\end{aligned} \tag{12}$$

Again, let $\Theta(i,j) := \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i,r}H_{r,s}U_{s-1,j-1}$. Now, using Eq. (7), we have

$$\begin{aligned}
\Theta(i,j) &= \sum_{r=1}^{n-1}\sum_{s=1}^{n-1}\big(L_{i-1,r-1}+L_{i-1,r}\big)H_{r,s}U_{s-1,j-1}\\
&= \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r-1}H_{r,s}U_{s-1,j-1} + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}H_{r,s}U_{s-1,j-1}\\
&= \sum_{r=2}^{n-1}\sum_{s=2}^{n-1} L_{i-1,r-1}H_{r,s}U_{s-1,j-1} + \sum_{r=1}^{n-1} L_{i-1,r-1}H_{r,1}U_{0,j-1}\\
&\quad + \sum_{s=2}^{n-1} L_{i-1,0}H_{1,s}U_{s-1,j-1} + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}H_{r,s}U_{s-1,j-1}\\
&= \sum_{r=2}^{n-1}\sum_{s=2}^{n-1} L_{i-1,r-1}H_{r,s}U_{s-1,j-1} + \sum_{r=1}^{n-1} L_{i-1,r-1}H_{r,1}U_{0,j-1}\\
&\quad + \sum_{s=2}^{n-1} L_{i-1,0}H_{1,s}U_{s-1,j-1} + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}H_{r,s}\big(U_{s,j}-U_{s,j-1}\big)\\
&\qquad \big(\text{by Eq. (8)}\big)\\
&= \sum_{r=2}^{n-1}\sum_{s=2}^{n-1} L_{i-1,r-1}H_{r-1,s-1}U_{s-1,j-1} + \sum_{r=1}^{n-1} L_{i-1,r-1}H_{r,1}U_{0,j-1}\\
&\quad + \sum_{s=2}^{n-1} L_{i-1,0}H_{1,s}U_{s-1,j-1} + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}H_{r,s}U_{s,j}\\
&\quad - \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}H_{r,s}U_{s,j-1} \qquad \big(\text{by the structure of } H\big)\\
&= \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}H_{r,s}U_{s,j-1} + \sum_{r=1}^{n-1} L_{i-1,r-1}H_{r,1}U_{0,j-1} + \sum_{s=2}^{n-1} L_{i-1,0}H_{1,s}U_{s-1,j-1}\\
&\quad + \sum_{r=1}^{n-1}\sum_{s=0}^{n-1} L_{i-1,r}H_{r,s}U_{s,j} - \sum_{r=1}^{n-1} L_{i-1,r}H_{r,0}U_{0,j} - \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}H_{r,s}U_{s,j-1}\\
&\quad \big(\text{note that } L_{i-1,n-1}=U_{n-1,j-1}=0\big)\\
&= \sum_{r=1}^{n-1} L_{i-1,r-1}H_{r,1}U_{0,j-1} + \sum_{s=2}^{n-1} L_{i-1,0}H_{1,s}U_{s-1,j-1} + \sum_{r=0}^{n-1}\sum_{s=0}^{n-1} L_{i-1,r}H_{r,s}U_{s,j}\\
&\quad - \sum_{s=0}^{n-1} L_{i-1,0}H_{0,s}U_{s,j} - \sum_{r=1}^{n-1} L_{i-1,r}H_{r,0}U_{0,j}\\
&= \sum_{r=1}^{n-1} L_{i-1,r-1}H_{r,1}U_{0,j-1} + \sum_{s=2}^{n-1} L_{i-1,0}H_{1,s}U_{s-1,j-1} + (L\cdot H\cdot U)_{i-1,j}\\
&\quad - \sum_{s=0}^{n-1} L_{i-1,0}H_{0,s}U_{s,j} - \sum_{r=1}^{n-1} L_{i-1,r}H_{r,0}U_{0,j} \quad \big(\text{by Eq. (9)}\big).
\end{aligned}$$

After having substituted this in Eq. (12), we obtain

$$\begin{aligned}\Omega(i,j) &= (L\cdot H\cdot U)_{i,j-1}+(L\cdot H\cdot U)_{i-1,j}\\ &\quad +\sum_{r=1}^{n-1}L_{i-1,r-1}H_{r,1}U_{0,j-1}+\sum_{s=2}^{n-1}L_{i-1,0}H_{1,s}U_{s-1,j-1}\\ &\quad -\sum_{s=0}^{n-1}L_{i-1,0}H_{0,s}U_{s,j}-\sum_{r=1}^{n-1}L_{i-1,r}H_{r,0}U_{0,j}\\ &\quad +\sum_{s=1}^{n-1}L_{i,0}H_{0,s}U_{s-1,j-1}-\sum_{r=0}^{n-1}L_{i,r}H_{r,0}U_{0,j-1}.\end{aligned}$$

Finally, if this is substituted in Eq. (11) and the sums are put together, then we obtain

$$(L\cdot H\cdot U)_{i,j} = (L\cdot T\cdot U)_{i-1,j}+(L\cdot T\cdot U)_{i,j-1}+\Psi(i,j),$$

where

$$\begin{aligned}\Psi(i,j) &= \sum_{r=0}^{n-1}L_{i,r}H_{r,0}U_{0,j}+\sum_{r=1}^{n-1}L_{i-1,r-1}H_{r,1}U_{0,j-1}+\sum_{s=2}^{n-1}L_{i-1,0}H_{1,s}U_{s-1,j-1}\\ &\quad -\sum_{s=0}^{n-1}L_{i-1,0}H_{0,s}U_{s,j}-\sum_{r=1}^{n-1}L_{i-1,r}H_{r,0}U_{0,j}+\sum_{s=1}^{n-1}L_{i,0}H_{0,s}U_{s-1,j-1}\\ &\quad -\sum_{r=0}^{n-1}L_{i,r}H_{r,0}U_{0,j-1}.\end{aligned}$$

But by an easy calculation one can show that

$$\begin{aligned}\sum_{r=0}^{n-1}L_{i,r}H_{r,0}U_{0,j}-\sum_{r=0}^{n-1}L_{i,r}H_{r,0}U_{0,j-1} &= 0\\ \sum_{r=1}^{n-1}L_{i-1,r-1}H_{r,1}U_{0,j-1}-\sum_{r=1}^{n-1}L_{i-1,r}H_{r,0}U_{0,j} &= -2i\\ \sum_{s=2}^{n-1}L_{i-1,0}H_{1,s}U_{s-1,j-1} &= 1-j\\ \sum_{s=0}^{n-1}L_{i-1,0}H_{0,s}U_{s,j} &= 2+j\\ \sum_{s=1}^{n-1}L_{i,0}H_{0,s}U_{s-1,j-1} &= 1\end{aligned}$$

and so

$$\Psi(i,j) = -2(i+j),$$

which completes the proof of proposition. □

## 4. Principal Minors of the Matrix $B(\infty)$

In this section, we consider the following Hessenberg matrix:

$$H(n) = [h_{i,j}]_{0\le i,j<n} = \left[\begin{array}{cc|cccc} \hat\lambda_0 & -1 & 0 & 0 & \cdots & 0 \\ \hat\lambda_1 & 1 & 1 & 0 & \cdots & 0 \\ \hline \hat\lambda_2 & -1 & & & & \\ \hat\lambda_3 & \hat\lambda_2-\hat\lambda_3 & & & & \\ \vdots & & & & T(n-2) & \\ \hat\lambda_{n-1} & \hat\lambda_{n-2}-\hat\lambda_{n-1} & & & & \end{array}\right]_{n\times n},$$

where

$$T(n-2) = T_{(\nu_0,\nu_1,\nu_2,\dots),(-1,1,0,0,0,\dots)}(n-2).$$

The matrix $H(6)$, for instance, is hence given by

$$H(6) = [h_{i,j}]_{0\le i,j\le 5} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ -2 & -1 & -1 & 1 & 0 & 0 \\ 6 & -8 & -17 & -1 & 1 & 0 \\ 32 & -26 & -60 & -17 & -1 & 1 \\ 104 & -72 & -170 & -60 & -17 & -1 \end{bmatrix}_{6\times 6}.$$

**Lemma 10.** *With the notation defined above, we have* $\det H(n) = L_{n-1}$.

*Proof.* First, we apply the following row operations:

$$\begin{aligned} H_1(n) &= \Big(\textstyle\prod_{i=1}^{n-1} O_{i,0}(h_{i,1})\Big) H(n), \\ H_2(n) &= \Big(\textstyle\prod_{i=2}^{n-1} O_{i,1}(-h_{i,2})\Big) H_1(n), \\ H_3(n) &= \Big(\textstyle\prod_{i=3}^{n-1} O_{i,2}(-h_{i,3})\Big) H_2(n), \\ &\vdots \\ H_{n-1}(n) &= \Big(\textstyle\prod_{i=n-1}^{n-1} O_{i,n-2}(-h_{i,n-1})\Big) H_{n-2}(n). \end{aligned}$$

In fact, step by step, the columns are "emptied" until finally the following matrix

$$H_{n-1}(n) = \begin{bmatrix} \tilde h_{0,0} & -1 & 0 & 0 & 0 & \cdots & 0 \\ \tilde h_{1,0} & 0 & 1 & 0 & 0 & \cdots & 0 \\ \tilde h_{2,0} & 0 & 0 & 1 & 0 & \cdots & 0 \\ \tilde h_{3,0} & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde h_{n-2,0} & 0 & 0 & 0 & 0 & \cdots & 1 \\ \tilde h_{n-1,0} & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n}$$

is obtained. Evidently $\tilde{h}_{0,0} = \hat{\lambda}_0 = 2 = L_0$. Now, we claim that

$$\tilde{h}_{i,0} = (-1)^{i-1} L_i, \qquad 1 \le i \le n-1. \tag{13}$$

We proceed by strong induction on $i$. The cases $i = 1, 2$ and 3 are trivial:

$$\begin{aligned}
\tilde{h}_{1,0} &= h_{1,0} + h_{1,1} \cdot 2 \\
&= -1 + 1 \cdot 2 = 1 = L_1, \\
\tilde{h}_{2,0} &= h_{2,0} + h_{2,1} \cdot 2 - h_{2,2} \cdot (1) \\
&= -2 + (-1) \cdot 2 + (-1) \cdot (1) = -3 = -L_2, \\
\tilde{h}_{3,0} &= h_{3,0} + h_{3,1} \cdot 2 - h_{3,2} \cdot 1 - h_{3,3} \cdot (-3) \\
&= 6 + (-8) \cdot 2 - (-17) \cdot (1) - (-1) \cdot (-3) = 4 = L_3, \\
\tilde{h}_{4,0} &= h_{4,0} + h_{4,1} \cdot 2 - h_{4,2} \cdot 1 + h_{4,3} \cdot (-3) - h_{4,4} \cdot 4 \\
&= 32 + (-26) \cdot 2 - (-60) \cdot 1 - (-17) \cdot (-3) - (-1) \cdot 4 = -7 = -L_4.
\end{aligned}$$

Assume now that $i \ge 5$ and Eq. (13) is true for $1, 2, \ldots, i-1$, that is, $\tilde{h}_{k,0} = -L_k$, $1 \le k \le i-1$. We now prove that Eq. (13) is also true for $i$. In fact, by the applied row operations and induction hypothesis, we have

$$\begin{aligned}
\tilde{h}_{i,0} &= -h_{i,i}(-1)^{i-2} L_{i-1} - h_{i,i-1}(-1)^{i-3} L_{i-2} - \sum_{k=2}^{i-2} h_{i,i-k}(-1)^{i-k-2} L_{i-(k+1)} \\
&\quad h_{i,1} \cdot 2 + h_{i,0} \\
&= (-\nu_0)(-1)^{i-2} L_{i-1} - \nu_1(-1)^{i-3} L_{i-2} - \sum_{k=2}^{i-2} \nu_k (-1)^{i-k-2} L_{i-(k+1)} \\
&\quad + (\hat{\lambda}_{i-1} - \hat{\lambda}_i) \cdot 2 + \hat{\lambda}_i \\
&= (-1)^{i-2} L_{i-1} + 17(-1)^{i-3} L_{i-2} + (-1)^{i-2} \sum_{k=2}^{i-2} 5 \cdot 2^{k-2}(5k+2)(-1)^k L_{i-(k+1)} \\
&\quad 2 \cdot \hat{\lambda}_{i-1} - \hat{\lambda}_i \qquad (\text{ by definition of } \nu_k) \\
&= (-1)^{i-2} L_{i-1} + 17(-1)^{i-3} L_{i-2} + 5(-1)^{i-2} \sum_{k=2}^{i-2} 2^{k-2}(5k+2)(-1)^k L_{i-(k+1)} \\
&\quad -5 \cdot 2^{i-2} \qquad (\text{by Lemma 6(c)}) \\
&= (-1)^{i-2} \{ L_{i-1} - 17 L_{i-2} + 5 \sum_{k=0}^{i-4} (-2)^k (5k+12) L_{i-(k+3)} - 5 \cdot (-2)^{i-2} \} \\
&= (-1)^{i-2} \big( L_{i-1} - L_{i+1} \big) \qquad (\text{by Lemma 7}) \\
&= (-1)^{i-1} L_i.
\end{aligned}$$

as claimed.

Evidently, $\det H(n) = \det H_{n-1}(n)$. The lemma follows now immediately, by expanding the determinant along the last row of $H_{n-1}(n)$. □

We can now prove the following proposition which is the second main result of this article.

**Proposition 11.** *Let $[a_{i,j}]_{i,j\geq 0}$ be the sequence given by the recurrence*

$$a_{i,j} = a_{i-1,j-1} + a_{i-1,j} - 2i, \qquad i,j \geq 1 \tag{14}$$

*and the initial conditions $a_{i,0} = \lambda_i$ and $a_{0,i} = \mu_i$, $i \geq 0$. Then, we have*

$$\det_{0\leq i,j<n} [a_{i,j}] = L_{n-1}. \tag{15}$$

*Proof.* Let $A(n)$ denote the matrix $[a_{i,j}]_{0\leq i,j<n}$. First, we claim that

$$A(n) = L(n) \cdot B(n) \cdot U(n), \tag{16}$$

where $L(n) = [L_{i,j}]_{0\leq i,j<n}$ with

$$L_{i,j} = \begin{cases} 0 & \text{if} \quad i < j \\ \binom{i}{j} & \text{if} \quad i \geq j, \end{cases}$$

$U(n) = L(n)^T$ and $B(n) = [B_{i,j}]_{0\leq i,j<n}$ with $B_{i,0} = \hat{\lambda}_i$, $B_{0,j} = \hat{\mu}_j$, $B_{1,1} = 1$, $B_{2,1} = -1$ and

$$B_{i,j} = B_{i-1,j-1} - B_{i,j-1}, \quad 1 \leq i,j < n, \quad (i,j) \neq (1,1),(2,1). \tag{17}$$

The matrix $B(6)$, for instance, is given by

$$B(6) = [B_{i,j}]_{0\leq i,j<6} = \begin{bmatrix} \hat{\lambda}_0 = \hat{\mu}_0 & \hat{\mu}_1 & \hat{\mu}_2 & \hat{\mu}_3 & \hat{\mu}_4 \\ \hat{\lambda}_1 & 1 & -2 & 5 & -14 \\ \hat{\lambda}_2 & -1 & 2 & -4 & 9 \\ \hat{\lambda}_3 & -8 & 7 & -5 & 1 \\ \hat{\lambda}_4 & -26 & 18 & -11 & 6 \\ \hat{\lambda}_5 & -72 & 46 & -28 & 17 \end{bmatrix}.$$

In what follows, for convenience, we will let $A = A(n)$, $L = L(n)$, $B = B(n)$ and $U = U(n)$. Now, for the proof of the claimed factorization we compute the $(i,j)$-entry of $L \cdot B \cdot U$, that is,

$$(L \cdot B \cdot U)_{i,j} = \sum_{r=0}^{n-1} \sum_{s=0}^{n-1} L_{i,r} B_{r,s} U_{s,j}. \tag{18}$$

In fact, so as to prove the theorem, we should establish

$$\mathrm{R}_0(L \cdot B \cdot U) = \mathrm{R}_0(A) = (\mu_0, \mu_1, \ldots, \mu_{n-1}),$$

$$\mathrm{C}_0(L \cdot B \cdot U) = \mathrm{C}_0(A) = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1}),$$

and

$$(L \cdot B \cdot U)_{i,j} = (L \cdot B \cdot U)_{i-1,j-1} + (L \cdot B \cdot U)_{i-1,j} - 2i, \tag{19}$$

for $1 \leq i, j < n$.

Let us do the required calculations. First, suppose that $i = 0$. Then, we have

$$(L{\cdot}B{\cdot}U)_{0,j} = \sum_{r=0}^{n-1}\sum_{s=0}^{n-1} L_{0,r}B_{r,s}U_{s,j} = \sum_{s=0}^{n-1} B_{0,s}U_{s,j} = \sum_{s=0}^{n-1}\binom{j}{s}\hat{\mu}_s = \mu_j, \quad \text{(by Eq. (2))}$$

and so $\mathrm{R}_0(L \cdot B \cdot U) = \mathrm{R}_0(A) = (\mu_0, \mu_1, \ldots, \mu_{n-1})$.

Next, assume that $j = 0$. In this case, we obtain

$$(L{\cdot}B{\cdot}U)_{i,0} = \sum_{r=0}^{n-1}\sum_{s=0}^{n-1} L_{i,r}B_{r,s}U_{s,0} = \sum_{r=0}^{n-1} L_{i,r}B_{r,0} = \sum_{r=0}^{n-1}\binom{i}{r}\hat{\lambda}_r = \lambda_i, \quad \text{(by Eq. (2))}$$

and hence we have $\mathrm{C}_0(L \cdot B \cdot U) = \mathrm{C}_0(A) = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$.

Finally, we must establish Eq. (19). Now, let us assume that $1 \leq i, j \leq n-1$. In this case we have

$$(L \cdot B \cdot U)_{i,j} = \sum_{r=0}^{n-1}\sum_{s=0}^{n-1} L_{i,r}B_{r,s}U_{s,j} = \sum_{r=0}^{n-1} L_{i,r}B_{r,0} + \sum_{r=0}^{n-1}\sum_{s=1}^{n-1} L_{i,r}B_{r,s}U_{s,j}. \tag{20}$$

Let $\Omega(i,j) := \sum_{r=0}^{n-1}\sum_{s=1}^{n-1} L_{i,r}B_{r,s}U_{s,j}$. Then, using Eqs. (8) and (18), we obtain

$$\begin{aligned}
\Omega(i,j) &= \sum_{r=0}^{n-1}\sum_{s=1}^{n-1} L_{i,r}B_{r,s}\big(U_{s-1,j-1} + U_{s,j-1}\big) \\
&= \sum_{r=0}^{n-1}\sum_{s=1}^{n-1} L_{i,r}B_{r,s}U_{s-1,j-1} + \sum_{r=0}^{n-1}\sum_{s=1}^{n-1} L_{i,r}B_{r,s}U_{s,j-1} \\
&= \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i,r}B_{r,s}U_{s-1,j-1} + \sum_{s=1}^{n-1} B_{0,s}U_{s-1,j-1} \\
&\qquad + (L \cdot B \cdot U)_{i,j-1} - \sum_{r=0}^{n-1} L_{i,r}B_{r,0}.
\end{aligned} \tag{21}$$

Again, let $\Theta(i,j) := \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i,r}B_{r,s}U_{s-1,j-1}$. Now, using Eq. (7), we have

$$\begin{aligned}
\Theta(i,j) &= \sum_{r=1}^{n-1}\sum_{s=1}^{n-1}\big(L_{i-1,r-1}+L_{i-1,r}\big)B_{r,s}U_{s-1,j-1} \\
&= \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r-1}B_{r,s}U_{s-1,j-1} + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}B_{r,s}U_{s-1,j-1} \\
&= \sum_{r=2}^{n-1}\sum_{s=2}^{n-1} L_{i-1,r-1}B_{r,s}U_{s-1,j-1} + \sum_{r=1}^{n-1} L_{i-1,r-1}B_{r,1} \\
&\quad + \sum_{s=2}^{n-1} B_{1,s}U_{s-1,j-1} + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}B_{r,s}\big(U_{s,j}-U_{s,j-1}\big)\ \big(\text{by Eq. (8)}\big) \\
&= \sum_{r=2}^{n-1}\sum_{s=2}^{n-1} L_{i-1,r-1}\left(B_{r-1,s-1}-B_{r,s-1}\right)U_{s-1,j-1} \\
&\quad + \sum_{r=1}^{n-1} L_{i-1,r-1}B_{r,1} + \sum_{s=2}^{n-1} B_{1,s}U_{s-1,j-1} \\
&\quad + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}B_{r,s}U_{s,j} - \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}B_{r,s}U_{s,j-1} \qquad (\text{by Eq. (17)}) \\
&= \sum_{r=2}^{n-1}\sum_{s=2}^{n-1} L_{i-1,r-1}B_{r-1,s-1}U_{s-1,j-1} - \sum_{r=2}^{n-1}\sum_{s=2}^{n-1} L_{i-1,r-1}B_{r,s-1}U_{s-1,j-1} \\
&\quad + \sum_{r=1}^{n-1} L_{i-1,r-1}B_{r,1} + \sum_{s=2}^{n-1} B_{1,s}U_{s-1,j-1} \\
&\quad + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}B_{r,s}U_{s,j} - \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}B_{r,s}U_{s,j-1} \\
&= \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}B_{r,s}U_{s,j-1} - \sum_{r=2}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r-1}B_{r,s}U_{s,j-1} \\
&\quad + \sum_{r=1}^{n-1} L_{i-1,r-1}B_{r,1} + \sum_{s=2}^{n-1} B_{1,s}U_{s-1,j-1} \\
&\quad + \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}B_{r,s}U_{s,j} - \sum_{r=1}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}B_{r,s}U_{s,j-1} \\
&\quad \big(\text{note that } L_{i-1,n-1} = U_{n-1,j-1} = 0\big) \\
&= -\sum_{r=2}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r-1}B_{r,s}U_{s,j-1} + \sum_{r=1}^{n-1} L_{i-1,r-1}B_{r,1} + \sum_{s=2}^{n-1} B_{1,s}U_{s-1,j-1} \\
&\quad + \sum_{r=0}^{n-1}\sum_{s=0}^{n-1} L_{i-1,r}B_{r,s}U_{s,j} - \sum_{s=0}^{n-1} B_{0,s}U_{s,j} - \sum_{r=1}^{n-1} L_{i-1,r}B_{r,0} \\
&= -\sum_{r=2}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r-1}B_{r,s}U_{s,j-1} + \sum_{r=1}^{n-1} L_{i-1,r-1}B_{r,1} + \sum_{s=2}^{n-1} B_{1,s}U_{s-1,j-1} \\
&\quad + (L\cdot B\cdot U)_{i-1,j} - \sum_{s=0}^{n-1} B_{0,s}U_{s,j} - \sum_{r=1}^{n-1} L_{i-1,r}B_{r,0}\ \ \big(\text{by Eq. (18)}\big).
\end{aligned}$$

Now, for convenience, we put

$$\Gamma(i,j) = \sum_{r=2}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r-1}B_{r,s}U_{s,j-1}.$$

But then

$$\begin{aligned}
\Gamma(i,j) &= \sum_{r=2}^{n-1}\sum_{s=1}^{n-1}(L_{i,r}-L_{i-1,r})B_{r,s}U_{s,j-1} \quad \text{(by Eq. (7))}\\
&= \sum_{r=2}^{n-1}\sum_{s=1}^{n-1} L_{i,r}B_{r,s}U_{s,j-1} - \sum_{r=2}^{n-1}\sum_{s=1}^{n-1} L_{i-1,r}B_{r,s}U_{s,j-1}\\
&= \sum_{r=0}^{n-1}\sum_{s=0}^{n-1} L_{i,r}B_{r,s}U_{s,j-1} - \sum_{r=0}^{1}\sum_{s=0}^{n-1} L_{i,r}B_{r,s}U_{s,j-1} - \sum_{r=2}^{n-1} L_{i,r}B_{r,0}\\
&\quad - \sum_{r=0}^{n-1}\sum_{s=0}^{n-1} L_{i-1,r}B_{r,s}U_{s,j-1} + \sum_{r=0}^{1}\sum_{s=0}^{n-1} L_{i-1,r}B_{r,s}U_{s,j-1} + \sum_{r=2}^{n-1} L_{i-1,r}B_{r,0}\\
&= (L\cdot B\cdot U)_{i,j-1} - \sum_{s=0}^{n-1} L_{i,1}B_{1,s}U_{s,j-1} - \sum_{r=2}^{n-1} L_{i,r}B_{r,0} - (L\cdot B\cdot U)_{i-1,j-1}\\
&\quad + \sum_{s=0}^{n-1} L_{i-1,1}B_{1,s}U_{s,j-1} + \sum_{r=2}^{n-1} L_{i-1,r}B_{r,0} \quad \big(\text{by Eq. (18)}\big).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\Theta(i,j) = {} & -(L\cdot B\cdot U)_{i,j-1} + \sum_{s=0}^{n-1} L_{i,1}B_{1,s}U_{s,j-1} + \sum_{r=2}^{n-1} L_{i,r}B_{r,0} + (L\cdot B\cdot U)_{i-1,j-1}\\
& - \sum_{s=0}^{n-1} L_{i-1,1}B_{1,s}U_{s,j-1} - \sum_{r=2}^{n-1} L_{i-1,r}B_{r,0} + \sum_{r=1}^{n-1} L_{i-1,r-1}B_{r,1}\\
& + \sum_{s=2}^{n-1} B_{1,s}U_{s-1,j-1} + (L\cdot B\cdot U)_{i-1,j} - \sum_{s=0}^{n-1} B_{0,s}U_{s,j} - \sum_{r=1}^{n-1} L_{i-1,r}B_{r,0}
\end{aligned}$$

After having substituted this in Eq. (21) and the sums are put together, then we obtain

$$\Omega(i,j) = (L\cdot B\cdot U)_{i-1,j-1} + (L\cdot B\cdot U)_{i-1,j} + \Psi(i,j)$$

where

$$\begin{aligned}
t\Psi(i,j) = {} & \sum_{s=0}^{n-1} L_{i,1}B_{1,s}U_{s,j-1} + \sum_{r=2}^{n-1} L_{i,r}B_{r,0} - \sum_{s=0}^{n-1} L_{i-1,1}B_{1,s}U_{s,j-1} - \sum_{r=2}^{n-1} L_{i-1,r}B_{r,0}\\
& + \sum_{r=1}^{n-1} L_{i-1,r-1}B_{r,1} + \sum_{s=2}^{n-1} B_{1,s}U_{s-1,j-1} - \sum_{s=0}^{n-1} B_{0,s}U_{s,j} - \sum_{r=1}^{n-1} L_{i-1,r}B_{r,0}\\
& + \sum_{s=1}^{n-1} B_{0,s}U_{s-1,j-1} - \sum_{r=0}^{n-1} L_{i,r}B_{r,0}.
\end{aligned}$$

Finally, if this is substituted in Eq. (20), then we obtain

$$(L\cdot B\cdot U)_{i,j} \quad = \quad \sum_{r=0}^{n-1} L_{i,r}B_{r,0} + (L\cdot B\cdot U)_{i-1,j-1} + (L\cdot B\cdot U)_{i-1,j} + \Psi(i,j),$$

In the sequel, we will show that

$$\sum_{r=0}^{n-1} L_{i,r}B_{r,0} + \Psi(i,j) = -2i.$$

In fact, by easy calculations one can show that

$$\begin{aligned}
&\sum_{r=2}^{n-1} L_{i,r}B_{r,0} - \sum_{r=2}^{n-1} L_{i-1,r}B_{r,0} + \sum_{r=1}^{n-1} L_{i-1,r-1}B_{r,1} - \sum_{r=1}^{n-1} L_{i-1,r}B_{r,0}\\
&= \sum_{r=2}^{n-1}\{L_{i,r} - L_{i-1,r}\}B_{r,0} + \sum_{r=1}^{n-1} L_{i-1,r-1}B_{r,1} - \sum_{r=2}^{n-1} L_{i-1,r-1}B_{r-1,0}\\
&= \sum_{r=2}^{n-1} L_{i-1,r-1}B_{r,0} + L_{i-1,0}B_{1,1} + \sum_{r=2}^{n-1} L_{i-1,r-1}\{B_{r,1} - B_{r-1,0}\}\\
&\quad \text{(by Eq. (7))}\\
&= L_{i-1,1}B_{2,0} + \sum_{r=3}^{n-1} L_{i-1,r-1}B_{r,0} + 1 + L_{i-1,1}\{B_{2,1} - B_{1,0}\}\\
&\quad - \sum_{r=3}^{n-1} L_{i-1,r-1}B_{r,0} \quad \text{( by Eq. (17))}\\
&= -2(i-1) + 1 + (i-1)\{-1-(-1)\} = -2i+3.
\end{aligned}$$

And also

$$\begin{aligned}
&\sum_{s=0}^{n-1} L_{i,1}B_{1,s}U_{s,j-1} - \sum_{s=0}^{n-1} L_{i-1,1}B_{1,s}U_{s,j-1} + \sum_{s=2}^{n-1} B_{1,s}U_{s-1,j-1}\\
&- \sum_{s=0}^{n-1} B_{0,s}U_{s,j} + \sum_{s=1}^{n-1} B_{0,s}U_{s-1,j-1}\\
&= \sum_{s=0}^{n-1}\{L_{i,1} - L_{i-1,1}\}B_{1,s}U_{s,j-1} + \sum_{s=1}^{n-1} B_{1,s+1}U_{s,j-1} - B_{0,0}U_{0,j}\\
&+ \sum_{s=1}^{n-1} B_{0,s}\{U_{s-1,j-1} - U_{s,j}\}\\
&= \sum_{s=0}^{n-1} B_{1,s}U_{s,j-1} + \sum_{s=1}^{n-1} B_{1,s+1}U_{s,j-1} - (2)(1) - \sum_{s=1}^{n-1} B_{0,s}U_{s,j-1}\\
&\quad \text{(by Eqs. (7) and (8))}\\
&= B_{1,0}U_{0,j-1} + \sum_{s=1}^{n-1}\{B_{1,s} + B_{1,s+1}\}U_{s,j-1} - 2 - \sum_{s=1}^{n-1} B_{0,s}U_{s,j-1}\\
&= (-1)(1) + \sum_{s=1}^{n-1} B_{0,s}U_{s,j-1} - 2 - \sum_{s=1}^{n-1} B_{0,s}U_{s,j-1} = -3.
\end{aligned}$$

Hence, we conclude that

$$\sum_{r=0}^{n-1} L_{i,r}B_{r,0} + \Psi(i,j) = \sum_{r=0}^{n-1} L_{i,r}B_{r,0} + (-2i+3) - 3 - \sum_{r=0}^{n-1} L_{i,r}B_{r,0} = -2i,$$

as desired.

Evidently, the claimed factorization of $A(n)$ immediately implies that

$$\det A(n) = \det B(n).$$

Finally, we apply the following column operations:

$$\begin{aligned}
B_1(n) &= B(n) \prod_{i=1}^{n-2} O'_{n-i,n-i-1}(3), \\
B_2(n) &= B_1(n) \prod_{i=1}^{n-3} O'_{n-i,n-i-1}(1), \\
B_3(n) &= B_2(n) \prod_{i=1}^{n-4} O'_{n-i,n-i-1}(1), \\
&\vdots \\
B_{n-2}(n) &= B_{n-3}(n) \prod_{i=1}^{1} O'_{n-i,n-i-1}(1).
\end{aligned}$$

Now, it is easy to see that $B_{n-2}(n) = H(n)$, which has already been introduced. Moreover, since

$$\det B(n) = \det B_1(n) = \ldots = \det B_{n-2}(n) = \det H(n),$$

by Lemma 10 we conclude that $\det B(n) = L_{n-1}$, which proves the proposition. □

*Proof of Theorem* 4. It is proved by Proposition 9 and Proposition 11. □

## 5. Conclusion

In this paper we have found two infinite-dimensional matrices whose entries are recursively defined and the sequence of their principal minors form the Lucas sequence. These matrices are generated by nonhomogeneous recurrence relations. It is not difficult to find other examples of matrices with the same property. The following question, however, is not so obvious.

**Problem.** Is there an infinite family of infinite-dimensional matrices whose entries are recursively defined and the sequence of their principal minors form the Lucas sequence?

**Acknowledgements** The authors are thankful to the referee for the careful reading of ant earlier version of this manuscript and its comments that improved the article. This article was prepared while the first author was visiting the Institute for Research in Fundamental Sciences (IPM). It would be a pleasure to thank IPM for its hospitality and facilities.

## References

[1] R. Bacher. Determinants of matrices related to the Pascal triangle. *J. Theorie Nombres Bordeaux,* 14 (2002), 19-41.

[2] N. D. Cahill and D. A. Narayan. Fibonacci and Lucas numbers as tridiagonal matrix determinants. *Fibonacci Quart.,* 42(3)(2004), 216-221.

[3] G. S. Cheon, S. G. Hwang, S. H. Rim and S. Z. Song, *Matrices determined by a linear recurrence relation among entries*, Linear Algebra Appl., 373 (2003), 89-99.

[4] A. R. Moghaddamfar, S. M. H. Pooya, S. Navid Salehy and S. Nima Salehy. Fibonacci and Lucas sequences as the principal minors of some infinite matrices. *Journal of Algebra and Its Applications,* 8(6)(2009), 869-883.

[5] A. R. Moghaddamfar, K. Moghaddamfar and H. Tajbakhsh. New families of integer matrices whose leading principal minors form some well-known sequences", *Electronic Journal of Linear Algebra*, 22(2011).

[6] Y. Yang and M. Leonard. Evaluating determinants of convolution-like matrices via generating functions. *International Journal of Information and Systems Sciences*, 3(4)(2007), 569-580.

[7] H. Zakrajšek and M. Petkovšek, Pascal-like determinants are recursive, *Adv. Appl. Math.*, 33(3) (2004), 431-450.