VARIATIONS OF THE POINCARÉ MAP

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Abstract

Nogueira, in 1995, presented a study of the Poincaré map and the related Parry-Daniels map. In this note some variants of this algorithm are presented which seem to have quite different ergodic behavior.

1. Introduction

A 2-dimensional *homogeneous* multidimensional continued fraction is given by a subset $B_F \subseteq (\mathbb{R}^+)^3$ and a map $F : B_F \to B_F$ with the following property. There is a partition $\{B(i) : i \in I\}$, I finite or countable, of B_F and a set of regular matrices $\{\alpha_F(i) : i \in I\}$ such that the map F is given by $\alpha_F(i)$ on every set $B(i)$. We denote the inverse matrices of $\alpha_F(i)$ by $M_F(i)$. A *cylinder* $B(i_1, ..., i_s)$ is the set $\{x \in B : F^{k-1}x \in B(i_k), 1 \leq k \leq s\}$. The cylinder is called *full* if $F^s B(i_1, ..., i_s) = B_F$. Let

$$\|x\| := x_0 + x_1 + x_2.$$

Then we associate a 2-dimensional *inhomogeneous* continued fraction $\widehat{F}$ on the 2-simplex

$$\Sigma_2 = \{(x_0, x_1, x_2) : 0 \leq x_0 \leq 1, 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, x_0 + x_1 + x_2 = 1\}$$

by the definition

$$\widehat{F}x := \frac{Fx}{\|Fx\|},\, x \in B \cap \Sigma_2.$$

The cylinders for the map $\widehat{F}$ will be denoted by the same symbols as for the map F. Instead of working with $\widehat{F}$ on Σ_2 we often use its projection on the x_0x_1-plane and we write (u, v) for its coordinates.

The map $\widehat{F}$ is a fractional linear map which explains the notion of a multidimensional continued fraction. If the Jacobians are defined by the equation

$$\int_{B(i_1,\dots,i_s)\cap \widehat{F}^{-s}E} d\lambda(x) = \int_E \omega(i_1, ..., i_s; x) d\lambda(x),$$

(λ denotes Lebesgue measure), then we say that $\widehat{F}$ satisfies a *Rényi condition* if there is a constant $C \geq 1$ such that $\omega(i_1, ..., i_s; x) \leq C\omega(i_1, ..., i_s; y)$ for all points $x, y \in \widehat{F}^s B(i_1, ..., i_s)$. More information on multidimensional continued fractions can be found in [7].

2. Poincaré's Algorithm

Poincaré's algorithm in dimension $d = 3$ can be described analytically as follows. Let $x = (x_0, x_1, x_2)$, $x_0, x_1, x_2 \geq 0$ and let $\pi \in \mathcal{S}_3$ a permutation such that $x_{\pi 0} \leq x_{\pi 1} \leq x_{\pi 2}$. Then define

$$(x_0', x_1', x_2') = P(x_0, x_1, x_2) := (x_{\pi 0}, x_{\pi 1} - x_{\pi 0}, x_{\pi 2} - x_{\pi 1}).$$

The six inverse branches can be described by the following matrices:

$$M_P(\epsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \; M_P(01) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$M_P(12) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \; M_P(021) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_P(012) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \; M_P(02) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Poincaré [5] presented this algorithm in a short paper in 1884. Schweiger [6] studied the related Parry-Daniels map and Nogueira [4] gave a very complete account on this interesting algorithm (see also Schweiger [7]). A good visualization can be given by using the Parry-Daniels map $\widehat{P}$ on the 2-simplex. As introduced before let

$$\|x\| = x_0 + x_1 + x_2.$$

Then for any $x \in \Sigma_2$ the Parry-Daniels map is given as

$$\widehat{P}(x) = \frac{P(x)}{\|P(x)\|}.$$

Then the cylinders can be illustrated as triangles on Σ_2. However, it must be pointed out that the ergodic behaviors of the Poincaré map P and the Parry-Daniels map $\widehat{P}$ are different.

A multidimensional continued fraction attaches to every point a sequence of digits $(i_1, i_2, ...)$. For many algorithms the set of points which miss a certain digit is a set of Lebesgue measure 0. This phenomenon is well-known for the 1-dimensional case (e.g., for continued fractions or digital expansions). For the Selmer algorithm some digits are transient. This means that for almost all points these digits appear only finitely many often. However, if we neglect these digits this algorithm remains as a map on a set with non-empty interior. The Parry-Daniels map is unusual since four digits are transient but the remnant set is more like a fractal but still has positive Lebesgue measure. In the next sections we will consider various modifications of the Poincaré map which show quite different ergodic behavior.

3. First Variation

The first variation is given by using full subtraction in the first coordinate. This means replacing the map P by the map Q, namely

$Q(x_0, x_1, x_2) = (x_{\pi 0}, x_{\pi 1} - a x_{\pi 0}, x_{\pi 2} - x_{\pi 1}), \quad a = [\frac{x_{\pi 1}}{x_{\pi 0}}]$.

It is easy to see that it is sufficient to consider only three permutations: $\pi = (01), (012)$ or (02). This gives us three matrices:

$$M_Q(01; a) = \begin{pmatrix} a & 1 & 0 \\ 1 & 0 & 0 \\ a & 1 & 1 \end{pmatrix}$$

$$M_Q(012; a) = \begin{pmatrix} a & 1 & 1 \\ 1 & 0 & 0 \\ a & 1 & 0 \end{pmatrix}$$

$$M_Q(02; a) = \begin{pmatrix} a & 1 & 1 \\ a & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since $x_{\pi 1} - a x_{\pi 0} \leq x_{\pi 0}$ the region $x_0 < x_1$ is transient. The closures of the cylinders $B(\pi; a)$, $\pi \in \{(01), (012), (02)\}$, $a \in \mathbb{N}$, are fully mapped onto $\{(x_0, x_1, x_2) : 0 \leq x_1 \leq x_0,\ 0 \leq x_2\} =: B_Q$.

The eigenvector of $M_Q(01; 1)$ is the positive vector

$$w := (-4 + 2\sqrt{5}, 7 - 3\sqrt{5}, -1 + \sqrt{5}).$$

Therefore the segment $[o, w]$ is invariant under P which suggested the existence of an attractor for the Parry-Daniels map. Furthermore $M_Q(01; a)$ also has an eigenvector in the set $B(01; a)$ which is related to the positive root of $\lambda^2 = a\lambda + 1$. We denote the restriction of the map Q to the simplex Σ_2 by $\widehat{Q}$, namely

$$\widehat{Q}(x) = \frac{Q(x)}{\|Q(x)\|}.$$

Theorem 1. *The set* $\Lambda = \{x \in B_{\widehat{Q}} : \widehat{Q}^N x \in \bigcup_{a=1}^{\infty} B(01;a)$ *for all* $N \geq 0\}$ *is a set of positive Lebesgue measure.*

Proof. The map $M_Q(01;a)$ corresponds to the linear fractional map

$$(u,v) \mapsto \left(\frac{au+v}{2au+v+1}, \frac{u}{2au+v+1} \right)$$

with the Jacobian

$$\frac{1}{(2au+v+1)^3}.$$

The associated matrix gives the recursion

$$(1, A_{n+1}, B_{n+1}) = (1, A_n, B_n) \begin{pmatrix} 1 & 2a_{n+1} & 1 \\ 0 & a_{n+1} & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Therefore we obtain

$$\lambda(B(a_1, ..., a_n)) = \frac{1}{2(1+A_n)(2+A_n+B_n)}.$$

The recursion relation $A_{n+1} = a_{n+1}A_n + A_{n-1} + 2a_{n+1} + 1$ shows

$$\begin{aligned} A_{n+1} + 3 \quad &= a_{n+1}(A_n + 3) + (A_{n-1} + 3) + 1 - a_{n+1} \\ &\leq a_{n+1}(A_n + 3) + (A_{n-1} + 3). \end{aligned}$$

We compare it with the recursion for regular continued fraction $q_{n+1} = a_{n+1}q_n + q_{n-1}$. Since $A_1 = 2a_1$ and $q_1 = a_1$, we obtain

$$\frac{q_n}{A_n + 3} \geq \frac{1}{5}.$$

If $I(a_1, ..., a_n)$ is the interval which contains all continued fractions with $a_1(x) = a_1, ..., a_n(x) = a_n$ we finally obtain

$$\frac{\lambda(B(a_1, ..., a_n))}{\lambda(I(a_1, ..., a_n))} \geq \frac{1}{50}.$$

□

4. A Further Variation

A further variation is done by using the second coordinate:

$$R(x_0, x_1, x_2) = (x_{\pi 0}, x_{\pi 1} - x_{\pi 0}, x_{\pi 2} - b x_{\pi 1}),\ b = \left[\frac{x_{\pi 2}}{x_{\pi 1}}\right].$$

Two typical matrices now look as follows:

$$M_R(\epsilon; b) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ b & b & 1 \end{pmatrix}, \; M_R((12); b) = \begin{pmatrix} 1 & 0 & 0 \\ b & b & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Since $x_{\pi 0} + (x_{\pi 1} - x_{\pi 0}) = x_{\pi 1} \geq x_{\pi 2} - b x_{\pi 1}$, the region $x_0 + x_1 < x_2$ is transient. The cylinders $B(\pi; b)$ are full for $\pi \in \{(12), (021), (012), (02)\}$ but the cylinders $B(\epsilon; b)$ and $B((01); b)$ are not full. Clearly, this concerns the restriction of the map to the region

$$B_R := \{(x_0, x_1, x_2) : \; x_0 + x_1 \geq x_2\}.$$

More precisely, $B(\epsilon; 1)$ and $B(01; 1)$ are the only cylinders with $\pi = \epsilon$ or (01). The digit $(\pi; 1)$ for any $\pi \in \mathcal{S}_3$ can be followed only by $(\pi; b), \pi = (021), (02), (012)$. Note that the vector

$$w = (-4 + 2\sqrt{5}, 7 - 3\sqrt{5}, -1 + \sqrt{5})$$

now lies in the transient region. Therefore it is expected that the ergodic behavior is different from the Poincaré map.
Let $\Sigma_2^{\#} = \{x = (x_0, x_1, x_2) \in \Sigma_2 : \; 0 \leq x_2 \leq x_0 + x_1\}$.

Lemma 2. *The algorithm connected with $\widehat{R}$ is convergent.*

Proof. Let $((B_{ij})) = M_R(\pi_1, b_1) \ldots M_R(\pi_s, b_s)$. We will exemplify the idea of the proof with a typical example. Let $B((\pi_1, b_1), ..., (\pi_s, b_s))$ be the "quadrangle" with vertices (B_{00}, B_{10}, B_{20}), (B_{01}, B_{11}, B_{21}), $(B_{01} + B_{02}, B_{11} + B_{12}, B_{21} + B_{22})$, and $(B_{00} + B_{02}, B_{10} + B_{12}, B_{20} + B_{22})$. The corresponding points on the simplex Σ_2 are given by dividing the coordinates by the sum $M_j := \sum_{i=0}^{2} B_{ij}$, $0 \leq j \leq 2$.

Let $(\pi_{s+1}, b_{s+1}) = ((02), b)$. The cylinder

$$B((\pi_1, b_1), ..., (\pi_s, b_s), ((02), b))$$

has the vertices $((B_{i0} b + B_{i1} + B_{i2}))$, $((B_{i0} b + B_{i1}))$, $((B_{i0}(b+1) + B_{i1}))$, and $((B_{i0}(b+1) + B_{i1} + B_{i2}))$, $i = 0, 1, 2$. We consider the 0-th coordinate and find

$$\left| \frac{\frac{B_{00} b + B_{01}}{M_0 b + M_1} - \frac{B_{00}(b+1) + B_{01}}{M_0(b+1) + M_1}}{\frac{B_{00}}{M_0} - \frac{B_{01}}{M_1}} \right| = \frac{M_0 M_1}{(M_0 b + M_1)(M_0(b+1) + M_1)} \leq \frac{1}{3}.$$

The line of intersection of the plane through $((B_{i0} b + B_{11}))$ and $((B_{i2} b))$ and the plane through $((B_{i1} + B_{i2}))$ and $((B_{i0} + B_{i2}))$ is given as $((B_{i0} b + B_{i1} + (b+1) B_{i2}))$. Therefore we now compare

$$\left| \frac{\frac{B_{00} b + B_{01}}{M_0 b + M_1} - \frac{B_{00} b + B_{01} + B_{02}}{M_0 b + M_1 + M_2}}{\frac{B_{00} b + B_{01}}{M_0 b + M_1} - \frac{B_{00} b + B_{01} + (b+1) B_{02}}{M_0 b + M_1 + (b+1) M_2}} \right| = \frac{M_0 b + M_1 + (b+1) M_2}{(b M_0 + M_1 + M_2)(b+1)}.$$

It is easy to see that $2M_2 \le M_0 + M_1$. Therefore we obtain

$$\frac{M_0 b + M_1 + (b+1)M_2}{(M_0 b + M_1 + M_2)(b+1)} \le \frac{M_0 + M_1 + 2M_2}{2(M_0 + M_1 + M_2)} \le \frac{2}{3}.$$

The other cases are treated in a similar way. This shows that the diameter of a cylinder shrinks if $\pi_s \in \{(02), (021), (012), (12)\}$. Note that $\pi = \epsilon$ or $\pi = (01)$ must be followed by $(02), (012)$ or (021). $\square$

Theorem 3. *The map $\widehat{R} : \Sigma_2^{\#} \to \Sigma_2^{\#}$ is ergodic and admits a finite invariant measure which is equivalent to Lebesgue measure.*

Proof. Clearly, the set of all points such that $\widehat{R}^s x \in B(12)$ for all $s \ge s(x)$ is a set of measure 0. Furthermore, observe that the digits $(\varepsilon; b)$ and $((01); b)$ must be followed by $(021), (02)$ or (012). This shows that one out of the following types of digits or blocks appears infinitely many often for almost all points:

$$(021); (02); \varepsilon(012); (01)(012); (12)(012); (012)(012).$$

We look at the Jacobians of the inverse branches of the iterates of $\widehat{R}$. Let

$$\omega((\pi_1, b_1), ..., (\pi_s, b_s); x) = \frac{1}{(B_0^{(s)} x_0 + B_1^{(s)} x_1 + B_2^{(s)} x_2)^3}.$$

We estimate this function on the quadrangle with vertices $(1,0,0)$, $(0,1,0)$, $(0, \frac{1}{2}, \frac{1}{2})$, and $(\frac{1}{2}, 0, \frac{1}{2})$. This amounts to the comparison of the four quantities $B_0^{(s)}, B_1^{(s)}, B_0^{(s)} + B_2^{(s)}$ and $B_1^{(s)} + B_2^{(s)}$.

If we consider two typical cases

$$M_R(021; b) \quad = \quad \begin{pmatrix} 1 & 1 & 0 \\ b & b & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$M_R(012; b) M_R(012; d) \quad = \quad \begin{pmatrix} bd + b + 1 & bd + 1 & b \\ d & d & 1 \\ d + 1 & d & 1 \end{pmatrix},$$

then we see that these four quantities bound each other. This implies a Rényi condition. Ergodicity and the existence of a σ-finite invariant measure now follow from the extended Rényi theorem for multidimensional continued fractions (see [7]).

We give a sketch of a proof that the invariant measure is finite. Let $W_n := \{x : \widehat{P}^j x \in B(12),\ 0 \le j \le n-1\}$. Letting F_n denote the n-th Fibonacci number, an easy induction shows

$$M_P(12)^n = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 & 0 \\ F_{n+2} - 2 & F_n & F_{n-1} \\ F_{n+1} - 1 & F_{n-1} & F_{n-2} \end{pmatrix}.$$

This shows that $\sum_{n=1}^{\infty} \lambda(W_n)$ is convergent. Therefore the invariant measure for $\widehat{R}$ is finite. □

5. Another Variation

The next variation is done by using both coordinates:

$$T(x_0, x_1, x_2) = (x_{\pi 0}, x_{\pi 1} - a x_{\pi 0}, x_{\pi 2} - b x_{\pi 1}), \quad a = \left[\frac{x_{\pi 1}}{x_{\pi 0}}\right], b = \left[\frac{x_{\pi 2}}{x_{\pi 1}}\right].$$

If one considers the projection of this algorithm onto the set $B(\epsilon)$ we obtain the Parusnikov algorithm (Bryuno & Parusnikov [2]).
The region $x_0 < x_1$ is transient and the basic set again is $B_T = B_Q$. To give one example, the cylinders $B((01); a, b)$ are the "quadrilaterals" with endpoints

$$(a, 1, ab), (a+1, 1, (a+1)b), (a, 1, a(b+1)), (a+1, 1, (a+1)(b+1)).$$

However, no cylinder is full. In fact $TB(\pi; a, b)$ is the "quadrilateral" with endpoints $(1, 0, 0), (1, 1, 0), (1, 0, a), (1, 1, a+1)$. Therefore $TB(\pi; a, b)$ contains the region $B(02) \cup B(012)$ but only a part of $B(01)$.

We consider $\Sigma_2^* = \{x = (x_0, x_1, x_2) \in \Sigma_2 : 0 \le x_1 \le x_0\}$. The ergodic behavior of $\widehat{T}$ is not known. A typical matrix is given as

$$M_T(01; a, b) = \begin{pmatrix} a & 1 & 0 \\ 1 & 0 & 0 \\ ab & b & 1 \end{pmatrix}.$$

The Jacobian of the projected algorithm is given as

$$\frac{1}{(1 + (a+ab)u + bv)^3}.$$

The Jacobian must be estimated on the quadrangle with vertices $(1, 0)$, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{a+1}, 0)$, and $(\frac{1}{a+3}, \frac{1}{a+3})$. Hence no Rényi condition is satisfied. The map $\widehat{T}$ is reminiscent of the Selmer division algorithm ([7]). However, since $\widehat{T}B(\pi; a, b)$ contains a full cylinder its ergodic behavior might be easier to be investigated.

6. A Short Digression

This is an opportunity to make a short digression to the division form of the Selmer algorithm. Take $B = \{(x_1, x_2) : 0 < x_2 \le x_1 \le 1\}$ and define the map

$$S(x_1, x_2) = \left(\frac{x_2}{x_1}, \frac{1 - k x_2}{x_1}\right), \, k = \left[\frac{1}{x_2}\right].$$

Note that this map is not the multiplicative acceleration of the Selmer algorithm. In fact, the behavior of S is unexpectedly complicated. The cylinders $B(k)$ of rank 1 are the quadrangles with vertices $(1, \frac{1}{k})$, $(1, \frac{1}{k+1})$, $(\frac{1}{k+1}, \frac{1}{k+1})$, and $(\frac{1}{k}, \frac{1}{k})$ for $k \geq 2$ and the triangle $(1,1)$, $(1, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$. No cylinder is full. The sets $SB(k)$ are the quadrangles $(\frac{1}{k}, 0)$, $(\frac{1}{k+1}, \frac{1}{k+1})$, $(1,1)$, $(1,0)$ and the triangle $(1,0)$, $(\frac{1}{2}, \frac{1}{2})$, $(1,1)$. Observe that $B(k) \subseteq SB(k)$ for any k.

The following lemma can be verified by straightforward calculations.

Lemma 4. *The sets $B(k,n)$ can be described as follows.*

(i) *If $k+1 \leq n$ then the cylinder $B(k,n)$ is the quadrangle with vertices $(1, \frac{n}{(n+1)k})$, $(1, \frac{n-1}{nk})$, $(\frac{n}{nk+1}, \frac{n}{nk+1})$, and $(\frac{n+1}{n(k+1)+1}, \frac{n+1}{n(k+1)+1})$.*

(ii) *If $2 \leq n \leq k$ then the cylinder $B(k,n)$ is the quadrangle with vertices $(\frac{n+1}{k+1}, \frac{1}{k+1})$, $(\frac{n}{k+1}, \frac{1}{k+1})$, $(\frac{n}{nk+1}, \frac{n}{nk+1})$, and $(\frac{n+1}{n(k+1)+1}, \frac{n+1}{n(k+1)+1})$.*

(iii) *If $n = 1$ then the cylinder $B(k,1)$ is the triangle with vertices $(\frac{2}{k+1}, \frac{1}{k+1})$, $(\frac{1}{k+1}, \frac{1}{k+1})$, and $(\frac{2}{k+2}, \frac{2}{k+2})$.*

Theorem 5. *Let $k+1 \leq n$. If $(m+1)k \leq n$ then $B(k,n,m) = \emptyset$ mod 0.*

Proof. If $k+1 \leq n$ then the set $S^2B(k,n)$ is the quadrangle with vertices $(\frac{k}{n}, \frac{k}{n})$, $(\frac{k}{n-1}, 0)$, $(\frac{1}{n}, 0)$, and $(\frac{1}{n+1}, \frac{1}{n+1})$. Only if $\frac{k}{n} = \frac{1}{m+1}$ the sets $S^2B(k,m)$ and $B(m)$ have the common vertex $(\frac{1}{m+1}, \frac{1}{m+1})$. □

This result shows that the conditions for admissible digits are more complicated than expected.

7. Another Idea

Another idea would be to consider the map

$$Z(x_0, x_1, x_2) = (x_{\pi 0}, x_{\pi 1} - x_{\pi 0}, x_{\pi 2} - x_{\pi 1} - d x_{\pi 0}),\ d = \left[\frac{x_{\pi 1} - x_{\pi 0}}{x_{\pi 0}}\right].$$

The map Z projects every cylinder $B(\pi; d)$ onto the "triangle" with vertices $(1,0,1)$, $(1,0,0)$, $(0,1,0)$, which corresponds to the condition $x_0 \geq x_2$. This algorithm is reminiscent of the Garrity triangle sequence (Assaf et al. [1]; Messaoudi, Nogueira, and Schweiger [3]). We find three branches on the triangle with vertices $(1,0)$, $(0,1)$, and $(\frac{1}{2})$:

$$\widehat{Z}(012; d)(u,v) = \frac{(1+du, u)}{1+(d+2)u+v}$$

$$\widehat{Z}(02;d)(u,v) = \frac{(1+du, u+v)}{1+(d+2)u+v}$$
$$\widehat{Z}(021;d)(u,v) = \frac{(u+v, 1+du)}{1+(d+2)u+v}.$$

A multidimensional continued fraction $\widehat{F}^*$ is called a *dual map* (or sometimes a *backward algorithm*) of $\widehat{F}$ if the following holds: a cylinder (of the dual map) $B^*(k_1, k_2, ..., k_s) \neq \emptyset$ if and only if the cylinder (of the given map) $B(k_s, k_{s-1}, ..., k_1) \neq \emptyset$. The main use of this notion is that it can be used in some cases to find an invariant measure.

The dual map $\widehat{Z}^*$ can be realized on the region $\{(\xi,\eta) : 0 \leq \eta \leq \xi\}$. Therefore a finite invariant measure can be found with the density function

$$h(u,v) = \frac{1}{u(u+v)}.$$

This can be also verified by using the Kuzmin equation (see [7]). Integration gives

$$\int\int \frac{dudv}{u(u+v)} = \frac{\pi^2}{12}.$$

Lemma 6. *If the sequence (d_n), $n \geq 1$, is rapidly increasing then the nested sequence of cylinders $B((012;d_1), ..., (012;d_n))$ does not shrink to a point.*

Proof. The associated matrices are of the form

$$M_Z(012;d) \quad = \quad \begin{pmatrix} d+1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} .$$

If the cylinder $B((012;d_1), ..., (012;d_n))$ has the vertices (B_{00}, B_{10}, B_{20}), $(B_{00} + B_{02}, B_{10} + B_{12}, B_{20} + B_{22})$, and (B_{01}, B_{11}, B_{21}), then the cylinder $B((012;d_1), \ldots, (012;d_n), (012;d))$ has the vertices $(((d+1)B_{j0} + B_{j1} + B_{j2}))$, $(((d+2)B_{j0} + B_{j1} + B_{j2}))$, and $((B_{j0} + B_{j2}))$, $j = 0, 1, 2$. We look at the 0-th coordinate and calculate

$$\frac{B_{00}}{M_0} - \frac{B_{00}+B_{02}}{M_0+M_2} = \frac{B_{00}M_2 - M_0B_{02}}{M_0(M_0+M_2)};$$

$$\frac{(d+1)B_{00}+B_{01}+B_{02}}{(d+1)M_0+M_1+M_2} - \frac{B_{00}+B_{02}}{M_0+M_2}$$
$$= \frac{d(B_{00}M_2 - M_0B_{02}) + M_0B_{01} - M_1B_{00} + M_2B_{01} - M_1B_{02}}{((d+1)M_0+M_1+M_2)(M_0+M_2)}.$$

If $d_{n+1} = d$ is sufficiently large the quotient of both quantities can be made greater than $1 - \frac{1}{n^2}$. Therefore this sequence does not shrink to an interval. □

Lemma 6 shows that this multidimensional continued fraction is not convergent in infinitely many points.

8. A Final Variation

The final variation is much more routine.

$$W(x_0, x_1, x_2) = (x_{\pi 0}, x_{\pi 1} - a x_{\pi 0}, x_{\pi 2} - b x_{\pi 1} - c x_{\pi 0})$$
$$a = [\frac{x_{\pi 1}}{x_{\pi 0}}], b = [\frac{x_{\pi 2}}{x_{\pi 1}}], c = [\frac{x_{\pi 2} - b x_{\pi 1}}{x_{\pi 0}}], 1 \leq a, 1 \leq b, 0 \leq c \leq a.$$

Note that this algorithm is different from the Bryuno algorithm as given in [2].

Since the partition for $\widehat{W}$ is a refinement of the partition for $\widehat{R}$ this algorithm is convergent. The basic set is

$$B_W = \{(x_0, x_1, x_2) : \max(x_1, x_2) \leq x_0\}.$$

This is the "quadrangle" with vertices $(1,0,0)$, $(1,1,0)$, $(1,1,1)$, and $(1,0,1)$. The cylinders $B(\pi; a, b, c)$ are full for $c < a$. For $c = a$ its image reduces to $WB(\pi; a, b, a) = B(02)$. The matrices are given as

$$M_W(02; a, b, c) = \begin{pmatrix} ab+c & b & 1 \\ a & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$M_W(012; a, b, c) = \begin{pmatrix} ab+c & b & 1 \\ 1 & 0 & 0 \\ a & 1 & 0 \end{pmatrix}.$$

Now it is easy to verify a Rényi condition. Therefore $\widehat{W}$ is ergodic and admits a finite invariant measure which is equivalent to Lebesgue measure.

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