A RELATION BETWEEN TRIANGULAR NUMBERS AND PRIME NUMBERS

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Abstract

We study a relation between factorials and their additive analog, the triangular numbers. We show that there is a positive integer k such that $n! = 2^kT$ where T is a product of triangular numbers. We discuss the primality of $T \pm 1$ and the primality of $|T - p|$ where p is either the smallest prime greater than T or the greatest prime less than T.

1. Introduction

There is a natural relation between triangular numbers and factorials. Triangular numbers are the additive analogs of factorials. We show that there is a positive integer k such that $n! = 2^kT$ where T is a product of triangular numbers. The number of factors of T depends on the parity of n.

There are many open questions about the relationship between prime numbers and factorials. For example, are there infinitely many primes of the form $n! \pm 1$? Erdös [4] asked if there are infinitely many primes p for which $p - k!$ is composite for each k such that $1 \leq k! \leq p$. Fortune's conjecture [5] asks whether the product of the first n consecutive prime numbers plus or minus one is a prime. Since T is a product of triangular numbers, it is natural to ask whether $T \pm 1$ is a prime. It is also natural to ask whether $|T - p|$ is a prime number, where p is either the smallest prime greater than T or the greatest prime less than T.

In this paper we prove that there are infinitely many cases for which $T \pm 1$ is not a prime. We also give both numerical and theoretical evidence for the primality of

$|T-p|$ where $p \neq T \pm 1$.

We now formally state the question. We denote by t_n the n^{th} *triangular* number where $n \geq 0$ with $t_0 = 0$ and $t_n = t_{n-1} + n$. We define $T(k) = \prod_{i=1}^{k} t_{2i-1}$ and $T'(k) = t_5 \prod_{i=3}^{k} t_{2i}$ for $k > 2$ an integer. If there is no ambiguity, we use T to mean either $T(k)$ or $T'(k)$.

Question 1. *If T is either $T(k)$ or $T'(k)$, and p is either the smallest prime greater than $T+1$ or the greatest prime less than $T-1$, then*

(1) *are there infinitely many primes of the form $T \pm 1$?*

(2) *Is $|T-p|$ a prime number?*

2. Preliminaries

In this section we introduce some notation. Throughout the paper we use k to represent a positive integer. We prove that $n! = 2^k \prod_{i=0}^{k-1}(t_k - t_i)$ if $n = 2k$ and $n! = 2^k \prod_{i=0}^{k-1}(t_{k+1} - t_i)$ if $n = 2k+1$. Proposition 2, part (2) is in [2, 3]. Proposition 2, part (1) is a natural relation. Therefore, we believe that it is known, but unfortunately we have not found this property in the mathematics literature.

Proposition 2. *If n is a positive integer, then*

(1) $n! = \begin{cases} 2^k T(k) & \text{if } n = 2k \\ 2^{k+1} T'(k) & \text{if } n = 2k+1. \end{cases}$

(2) $T(k) = \prod_{i=0}^{k-1}(t_k - t_i)$.

(3) $2T'(k) = \prod_{i=0}^{k-1}(t_{k+1} - t_i)$.

Proof. We prove part (1) for $n = 2k$, the other case is similar.

$$\begin{aligned} 2^k T(k) &= 2^k \cdot t_1 \cdot t_3 \, \ldots \, t_{2k-1} \\ &= 2^k \cdot \frac{1 \cdot 2}{2} \cdot \frac{3 \cdot 4}{2} \cdots \frac{(2k-1) \cdot 2k}{2} \\ &= (2k)! = n! \, . \end{aligned}$$

We now prove part (2). We suppose that $n = 2k$. From part (1) we know that $n! = 2^k T(k)$. So,

$$\begin{aligned} 2^k T(k) &= 1 \cdot 2 \cdot 3 \cdot 4 \, \ldots \, k \cdot (k+1) \, \ldots \, (2k-3) \cdot (2k-2) \cdot (2k-1) \cdot 2k \\ &= [1 \cdot 2k] \cdot [2 \cdot (2k-1)] \cdot [3 \cdot (2k-2)] \, \ldots \, [k \cdot (k+1)] \\ &= [k \cdot (k+1)] \, \ldots \, [3 \cdot (2k-2)] \cdot [2 \cdot (2k-1)] \cdot [1 \cdot (2k)] \\ &= \prod_{i=0}^{k-1} (k-i) \cdot (k+i+1) \end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=0}^{k-1} (k^2 + k - i^2 - i) \\
&= \prod_{i=0}^{k-1} \left(k(k+1) - i(i+1)\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
T(k) &= \frac{1}{2^k} \prod_{i=0}^{k-1} \left(k(k+1) - i(i+1)\right) \\
&= \prod_{i=0}^{k-1} \left(\frac{k(k+1)}{2} - \frac{i(i+1)}{2}\right) \\
&= \prod_{i=0}^{k-1} (t_k - t_i).
\end{aligned}
$$

We prove part (3). We suppose that $n = 2k + 1$. It is easy to see that $2T'(k) = \dfrac{T(k+1)}{(k+1)}$. Thus,

$$
2T'(k) = \frac{T(k+1)}{k+1} = \frac{1}{k+1} \prod_{i=0}^{k} (t_{k+1} - t_i) = \prod_{i=0}^{k-1} (t_{k+1} - t_i). \qquad \square
$$

Notice that $2T'(k) = \prod_{i=1}^{k} t_{2i}$. Therefore, we can ask Question 1 replacing $T'(k)$ by $2T'(k)$. Numerical calculations show that Question 1, part (2) is true for $2T'(k)$ with $k \leq 1000$. We have found that there are only 9 prime numbers of the form $2T'(k) - 1$ for $k \leq 1000$ and 12 prime numbers of the form $2T'(k) + 1$ for $k \leq 1000$.

Since $t_k = \binom{k+1}{2}$, Proposition 2, part (1) can be restated as

$$
n! = 2^k \prod_{i=1}^{k} \binom{2i}{2} = 2^k \prod_{i=0}^{k-1} \left(\binom{k+1}{2} - \binom{i+1}{2} \right) \text{ if } n = 2k
$$

and

$$
n! = 2^k \prod_{i=1}^{k} \binom{2i+1}{2} = 2^k \prod_{i=0}^{k-1} \left(\binom{k+2}{2} - \binom{i+1}{2} \right) \text{ if } n = 2k+1.
$$

We use Theorem 3 to prove Propositions 6 and 7. These propositions give upper bounds for the number of primes in an interval.

Let f be a real function and g be a positive function. We use $f \ll g$ to mean that there is a constant $c > 0$ such that $|f(x)| \le cg(x)$ for all x in the domain of f. This is also denoted by $f = O(g)$. For the following two theorems q is a prime. If N is a positive even integer, we write $\pi_N(x)$ to denote the number of primes b up to x such that $N + b$ is also prime, and, we write $r(N)$ to denote the number of representations of N as the sum of two primes.

Theorem 3. [6, Theorems 7.2 and 7.3] *If N is a positive even integer, then*

(1) $\pi_N(x) \ll \dfrac{x}{(\ln x)^2} \displaystyle\prod_{q|N} \left(1 + \frac{1}{q}\right)$.

(2) $r(N) \ll \dfrac{N}{(\ln N)^2} \displaystyle\prod_{q|N} \left(1 + \frac{1}{q}\right)$.

3. Evidences for Primality of $|T - p|$

In this section we provide strong evidence that Question 1, part (2) is probably true. We use the prime number theorem to give a first approach for the validity of this question, and construct several examples that show that $|T - l|$ is a prime where l is a prime number. We found that if l is in a specific interval, then $|T - l|$ is a prime (we give a detailed description of this interval below.) We give an upper bound for the number of primes in this interval.

Propositions 4 and 6 give a theoretical support to believe that the facts shown in the following examples may be true in general. In Section 5 there are 2 tables that show some primes of the form $Q - T$ and $T - q$, where Q is the smallest prime greater than T and q greatest prime less than T. We have observed that Q is in the interval $(T, T + p^2)$ where p is either the smallest prime greater than $2k$ if $T = T(k)$ or is the smallest prime greater than $2k+1$ if $T = T'(k)$. From Table 4 we can verify that either $p \le Q - T < p^2$ or $Q - T = 1$. From Table 1 we can verify that either $T - p^2 < q \le T - p$ or $T - q = 1$. Using a computer program the authors verified that this fact is also true for all $k \le 10^3$. Since every number in $(T + 1, T + p)$ is composite, we are going to analyze the behavior of Q in $[T + p, T + p^2)$ and $Q = T + 1$. In Proposition 4 we show that if $T + p \le Q < T + p^2$, then it proves Question 1, part (2).

We first give a heuristic argument to show that if $Q \ne T + 1$, then $T + p \le Q < T + p^2$. It is known from prime number theorem that if q is the next prime greater than a number $m + 1$, then q is near $m + \ln m$. So, Q is near $T + \ln T$. If p is the next prime greater than n, then

$$\ln(T) = \ln\left(\frac{n!}{2^k}\right) \sim n \ln n - n - k \ln 2 + 1 < p^2.$$

Therefore, if $Q \neq T+1$ and $Q < T + \ln T$, then $T + p \leq Q < T + p^2$.

We now give some examples that show that there are several primes l that satisfy $T+p \leq l < T+p^2$. Proposition 6 gives a general upper bound for the total number of primes of the form $T+b$ in $[T+p, T+p^2)$ where b is a prime.

If $k = 3$, then $T(3) = 90$, $2k = 6$ and $p = 7$. So, $p^2 = 49$. These give rise to the interval $[T+p, T+p^2) = [97, 139)$. In this interval there are 9 primes. Thus, $Q-T(3)$ is prime where Q is a prime with $97 \leq Q < 139$. Indeed, all possible outcomes for $Q-T(3)$ are: $97-90 = 7$; $101-90 = 11$; $103-90 = 13$; $107-90 = 17$; $109-90 = 19$; $113-90 = 23$; $127-90 = 37$; $131-90 = 41$; $137-90 = 47$. Note that 139 is a prime, but $139 - 90 = 49 = 7^2$.

For the next example we need $k > 3$. If we take $k = 4$, then $T'(4) = 11340$, $2k+1 = 9$ and $p = 11$. So, these give rise to the interval $[T+p, T+p^2) = [11351, 11461)$. For every prime Q in $[11351, 11461)$, it holds that $Q - T'(4)$ is a prime. That is, $11351 - 11340 = 11$; $11353 - 11340 = 13$; $11369 - 11340 = 29$; $11383-11340 = 43$; $11393-11340 = 53$; $11399-11340 = 59$; $11411-11340 = 71$; $11423-11340 = 83$; $11437-11340 = 97$; $11443-11340 = 103$; $11447-11340 = 107$.

We have observed that $Q - T$ is also a prime for some primes Q greater than $T + p^2$. That is, if there is no prime number between T and $T + p^2$, this does not automatically mean that Question 1, part (2) will fail. For example, if $k = 5$, then $T(5) = 113400$, $2k = 10$ and $p = 11 > 2k$. So, $p^2 = 121$. These give rise to the interval $[T+p, T+p^2) = [113411, 113521)$. The number $T(5)+121 = 113400+121 = 113521 = 61 \cdot 1861$. We analyze the behavior of $Q - T(5)$, for consecutive primes Q beyond of $T(5) + 11^2$. The outcomes for $Q - T(5)$ are: $113537 - 113400 = 137$; $113539 - 113400 = 139$; $113557 - 113400 = 157$; $113567 - 113400 = 167$; $113591 - 113400 = 191$.

This example shows that if we take a prime Q beyond $T + p^2$, then $Q - T$ is not automatically composite. Thus, even if there is no prime number between T and $T+p^2$, we can expect that $Q-T$ may be a prime. Notice, if the next prime greater than T is $Q = T + p^2$, then the question fails.

The following example shows that there are several primes q such that $T(k) - q$ is either one or a prime with $T(k) - p^2 < q < T(k)$.

If $k = 3$, then $T(3) = 90$, $2k = 6$ and $p = 7$. So, $p^2 = 49$. These give rise to the interval $(T-p^2, T-p] = (41, 83]$. In this interval there are 10 primes q. All possible outcomes for $T(3) - q$ are: $90 - 83 = 7$; $90 - 79 = 11$; $90 - 73 = 17$; $90 - 71 = 19$; $90-67 = 23$; $90-61 = 29$; $90-59 = 31$; $90-53 = 47$; $90-47 = 43$; $90-43 = 47$. In this example, 41 is prime, but $90 - 41 = 49 = 7^2$. Note that $T(3) - 1 = 89$ is prime. In Table 3 there are some k values for which $T(k) - 1$ is prime.

We now give some notation needed for Propositions 4 and 6. We use p_r to mean the smallest prime greater than n when n is either $2k$ if $T = T(k)$ or $2k+1$ if $T = T'(k)$. The subscript r takes a special role: $r - 1$ counts the number of primes less than or equal to n.

Propositions 6 and 7 are a direct application of Theorem 3. We obtain an upper bound for the number of primes in the intervals $[T+p_r, T+p_r^2)$ and $(T-p_r^2, T+p_r]$. If there is a prime in the intervals $[T+p_r, T+p_r^2)$ then it gives a positive answer for Question 1, part (2). If Cramer's Conjecture [1] is true, then there is a prime in $[T+p_r, T+p_r^2)$.

Proposition 4. *Let l be a prime and $k > 3$.*

(1) If $T + p_r \leq l < T + p_r^2$, then $l - T$ is prime.

(2) If $T - p_r^2 < l \leq T - p_r$, then $T - l$ is prime.

Proof. We prove part (1) for $T = T(k)$, the other case and part (2) are similar. Suppose that $T + p_r \leq l < T + p_r^2$. Since $T(k) = \dfrac{(2k)!}{2^k}$, every prime $t < 2k$ divides $T(k)$. Thus, if $t < 2k$ is a prime, then t does not divide $l - T(k)$. We know that $p_r \leq l - T(k) < p_r^2$. Since p_r^2 is the smallest composite number that satisfies that $T(k)$ and p_r^2 are relatively prime, $l - T$ is a prime number. □

Corollary 5. *If p is a prime and $k > 3$, then*

(1) *if $p \in [T + p_r, T + p_r^2)$, then p has the form $T + b$ where b is a prime.*

(2) *If $p \in (T - p_r^2, T - p_r]$, then p has the form $T - b$ where b is a prime.*

Proof. We prove part (1); part (2) is similar. Suppose that $p \in [T + p_r, T + p_r^2)$, by Proposition 4, $p - T$ is prime. Therefore, $p = T + (p - T)$. □

Proposition 6. *The number of primes in $[T + p_r, T + p_r^2)$ is $O((n + 1)r^2)$.*

Proof. We prove the case $n = 2k$, the other case is similar. By Corollary 5 the number of primes in $[T + p_r, T + p_r^2)$ is $\pi_T(p_r^2)$ as in Theorem 3, part (1). Thus,

$$\pi_T(p_r^2) \ll \frac{p_r^2}{(\ln p_r^2)^2} \prod_{p|T} \left(1 + \frac{1}{p}\right).$$

$$\pi_T(p_r^2) \ll \frac{p_r^2}{4(\ln p_r)^2} \prod_{t=1}^{n} \frac{t+1}{t} = \left(\frac{p_r}{\ln p_r}\right)^2 \frac{n+1}{4}.$$

If r tends to infinity, then by the Prime Number Theorem $r \sim \dfrac{p_r}{\ln p_r}$. This implies that $\pi_T(p_r^2) = O(r^2(n + 1))$. □

Proposition 7. *The number of primes in $(T - p_r^2, T - p_r]$ is $O\left(\frac{T}{(\log T)^2}(n + 1)\right)$.*

Proof. Let $S_T(p_r)$ be the number of primes of the form $T-l$ where $l < p_r^2$ is prime. By Corollary 5 the number of primes in $(T-p_r^2, T-p_r]$ is $S_T(p_r)$. If $T-l$ is a prime where $l < p_r^2$ is a prime, then T can be written as a sum of two primes. Indeed, $T = (T-l)+l$. This and Theorem 3, part (2), imply that

$$S_T(p_r) \le r(T) \ll \frac{T}{(\log T)^2}\prod_{q|T}\left(1+\frac{1}{q}\right) \le \frac{T}{(\log T)^2}\prod_{t=1}^{n}\left(\frac{t+1}{t}\right) = \frac{T}{(\log T)^2}(n+1).$$

This proves that $S_T(p_r)$ is $O\left(\frac{T}{(\log T)^2}(n+1)\right)$. □

4. Primality of $T \pm 1$

We are going to discuss whether a number of the form $T \pm 1$ is not a prime. From Tables 4 and 1 we observe that there are few primes of the form $T \pm 1$. For example, in our search we have found only 6 primes of the form $T(k)-1$, for $2 \le k \le 2000$ (see Table 2). Table 3 shows all k values for which $T \pm 1$ is prime, for $k \le 2000$. Note that $T(2000) \sim 1.59 \times 10^{12072}$.

Propositions 8, 9 and 10 prove that there are infinitely many k such that $T \pm 1$ is not a prime. These results give rise to another question. Are there infinitely many primes of the form $T \pm 1$? We now formally state the propositions.

Proposition 8. *If $p > 7$ is a prime number with p equal to either $2k+1$ or $2k+3$, then*

(1) $p \equiv \pm 1 \bmod 8$ if and only if p is a proper divisor of $T(k)+1$.

(2) $p \equiv \pm 3 \bmod 8$ if and only if p is a proper divisor of $T(k)-1$.

Proof. We suppose that $p \equiv \pm 1 \bmod 8$ and prove that p divides $T(k)+1$. If $k = \frac{p-1}{2}$, then

$$(2k)! = \left(2\,\frac{p-1}{2}\right)! = (p-1)!$$

Therefore, by Wilson's theorem $(2k)! \equiv -1 \bmod p$. Since $p \equiv \pm 1 \bmod 8$, by the law of quadratic reciprocity 2 is a quadratic residue modulo p. Therefore, by Euler's criterion $2^k = 2^{\frac{p-1}{2}} \equiv 1 \bmod p$. This and Proposition 2 imply that

$$T(k) = \frac{(2k)!}{2^k} = \frac{(p-1)!}{2^{\frac{p-1}{2}}} \equiv -1 \bmod p.$$

Thus, p divides $T(k)+1$.

We suppose that $p = T(k) + 1$. That is,

$$p = T(k) + 1 = \frac{(p-1)!}{2^{\frac{p-1}{2}}} + 1.$$

Therefore, $(p-1)! = (p-1)2^{\frac{p-1}{2}}$. This implies that $(p-2)! = 2^{\frac{p-1}{2}}$. That is a contradiction. This proves that p is a proper divisor of $T(k) + 1$.

We now suppose that $k = \dfrac{p-3}{2}$. Since

$$T(k) = \frac{(p-3)!}{2^{\frac{p-3}{2}}} = \frac{(p-3)!(-2)(-1)}{2^{\frac{p-1}{2}}(2^{-1})(2)},$$

$$\frac{(p-3)!}{2^{\frac{p-3}{2}}} \equiv \frac{(p-3)!(p-2)(p-1)}{2^{\frac{p-1}{2}}} \equiv -1 \bmod p.$$

Thus, p divides $T(k) + 1$. If $p = T(k) + 1$, then

$$p - 1 = \frac{(p-3)!}{2^{\frac{p-3}{2}}}. \tag{1}$$

Since $p > 7$, $p - 3 = 2t$ for some $t \geq 4$. Thus,

$$\begin{aligned}(p-3)! \;=\; (2t)! &= 2 \cdot 4 \ldots (2t) \cdot 1 \cdot 3 \ldots (2t-1)\\ &= 2^t(1 \cdot 2 \ldots t) \cdot (1 \cdot 3 \ldots (2t-1))\\ &= 2^t \cdot t! \cdot (1 \cdot 3 \ldots (2t-1)).\end{aligned}$$

Therefore, $(p-3)!/2^t = t! \cdot (1 \cdot 3 \ldots (2t-1))$. This, (1) and $p - 3 = 2t$ imply that $2(t+1) = t! \cdot (1 \cdot 3 \ldots (2t-1))$. That is a contradiction, since $2(t+1) < t!$ for $t \geq 4$. This proves that p is a proper divisor of $T(k) + 1$.

Conversely, we assume that p is a proper divisor of $T(k) + 1$ and prove that $p \equiv \pm 1 \bmod 8$. We suppose that $k = \dfrac{p-1}{2}$. Since p is a proper divisor of $T(k)+1$, $T(k) \equiv -1 \bmod p$. So, $(2k)! \equiv -2^k \bmod p$. Therefore, $(p-1)! \equiv -2^{\frac{p-1}{2}} \bmod p$. This and the Wilson's theorem imply that $2^{\frac{p-1}{2}} \equiv 1 \bmod p$. By the law of quadratic reciprocity 2 is a quadratic residue modulo p. This implies that $p \equiv \pm 1 \bmod 8$.

We now suppose that $k = \dfrac{p-3}{2}$. Since p divides $T(k) + 1$, $T(k) \equiv -1 \bmod p$. So, $(2k)! \equiv -2^k \bmod p$. Therefore, $\left(2\dfrac{(p-3)}{2}\right)! \equiv -2^{\frac{p-3}{2}} \bmod p$. Thus,

$$(p-3)!(p-2)(p-1) \equiv -2^{\frac{p-3}{2}}(-2)(-1) \bmod p.$$

This implies that

$$(p-1)! \equiv -2^{\frac{p-1}{2}}(2^{-1})(-2)(-1) \bmod p.$$

Since $(p-1)! \equiv -1 \bmod p$, $2^{\frac{p-1}{2}} \equiv 1 \bmod p$. This implies that $p \equiv \pm 1 \bmod 8$.

Proof of part (2). We prove that p divides $T(k)-1$. Suppose that $p \equiv \pm 3 \bmod 8$. Wilson's theorem and $k = \dfrac{p-1}{2}$ imply that $(2k)! \equiv -1 \bmod p$. Since $p \equiv \pm 3 \bmod 8$, by the quadratic reciprocity law, 2 is not a quadratic residue modulo p. Therefore, by Euler's criterion, $2^k = 2^{\frac{p-1}{2}} \equiv -1 \bmod p$. This implies that $T(k) \equiv 1 \bmod p$. So, p divides $T(k)-1$. We suppose $p = T(k)-1$. That is, $p = \dfrac{(p-1)!}{2^{\frac{p-1}{2}}} - 1$. So, $(p-1)! = (p+1)2^{\frac{p-1}{2}}$. That is a contradiction.

If $k = \dfrac{p-3}{2}$, then

$$T(k) = \frac{(p-3)!}{2^{\frac{p-3}{2}}} \equiv \frac{(p-3)!(p-2)(p-1)}{2^{\frac{p-1}{2}}} \equiv 1 \bmod p.$$

So, the proof follows as above, proving that p is a proper divisor of $T(k)-1$.

We prove that $p \equiv \pm 3 \bmod 8$. Suppose that $k = \dfrac{p-1}{2}$. Since p divides $T(k)-1$, $T(k) \equiv 1 \bmod p$. So, $(2k)! \equiv 2^k \bmod p$. Therefore, $(p-1)! \equiv 2^{\frac{p-1}{2}} \bmod p$. This and Wilson's theorem imply that $2^{\frac{p-1}{2}} \equiv -1 \bmod p$. By the law of quadratic reciprocity, 2 is not a quadratic residue modulo p. This implies that $p \equiv \pm 3 \bmod 8$.

We now suppose that $k = \dfrac{p-3}{2}$. Since p divides $T(k)-1$, $T(k) \equiv 1 \bmod p$. So, $(2k)! \equiv 2^k \bmod p$. Therefore, $\left(2\dfrac{(p-3)}{2}\right)! \equiv 2^{\frac{p-3}{2}} \bmod p$. Thus,

$$(p-3)!(p-2)(p-1) \equiv 2^{\frac{p-3}{2}}(-2)(-1) \bmod p.$$

This implies that $(p-1)! \equiv 2^{\frac{p-1}{2}}(2^{-1})(-2)(-1) \bmod p$. This and Wilson's theorem imply that $2^{\frac{p-1}{2}} \equiv -1 \bmod p$. Thus, $p \equiv \pm 3 \bmod 8$. $\square$

Proposition 9. *If $p > 3$ is a prime number with $p = 2k+3$, then*

(1) $p \equiv \pm 1 \bmod 8$ *if and only if p is a proper divisor of $T'(k)-1$.*

(2) $p \equiv \pm 3 \bmod 8$ *if and only if p is a proper divisor of $T'(k)+1$.*

Proof. The proofs of parts (1) and (2) are similar to the proofs of Proposition 8, parts (1) and (2), respectively. $\square$

Proposition 10. *Let p be a prime number such that $p = 4k+1$. Then $p \equiv 5 \bmod 8$ if and only if p is a proper divisor of either $T(k)+1$ or $T(k)-1$.*

Proof. We first prove that $\left[\left(\frac{p-1}{2}\right)!\right]^2 \equiv -1 \bmod p$. Obviously,

$$(p-1)! = (1)(p-1)(2)(p-2)\dots\left(\frac{p-1}{2}\right)\left(p-\frac{p-1}{2}\right).$$

Therefore,

$$(p-1)! \equiv (1)(-1)(2)(-2)\dots\left(\frac{p-1}{2}\right)\left(-\frac{p-1}{2}\right) \mod p.$$

So,

$$(p-1)! \equiv \left(\frac{p-1}{2}\right)!\left(\frac{p-1}{2}\right)!(-1)^{\frac{p-1}{2}} \mod p.$$

Since $p = 4k+1$, $(-1)^{\frac{p-1}{2}} = 1$. These and Wilson's theorem imply that

$$\left[\left(\frac{p-1}{2}\right)!\right]^2 \equiv -1 \mod p. \quad (2)$$

We now prove that $p \equiv 5 \mod 8$ if and only if p is a proper divisor of either $T(k)-1$ or $T(k)+1$.

$(T(k))^2 \equiv 1 \mod p$ if and only if $\left[\frac{(2k)!}{2^k}\right]^2 \equiv 1 \mod p$ if and only if $\frac{\left[\left(\frac{p-1}{2}\right)!\right]^2}{2^{\frac{p-1}{2}}} \equiv 1 \mod p$.

This and (2), imply that

$$(T(k))^2 \equiv 1 \mod p \text{ if and only if } 2^{\frac{p-1}{2}} \equiv -1 \mod p \text{ if and only if } p \equiv \pm 3 \mod 8.$$

Since $p = 4k+1$, $(T(k))^2 \equiv 1 \mod p$ if and only if $p \equiv 5 \mod 8$.

It is easy to see that if p is a divisor of either $T(k)+1$ or $T(k)-1$, then p is a proper divisor of either $T(k)+1$ or $T(k)-1$, respectively. $\square$

5. Tables

k	$T(k) - q =$ prime or 1	$T'(k) - q =$ prime or 1
2	$6 - 5 = 1$	$15 - 13 = 2$
3	$90 - 89 = 1$	$315 - 313 = 2$
4	$2520 - 2503 = 17$	$11340 - 11329 = 11$
5	$113400 - 113383 = 17$	$623700 - 623699 = 1$
6	$7484400 - 7484383 = 17$	$48648600 - 48648583 = 17$
7	$681080400 - 681080383 = 17$	$5108103000 - 5108102983 = 17$
8	$81729648000 - 81729647983 = 17$	$694702008000 - 694702007959 = 41$
9	$12504636144000 - 12504636143963$ $= 37$	$118794043368000 - 118794043367959$ $= 41$

Table 1: Some primes of the form $T - q$.

k	Primes of the form $T(k)-1$ for $1 < k \leq 2000$
2	5
3	89
56	274017871895886614355245021851226872507509096980847975994844266521420 299245431500324696494845549659356284618231033652966211387635562226647 0399999999999999999999999999
92	450018843569393882276227680596716006487089310681842539412514262048834 586837442952353379844205073472685159662546130153568890072873003795362 844451732581991505888011382020736335842085227184693441046947485669624 634485050019491730954221690926915254316208777513302761668607999999999 99999999999999999999999999999999999
162	391548904515671716051346787260500894329100804861599843863236605693157 753938515286639559527448080180307092749222111738171154934229102563766 290007325839516166193652888106370272813680446264582621040916668979828 580909916493415772072696168113862960117719779637815600306771585482508 107493783060331912640281361853801867542860886655307894329862579460676 242332750442838738797300511969290692778986492294540611691256473129914 302664438196211535426598076748503430292227233813396104059956047273991 7 745073510746720620786978825877351293154441445603700969180904816639999 999999999999999999999999999999999999999999999999999999999999999999999 99999
170	340835263800046398325677066929789037599272966910781694237134220511694 592407221674541257352326694161941173174852612734995048749948298785427 864201761896754518975857870525407100505502667584445509342421176972834 591260193220046550390720555465344872560673854426589683541035239901055 283433221132729908219748626265401668191417034808684514905620110985521 966631215768857310684931442273323569549523637187288201582664169777656 534508255699021660672565431211046992785044507318407554205409308573862 694583409249597473614199749407605708422218605584741173228268059043735 766736030844019883753959587839999999999999999999999999999999999999999 9999999999999999999999999999999999999999999

Table 2: Some primes of the form $T(k)-1$.

Form	k values for which $T \pm 1$ is prime	Search limit
$T(k)+1$	$2, 4, 6, 70, 146, 448, 978$	2000
$T(k)-1$	$2, 3, 56, 92, 162, 170$	2900
$T'(k)+1$	$7, 16, 18, 24, 38, 44, 194, 286, 382, 895$	1000
$T'(k)-1$	$5, 12, 16, 24, 41, 46, 75, 337, 904, 2485$	3200

Table 3: Some k values for which $T \pm 1$ is prime.

k	$Q - T(k) =$ prime or 1	$Q - T'(k) =$ prime or 1
2	$7 - 6 = 1$	$17 - 15 = 2$
3	$97 - 90 = 7$	$317 - 315 = 2$
4	$2521 - 2520 = 1$	$11351 - 11340 = 11$
5	$113417 - 113400 = 17$	$623717 - 623700 = 17$
6	$7484401 - 7484400 = 1$	$48648617 - 48648600 = 17$
7	$681080429 - 681080400 = 29$	$5108103001 - 5108103000 = 1$
8	$81729648019 - 81729648000 = 19$	$694702008041 - 694702008000 = 41$
9	$12504636144029 - 12504636144000 = 29$	$118794043368047 - 118794043368000 = 47$

Table 4: Some primes of the form $Q - T$.

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