#A63 INTEGERS 11 (2011)

SUBPRIME FACTORIZATION AND THE NUMBERS OF BINOMIAL COEFFICIENTS EXACTLY DIVIDED BY POWERS OF A PRIME

William B. Everett
Chernogolovka, Moscow Oblast, Russia
bill@chgnet.ru

Received: 9/4/10, Revised: 7/26/10, Accepted: 9/29/11, Published: 10/10/11

Abstract

We use the notion of subprime factorization to establish recurrence relations for the number of binomial coefficients in a given row of Pascal's triangle that are divisible by p^j and not divisible by p^{j+1}, where p is a prime. Using these relations to compute this number can provide significant savings in the number of computational steps.

1. Introduction

Subprime factorization converts the prime divisibility of Pascal's triangle of binomial coefficients into a set of subprime-divisibility triangles. Each subprime-divisibility triangle is a very simple regular tiling with two basic triangular tiles: one tile containing only zeros and the other containing only ones. Two different subprime-divisibility triangles for a prime p differ only in scale and resolution, and the relative scale and resolution factor is a power of p. This set of regular tilings of Pascal's triangle provides the most detailed information about the prime divisibility of binomial coefficients in a simple form that allows easily obtaining various general results.

Let n be a nonnegative integer and p be a prime. Let $\theta_j(n,p)$ denote the number of binomial coefficients $\binom{n}{k}$, $0 \le k \le n$, such that $p^j \parallel \binom{n}{k}$, i.e., $p^j \mid \binom{n}{k}$ and $p^{j+1} \nmid \binom{n}{k}$. We represent n in the base p: $n = c_0 + c_1 p + c_2 p^2 + \cdots + c_r p^r$, $0 \le c_i < p$, $i = 0, 1, \ldots, r$, $c_r > 0$ for $n \ne 0$.

For $j = 0$ and $p = 2$, Glaisher, in 1899 [4], established the formula

$$\theta_0(n,2) = 2^{\sum_{i=0}^{r} c_i} = \prod_{i=0}^{r}(c_i + 1).$$

Fine generalized this result to any prime p in 1947 [3]: $\theta_0(n,p) = \prod_{i=0}^{r}(c_i+1)$. For $j = 1$, Carlitz gave the specific formula

$$\theta_1(n,p) = \sum_{i=0}^{r-1}(c_0+1)\cdots(c_{i-1}+1)(p-c_i-1)c_{i+1}(c_{i+2}+1)\cdots(c_r+1)$$

in 1967 [1]. Howard gave formulas for $\theta_j(n,2)$ and for $\theta_2(n,p)$ in 1971 and 1973 [5, 6]. We recently gave the following general formula for $\theta_j(n,p)$ [2].

We let W be the set of r-bit binary words, i.e.,

$$W = \{\mathbf{w} = w_1 w_2 \dots w_r \colon w_i \in \{0,1\},\ 1 \le i \le r\},$$

and partition W into $r{+}1$ subsets W_j, $0 \le j \le r$, where each subset consists of all the words containing exactly j ones:

$$W_j = \left\{\mathbf{w} \in W \colon \sum_{i=1}^{r} w_i = j\right\}.$$

We sum over the words in the subset W_j to obtain $\theta_j(n,p)$:

$$\theta_j(n,p) = \sum_{\mathbf{w} \in W_j} F(\mathbf{w}) L(\mathbf{w}) \prod_{i=1}^{r-1} M(\mathbf{w}, i), \tag{1}$$

where the functions $F(\mathbf{w})$, $L(\mathbf{w})$, and $M(\mathbf{w}, i)$ giving the $r{+}1$ factors in each summand are defined as

$$F(\mathbf{w}) = \begin{cases} c_0 + 1 & \text{if } w_1 = 0, \\ p - c_0 - 1 & \text{if } w_1 = 1, \end{cases}$$

$$L(\mathbf{w}) = \begin{cases} c_r + 1 & \text{if } r > 0 \text{ and } w_r = 0, \\ c_r & \text{if } r > 0 \text{ and } w_r = 1, \\ 1 & \text{if } r = 0, \end{cases}$$

$$M(\mathbf{w}, i) = \begin{cases} c_i + 1 & \text{if } w_i = 0 \text{ and } w_{i+1} = 0, \\ p - c_i - 1 & \text{if } w_i = 0 \text{ and } w_{i+1} = 1, \\ c_i & \text{if } w_i = 1 \text{ and } w_{i+1} = 0, \\ p - c_i & \text{if } w_i = 1 \text{ and } w_{i+1} = 1. \end{cases}$$

We take $\prod_{i=a}^{b}(\dots) = 1$ if $a > b$. Oversights in formula (1) in [2] in the cases $r = 0, 1$ are corrected here.

The sum in (1) contains $\binom{r}{j}$ terms, which means that to compute the complete set of $\theta_j(n,p)$, $0 \le j \le r$, the number of computational steps as a function of r (which, of course, depends on n and p) basically increases as $\sum_{j=0}^{r} \binom{r}{j} = 2^r$. We obtain recurrence relations that allow computing the set of $\theta_j(n,p)$ with the number of steps basically increasing as r^2.

To simplify obtaining values of $\theta_j(n,p)$ in practice, it is convenient to not calculate the individual values separately but treat the sequence $\{\theta_0(n,p), \theta_1(n,p), \dots\}$ as an entity. We let capital letters denote sequences, for example, $A = \{a_0, a_1, \dots\}$ and $B = \{b_0, b_1, \dots\}$. We have the usual operations of a linear space over integers. Obviously, $k(A \pm B) = kA \pm kB$. We also define a "shifted" sequence:

$$\vec{A} = \{\vec{a}_0 = 0, \vec{a}_1 = a_0, \vec{a}_2 = a_1, \dots\}.$$

Obviously, $\overrightarrow{(A \pm B)} = \vec{A} \pm \vec{B}$ and $\overrightarrow{(kA)} = k\vec{A}$.

Let $\Theta(n,p) = \{\theta_0(n,p), \theta_1(n,p), \dots\}$. For the recursion base, we have two "lowest value" cases: $n < p$ (i.e., $n = c_0$) or $p \le n < 2p$ (i.e., $n = c_o + p$). For $r = 0$,

$$\Theta(n,p) = \{c_0 + 1, 0, 0, \dots\}, \tag{2}$$

and for $r = 1$ and $c_1 = 1$,

$$\Theta(n,p) = \{2(c_0 + 1), p - c_0 - 1, 0, 0, \dots\}. \tag{3}$$

In Section 4, we prove the following theorems.

Theorem 1 (General recurrence relations). *For $r > 1$ and $c_r = 1$,*

$$\begin{aligned} \Theta(n,p) = 2\Theta(n - p^r, p) &+ (p - c_{r-1})\vec{\Theta}(n - p^r - (c_{r-1} - 1)p^{r-1}, p) \\ &- (p - c_{r-1} + 1)\vec{\Theta}(n - p^r - c_{r-1}p^{r-1}, p), \end{aligned} \tag{4}$$

and for $r > 0$ and $c_r > 1$,

$$\Theta(n,p) = c_r\Theta\big(n - (c_r - 1)p^r, p\big) - (c_r - 1)\Theta(n - c_r p^r, p). \tag{5}$$

Theorem 2 (Special recurrence relation). *If $n = p^k m - 1$ (i.e., the k least significant digits in the base-p representation of n are $p - 1$), then*

$$\Theta(n,p) = p^k\Theta(m - 1, p). \tag{6}$$

Remark. If $\theta_j(n,p)$ is desired for only one value of j, then we can write the recurrence relations in "component" form with the convention that $\theta_j(n,p) \equiv 0$ for $j < 0$. For (4) and (5), we have

$$\begin{aligned} \theta_j(n,p) &= 2\theta_j(n - p^r, p) + (p - c_{r-1})\theta_{j-1}(n - p^r - (c_{r-1} - 1)p^{r-1}, p) \\ &\qquad - (p - c_{r-1} + 1)\theta_{j-1}(n - p^r - c_{r-1}p^{r-1}, p), \quad r > 1, \quad c_r = 1, \\ \theta_j(n,p) &= c_r\theta_j\big(n - (c_r - 1)p^r, p\big) - (c_r - 1)\theta_j(n - c_r p^r, p), \quad r > 0, \quad c_r > 1. \end{aligned}$$

The recurrence relation $\theta_0(n,p) = 2\theta_0(n - p^r, p)$ follows obviously from Fine's formula for $j = 0$ with $c_r = 1$, but the recurrence relations in Theorem 1 are less obvious for $j > 0$.

Remark. Our motivation for seeking the general recurrence relations presented here was Shevelev's notion of "binomial predictors," which provides a recursive definition of $\Theta(n,p)$ for a very special class of n [9]. Special recurrence relation (6) generalizes Shevelev's result and is applicable to $1/p$ of all n. The "binomial predictor" notion, in addition to strongly restricting the least significant digit(s) in the base-p representation of n exactly as in the applicability condition for (6), also restricts the more significant digit(s). As an example, we consider Example 3 in [9]: $p = 3$, $n = 23$, $\theta_0(23,3) = 18$, $\theta_1(23,3) = 6$. Here n is the second member of the sequence $7, 23, 71, 215, \ldots$ for which we have the corresponding sequences of θ_0 $6, 18, 54, 162, \ldots$ and of θ_1 $2, 6, 18, 54, \ldots$. This, for instance, immediately suggests considering the sequence $6, 20, 62, 188, \ldots$, for which we have the corresponding sequences of θ_0 $3, 9, 27, 81, \ldots$ and of θ_1 $4, 12, 36, 108, \ldots$. The situation with the sequence of binomial predictors (beginning with 7) is prettier than the situation with the second sequence. With the binomial predictors, we can read the θ values directly from the base-p representation of $n + 1$. In both cases, if we know the θ values for the first member of the sequence, then we obtain the θ values for any subsequent member by multiplying by the appropriate power of p. This suggested the existence of a more general recurrence relation.

Remark. While we had supposed that Kummer's theorem on carries [7] could be used to obtain general formula (1) presented in [2] and the recurrence relations in Theorem 1 in this paper, the notion of subprime factorization leads directly and obviously to the general formula and the recurrence relations. After the first draft of this paper was completed, we became aware of [8], where a general formula was directly obtained using Kummer's theorem. The formula for a_{p^α}, the number of nonzero entries modulo p^α in row n of Pascal's triangle, given in [8] is essentially $\sum_{j=0}^{\alpha-1} \theta_j(n, p)$ in our notation.

In Section 2, we explain the notion of subprime factorization and indicate how it results in general formula (1). This notion simplifies the problem of finding a general formula for the number of binomial coefficients exactly divided by a fixed power of a prime and might be applicable or adaptable to similar problems. Moreover, it is central in proving the recurrence relations. In Section 3, we give a few simple numerical examples using the recurrence relations. In Section 4, we prove the recurrence relations. In the conclusion, we briefly discuss the relative computational efficiency of the methods for determining the number of binomial coefficients exactly divided by a fixed power of a prime.

2. Subprime Factorization

We used the notion of subprime (*sub*scripted *prime* symbol) factorization to obtain general formula (1). The first few natural numbers can be represented with ordinary prime factorization as 1, 2, 3, 2^2, 5, $2 \cdot 3$, 7, 2^3, 3^2. With subprime factorization, we can represent these numbers as 1, 2_1, 3_1, $2_1 2_2$, 5_1, $2_1 3_1$, 7_1, $2_1 2_2 2_3$, $3_1 3_2$. We emphasize that a subprime is not a number and we do not actually perform arithmetic operations with subprimes. This notion seems unnecessary and useless when looking at the sequence of natural numbers, but it is quite useful when considering the question of how many binomial coefficients are exactly divided by a given power of a prime. The question reduces to considering *which subprimes appear* in the subprime factorization of the binomial coefficients because *there are no powers of subprimes.* And eliminating "powers" drastically simplifies the problem.

The following facts are well known and easily established. These properties are more often stated in terms of Pascal's triangle modulo a prime. The statement $p \mid \binom{n}{k}$ and $\binom{n}{k} \cong 0 \bmod p$ are obviously equivalent.

P1. If $n < p$, then $p \nmid \binom{n}{k}$ for any k.

If p is a prime factor of $n!$, then $p \leq n$. For $n < p$, $p \mid \binom{n}{k}$ would then imply the contradiction $p < p$.

P2. For $0 < k < n+1$, if $p \mid \binom{n}{k-1}$ and $p \mid \binom{n}{k}$, then $p \mid \binom{n+1}{k}$, and if $p \nmid \binom{n}{k-1}$ and $p \nmid \binom{n}{k}$, then $p \nmid \binom{n+1}{k}$ *except* if $n+1$ is a multiple of p.

The first statement holds because multiplication is distributive over addition. The second statement is equivalent to if $\binom{n}{k-1} \ncong 0 \bmod p$ and $\binom{n}{k} \ncong 0 \bmod p$ and $\binom{n+1}{k} \cong 0 \bmod p$, then $n+1 \cong 0 \bmod p$. Now, $\binom{n}{k-1} \ncong 0 \bmod p$ and $\binom{n}{k} \ncong 0 \bmod p$ and $\binom{n+1}{k} \cong 0 \bmod p$ implies $\binom{n}{k-1} \cong a \bmod p$ and $\binom{n}{k} \cong p - a \bmod p$ for some a, $0 < a < p$. But such a pair of adjacent binomial coefficients occurs in Pascal's triangle modulo p only in a row n for which $n \cong p-1 \bmod p$.

P3. If n is a multiple of p^i for $i > 0$, i.e., $n = mp^i$, $0 < m \leq p$, then $p \mid \binom{n}{k}$ for all $0 < k < n$ such that k is not a multiple of p^i *except* if n is a power of p, in which case $p \mid \binom{n}{k}$ for *all* $0 < k < n$.

This property can be established by direct calculation of the powers of p in the prime factorization of $n!$, $k!$, and $(n-k)!$ in the relevant cases for $m < p$ and $m = p$.

We take corresponding statements to define the subprime factorization of the binomial coefficients. The basic differences are that p is replaced with $p_1 \dots p_i$ in the condition of the first fact and that "divides" and "is a multiple" are replaced with "appears in the subprime factorization of" and "the subprime factorization of n contains."

S1. If $n < p_1 \dots p_i$, then p_i does not appear in the subprime factorization of $\binom{n}{k}$ for any k.

S2. For $0 < k < n+1$ and $n+1$ not a multiple of p, p_i appears in the subprime factorization of $\binom{n+1}{k}$ if and only if p_i appears in the subprime factorizations of both $\binom{n}{k-1}$ and $\binom{n}{k}$.

S3. If the subprime factorization of n contains $p_1 \dots p_i$, then p_i appears in the subprime factorization of $\binom{n}{k}$ for all $0 < k < n$ such that $p_1 \dots p_i$ does not appear in the subprime factorization of k *with no exceptions.*

The absence of exceptions in S2 and S3 simplifies things.

We consider representations of Pascal's triangle. In the limit of an infinite number of rows, Pascal's triangle of binomial coefficients modulo a prime is self-similar; for

$p = 2$, it has the fractal dimension $\log_2 3$ (see, e.g., [10]). The "prime divisibility" function

$$f_p\left(\binom{n}{k}\right) = \begin{cases} 1 & \text{if } \binom{n}{k} = 0 \mod p, \\ 0 & \text{if } \binom{n}{k} \neq 0 \mod p, \end{cases}$$

maps the self-similar Pascal's triangle modulo p into the corresponding self-similar p-divisibility triangle. Passing from prime to subprime factorization converts the self-similarity to self-congruency in a sense. The fractal object becomes a set of regular tilings. Taking $p = 2$ as an example, the first few rows of the 2-divisibility triangle are

Row	
0	0
1	00
2	010
3	0000
4	01110
5	001100
6	0101010
7	00000000
8	011111110
9	0011111100
10	01011111010
11	000011110000
12	0111011101110
13	00110011001100
14	010101010101010
15	0000000000000000.

Here, the rows 0 and 1 correspond to P1, even numbered rows correspond to P3, and the other rows correspond to P2. The corresponding subprime-divisibility triangles for 2_1, 2_2, and 2_3 are

Row	2_1	2_2	2_3
0	0	0	0
1	00	00	00
2	010	000	000
3	0000	0000	0000
4	01010	01110	00000
5	000000	001100	000000
6	0101010	0001000	0000000
7	00000000	00000000	00000000
8	010101010	011101110	011111110
9	0000000000	0011001100	0011111100
10	01010101010	00010001000	00011111000
11	000000000000	000000000000	000011110000
12	0101010101010	0111011101110	0000011100000
13	00000000000000	00110011001100	00000011000000
14	010101010101010	000100010001000	000000010000000
15	0000000000000000	0000000000000000	0000000000000000.

In these subprime-divisibility triangles, the rows numbered 0 through $2^i - 1$ follow from S1, rows that are multiples of 2^i follow from S3, and all other rows follow from S2.

We give a simple example of observations that can follow from the subprime factorization of Pascal's triangle. If we replace $p^i \to p_1 p_2 \ldots p_i$ for all primes in the factorization of the natural numbers, then 2_1 divides every second number exactly once, 2_2 divides every fourth number exactly once, 2_3 divides every eighth number exactly once (here and hereafter, we use terms like "divide" as shorthand for "appears in the subprime factorization of"). This does not seem to gain us much beyond eliminating the need to keep track of the powers in the ordinary prime factorization. But if we define the triangular numbers as the sequence of binomial coefficients $\binom{n}{2}$, $n = 2, 3, \ldots$, then we see from the 2_i-divisibility triangles that 2_1 does not divide any triangular number, that 2_2 divides the triangular numbers for n of the forms $4m$ and $4m+1$, that 2_3 divides the triangular numbers for n of the forms $8m$ and $8m+1$, and so on. It is already visually obvious that p_i for $p+i > 3$ divides triangular numbers thus defined for n of the forms $p^i m$ and $p^i m + 1$. And it is clear why 2_1 is the only exception. The same kind of reasoning can be applied to the sequence of tetrahedral numbers defined as $\binom{n}{3}$, $n = 3, 4, \ldots$, and so on.

We can view the subprime-divisibility triangles as tilings with a triangle A of zeros and a triangle V of ones as the two basic tile types. Viewed as triangles consisting of zeros and ones, the subprime-divisibility triangles differ for different subprimes (and for different primes), but they are basically identical when viewed as tilings. The kth tile row, $k \geq 0$, consists of a triangle A followed by k copies of the rhombus formed by concatenating a triangle V and a triangle A.

If we take one element (0 or 1) as the unit length, then the tilings differ in scale (the size of the tiles). If we take a characteristic tile length (p^i for example) as the unit length, then the tilings only differ in resolution (elements per unit length). If we plotted the triangle for 2_9 at a scale of one inch equals the tile length with a white pixel for 0 and a black pixel for 1, then the "line" between the triangles A and V would look very much like a straight line at 512 ppi (pixels per inch). Moreover, if we plotted the similar triangle for 5_4 at a scale of one inch equals the tile length, then it would be difficult to distinguish the picture at 512 ppi from the picture at 625 ppi. In other words, we can view the subprime-divisibility triangles as all the same tilings except for their resolution. In the limit as $p^i \to \infty$, the basic tiles form a perfect square with the lower-left white triangle equal in area to the upper-right black triangle (we take the diagonal of the square to belong to the lower-left triangle, but this line has no area). In the discrete case (the digitized picture), the triangle A has the "area" $\sum_{k=1}^{p^i} k$ and the triangle V has the "area" $\sum_{k=1}^{p^i-1} k$ (and the difference in area is the number p^i of elements on the diagonal).

Letting $\sqcup$ denote concatenation, we can write $\mathrm{R} = \mathrm{V} \sqcup \mathrm{A}$ for the rhombus formed by concatenating a triangle V and a triangle A. Similarly, kR, for example, denotes the concatenation of k copies of R. We indicate the size of of these objects with subscripts. For example, A_{p^i} is the triangle A in the subprime-divisibility triangle for p_i, also called the p_i-divisibility triangle. The row lengths (number of zeros) of A_{p^i} from top to bottom range from 1 to p^i. The row lengths (number of ones) of the triangle V_{p^i} from top to bottom range from $p^i - 1$ to 0 (it is convenient to allow a row of zero length).

Viewing the p_i-divisibility triangle as a tiling, we can easily see that the region from p^i to p^{i+1} contains p tile rows $\mathrm{A}_{p^i} \sqcup k\mathrm{R}_{p^i}$, $0 \leq k < p$. The kth tile row in this region "overlays" the rows of Pascal's triangle numbered by $kp^i \leq n < (k+1)p^i$. Furthermore, it is easily seen that $\mathrm{A}_{p^{i+1}}$ and $\mathrm{V}_{p^{i+1}}$ respectively correspond to p tile rows of the p_i-divisibility triangle as $\mathrm{A}_{p^i} \sqcup k\mathrm{R}_{p^i}$ and $(p-k-1)\mathrm{R}_{p^i} \sqcup \mathrm{V}_{p^i}$, $0 \leq k < p$. As an example, we show the tiles A_5 and V_5 (with omitted subscripts) in the range $25 \leq n < 50$ with the separations between the corresponding A_{25} and V_{25} indicated by spaces and with the range of n shown for each tile row:

25 to 29	A	VAVAVAVAV	A
30 to 34	AVA	VAVAVAV	AVA
35 to 39	AVAVA	VAVAV	AVAVA
40 to 44	AVAVAVA	VAV	AVAVAVA
45 to 49	AVAVAVAVA	V	AVAVAVAVA.

This example is obviously generalizable to any prime p: we simply have p tile rows and p V triangles in the first of those rows.

We now consider some simple summaries of the information contained in the set of subprime-divisibility triangles for a given prime p. Treating the elements zero and one of the subprime-divisibility triangles as Boolean truth values and sum these triangles using Boolean addition, then we obtain the self-similar prime-divisibility triangle. If we treat the elements of the subprime-divisibility triangles as ordinary integers and sum these triangles using ordinary addition, we obtain a triangle of the degrees of the highest powers of the prime dividing the binomial coefficients. For example, summing the subprime-divisibility triangles for 2_i, $i = 1, 2, 3$, we obtain

Row	
0	0
1	00
2	010
3	0000
4	02120
5	001100
6	0102010
7	00000000
8	032313230
9	0022112200
10	01031213010
11	000011110000
12	0212032302120
13	00110022001100
14	010101030101010
15	0000000000000000.

Formula (1) is a slightly more complicated summary of information from the set of subprime-divisibility triangles. We illustrate the statement at the beginning of this section that the question of how many binomial coefficients are exactly divided by a given power of a prime reduces to considering *which subprimes appear* in the subprime factorization of the binomial coefficients. We give some specific terms $F(\mathbf{w})L(\mathbf{w})\prod_{i=1}^{r-1} M(\mathbf{w},i)$ in the summation in (1) for large r and $j=2$: for $\mathbf{w}=110\ldots$, the term

$$(p-c_0-1)(c_r+1)(p-c_1)c_2(c_3+1)\cdots(c_{r-1}+1)$$

counts the number of $\binom{n}{k}$ that contain p_1 and p_2 and no other p_i (this term is zero if $c_0=p-1$ or $c_2=0$); for $\mathbf{w}=1010\ldots$, the term

$$(p-c_0-1)(c_r+1)c_1(p-c_2-1)c_3(c_4+1)\cdots(c_{r-1}+1)$$

counts the number of $\binom{n}{k}$ that contain p_1 and p_3 and no other p_i (this term is zero if $c_0=p-1$ or $c_1=0$ or $c_2=p-1$ or $c_3=0$); and for $\mathbf{w}=0110\ldots$, the term

$$(c_0+1)(c_r+1)(p-c_1-1)(p-c_2)c_3(c_4+1)\cdots(c_{r-1}+1)$$

counts the number of $\binom{n}{k}$ that contain p_2 and p_3 and no other p_i (this term is zero if $c_1=p-1$ or $c_3=0$). Adding these terms does not strictly make sense from the subprime standpoint (it is like adding a count of apples and a count of oranges and a count of pears). But passing from subprime factorization back to ordinary prime factorization allows summing the terms (somewhat like considering apples, oranges, and pears to be fruits).

In summary, the notion of subprime factorization simplifies problems concerning powers of primes that divide binomial coefficients by, first, passing from primes and powers of primes to an infinite set of subprimes *without powers* (identical except for scaling) with each subprime associated with a specific power of the prime and, second, passing from a self-similar "fractal object" to an infinite set of regular tilings (identical except for scaling and resolution). Such an approach might be applicable or adaptable to other problems.

3. Examples

We give a few simple examples to help in visualizing the formulas. We take $p=5$ and consider a few four-digit numbers ($r=3$) in base 5. We note that the Θ sequences have a finite positive portion: there exists a nonnegative integer k such that $\theta_k(n,p)>0$ and $\theta_j(n,p)=0$ for all $j>k$. For brevity in the numerical examples, we let $\langle\theta_0(n,p),\ldots,\theta_k(n,p)\rangle$ denote the finite positive portion of the sequence $\Theta(n,p)$.

Example. $n = 586 = 4321_5$. To determine $\Theta(586, 5) \equiv \Theta(4321_5)$, we must calculate $\Theta(1321_5)$, $\Theta(321_5)$, $\Theta(121_5)$, and $\Theta(21_5)$ using the general recurrence relations and $\Theta(11_5)$ and $\Theta(1)$ using the base formulas. From the base formulas, we obtain

$$\Theta(11_5) \overset{(3)}{=} \langle 4, 3\rangle, \qquad \Theta(1) \overset{(2)}{=} \langle 2\rangle.$$

From the general recurrence relations, we obtain the additional intermediate values

$$\begin{aligned}
\Theta(21_5) &\overset{(5)}{=} 2\Theta(11_5) - \Theta(1) = 2\langle 4, 3\rangle - \langle 2\rangle = \langle 6, 6\rangle,\\
\Theta(121_5) &\overset{(4)}{=} 2\Theta(21_5) + 3\vec{\Theta}(11_5) - 4\vec{\Theta}(1)\\
&= 2\langle 6, 6\rangle + 3\langle 0, 4, 3\rangle - 4\langle 0, 2\rangle = \langle 12, 16, 9\rangle,\\
\Theta(321_5) &\overset{(5)}{=} 3\Theta(121_5) - 2\Theta(21_5)\\
&= 3\langle 12, 16, 9\rangle - 2\langle 6, 6\rangle = \langle 24, 36, 27\rangle,\\
\Theta(1321_5) &\overset{(4)}{=} 2\Theta(321_5) + 2\vec{\Theta}(121_5) - 3\vec{\Theta}(21_5)\\
&= 2\langle 24, 36, 27\rangle + 2\langle 0, 12, 16, 9\rangle - 3\langle 0, 6, 6\rangle = \langle 48, 78, 68, 18\rangle.
\end{aligned}$$

We then obtain

$$\begin{aligned}
\Theta(4321_5) &\overset{(5)}{=} 4\Theta(1321_5) - 3\Theta(321_5)\\
&= 4\langle 48, 78, 68, 18\rangle - 3\langle 24, 36, 27\rangle = \langle 120, 204, 191, 72\rangle,
\end{aligned}$$

i.e., $\theta_0(586, 5) = 120$, $\theta_1(586, 5) = 204$, $\theta_2(586, 5) = 191$, and $\theta_3(586, 5) = 72$.

Example. $n = 156 = 1111_5$. In addition to the same two base values as in the above example, we must calculate the intermediate value

$$\begin{aligned}
\Theta(111_5) &\overset{(4)}{=} 2\Theta(11_5) + 4\vec{\Theta}(11_5) - 5\vec{\Theta}(1)\\
&= 2\langle 4, 3\rangle + 4\langle 0, 4, 3\rangle - 5\langle 0, 2\rangle = \langle 8, 12, 12\rangle.
\end{aligned}$$

We then obtain

$$\begin{aligned}
\Theta(1111_5) &\overset{(4)}{=} 2\Theta(111_5) + 4\vec{\Theta}(111_5) - 5\vec{\Theta}(11_5)\\
&= 2\langle 8, 12, 12\rangle + 4\langle 0, 8, 12, 12\rangle - 5\langle 0, 4, 3\rangle = \langle 16, 36, 57, 48\rangle,
\end{aligned}$$

i.e., $\theta_0(156, 5) = 16$, $\theta_1(156, 5) = 36$, $\theta_2(156, 5) = 57$, and $\theta_3(156, 5) = 48$.

Example. $n = 149 = 1044_5$. We note that $149 = 5^2 \cdot 6 - 1$. Therefore, we can use the special recurrence relation $\Theta(149, 5) \overset{(6)}{=} 25\Theta(5, 5)$. In this case, we obtain

$$\Theta(1044_5) \overset{(6)}{=} 25\Theta(10_5) = 25\langle 2, 4\rangle = \langle 50, 100\rangle,$$

i.e., $\theta_0(149, 5) = 50$ and $\theta_1(149, 5) = 100$.

4. Proofs of the Recurrence Relations

In proving Eqs. (2)–(6), in addition to the A_{p^i} notation in Section 2, we use subscripts indicating the the first and last rows overlaid by the object. For example, $\mathrm{A}_{2p^i,3p^i-1} = \mathrm{A}_{p^i}$. We also use the notion of a "restricted" $\Theta(n,p)\big|_{\text{object}}$ containing the counts for the portion of the nth row of Pascal's triangle "overlaid" by the object. Thus, for $p \le n < 2p$, we can write

$$\begin{aligned}\Theta(n,p) \equiv \Theta(n,p)\big|_{\mathrm{A}_p \sqcup \mathrm{V}_p \sqcup \mathrm{A}_p} &= \Theta(n,p)\big|_{\mathrm{A}_p} + \Theta(n,p)\big|_{\mathrm{V}_p} + \Theta(n,p)\big|_{\mathrm{A}_p} \\ &= 2\Theta(n,p)\big|_{\mathrm{A}_p} + \Theta(n,p)\big|_{\mathrm{V}_p}.\end{aligned}$$

Proof of base formula (2). Base formula (2) follows directly from fact P1 in Section 2 because the number of binomial coefficients in the nth row of Pascal's triangle is $n+1$. □

Proof of base formula (3). For numbers $n = p + c_0$, we have

$$\Theta(n,p) = 2\Theta(n,p)\big|_{\mathrm{A}_{p,2p-1}} + \Theta(n,p)\big|_{\mathrm{V}_{p,2p-1}}.$$

Because $\mathrm{A}_{p,2p-1} = \mathrm{A}_{0,p-1}$, the first term in the right-hand side is twice the sequence given by base formula (2). From the discussion in Section 2, p divides every binomial coefficient overlaid by $\mathrm{V}_{p,2p-1} \equiv \mathrm{V}_p$. Therefore, the first term in $\Theta(n,p)\big|_{\mathrm{V}_p}$ must be zero. No p^i divides any $\binom{n}{k}$ for $i > 1$ because $n < p^i$. Therefore, only the second term in $\Theta(n,p)\big|_{\mathrm{V}_p}$ is nonzero, and the value of this term must be $n+1-2(c_0+1) = p - c_0 - 1$. Base formula (3) is proved. □

Proof of general recurrence relation (4). We first examine the objects that overlay the rows of Pascal's triangle from p^r to $2p^r - 1$. These rows are overlaid by

$$\mathrm{A}_{p^r,2p^r-1} \sqcup \mathrm{V}_{p^r,2p^r-1} \sqcup \mathrm{A}_{p^r,2p^r-1}.$$

From the discussion in Section 2, $\Theta(n,p)\big|_{\mathrm{A}_{p^r,2p^r-1}}$ is clearly identical to $\Theta(n-p^r,p)$. Hence, $\Theta(n,p) = 2\Theta(n-p^r,p) + \Theta(n,p)\big|_{\mathrm{V}_{p^r,2p^r-1}}$.

We examine $\mathrm{V}_{p^r,2p^r-1}$ in relation to the subprime-divisibility triangles. The portion of the triangle of the degrees of the highest powers of p dividing the binomial coefficients overlaid by $\mathrm{V}_{p^r,2p^r-1}$ contains the arithmetic sums of the corresponding portions of the subprime-divisibility triangles for p_1 through p_{r-1} plus one ($\mathrm{V}_{p^r,2p^r-1}$ contains only ones in the subprime-divisibility triangle for p_r). The "plus one" for all the powers of p in this overlaid region corresponds to the shift operation on sequences. This region contains p^{r-1} rows corresponding to $(p-1)\mathrm{R}_{p^{r-1},2p^{r-1}-1} \sqcup \mathrm{V}_{p^{r-1},2p^{r-1}-1}$ followed by p^{r-1} rows corresponding to

$$(p-2)\mathrm{R}_{p^{r-1},2p^{r-1}-1} \sqcup \mathrm{V}_{p^{r-1},2p^{r-1}-1},$$

and so on down to the last p^{r-1} rows corresponding to $\mathrm{V}_{p^{r-1},2p^{r-1}-1}$. The top set of rows corresponds to $p^r \leq n < p^r + p^{r-1}$, the second set corresponds to $p^r + p^{r-1} \leq n < p^r + 2p^{r-1}$, and so on. Altogether, the p sets of p^{r-1} rows correspond to $p^r \leq n < 2p^r$. We now have

$$\begin{aligned}\Theta(n,p)\big|_{\mathrm{V}_{p^r,2p^r-1}} &= (p-c_{r-1}-1)\vec{\Theta}(n-p^r-(c_{r-1}-1)p^{r-1},p)\big|_{\mathrm{R}_{p^{r-1},2p^{r-1}-1}} \\ &\quad + \vec{\Theta}(n-p^r-(c_{r-1}-1)p^{r-1},p)\big|_{\mathrm{V}_{p^{r-1},2p^{r-1}-1}}. \end{aligned} \tag{7}$$

We now seek the restricted Θ in terms of unrestricted Θ. Obviously,

$$\begin{aligned}\Theta(n-p^r-(c_{r-1}-1)p^{r-1},p) &= \Theta(n-p^r-(c_{r-1}-1)p^{r-1},p)\big|_{\mathrm{A}_{p^{r-1},2p^{r-1}-1}} \\ &\quad + \Theta(n-p^r-(c_{r-1}-1)p^{r-1},p)\big|_{\mathrm{R}_{p^{r-1},2p^{r-1}-1}} \\ &= 2\Theta(n-p^r-(c_{r-1}-1)p^{r-1},p)\big|_{\mathrm{A}_{p^{r-1},2p^{r-1}-1}} \\ &\quad + \Theta(n-p^r-(c_{r-1}-1)p^{r-1},p)\big|_{\mathrm{V}_{p^{r-1},2p^{r-1}-1}}.\end{aligned}$$

Moreover, we have

$$\Theta(n-p^r-(c_{r-1}-1)p^{r-1},p)\big|_{\mathrm{A}_{p^{r-1},2p^{r-1}-1}} = \Theta(n-p^r-c_{r-1}p^{r-1},p).$$

Therefore,

$$\begin{aligned}&\Theta(n-p^r-(c_{r-1}-1)p^{r-1},p)\big|_{\mathrm{R}_{p^{r-1},2p^{r-1}-1}} \\ &\qquad = \Theta(n-p^r-(c_{r-1}-1)p^{r-1},p) - \Theta(n-p^r-c_{r-1}p^{r-1},p)\end{aligned}$$

and

$$\begin{aligned}&\Theta(n-p^r-(c_{r-1}-1)p^{r-1},p)\big|_{\mathrm{V}_{p^{r-1},2p^{r-1}-1}} \\ &\qquad = \Theta(n-p^r-(c_{r-1}-1)p^{r-1},p) - 2\Theta(n-p^r-c_{r-1}p^{r-1},p).\end{aligned}$$

Substituting the expressions with unrestricted Θ in (7) and combining "like" terms, we obtain general recurrence relation (4). □

Proof of general recurrence relation (5). It is now easy to prove general recurrence relation (5). Clearly, for $r > 0$ and $c_r > 1$,

$$\begin{aligned}\Theta(n,p) &= \Theta(n-(c_r-1)p^r,p)|_{\mathrm{A}_{p^r,2p^1-1}} + c_r\Theta(n-(c_r-1)p^r,p)|_{\mathrm{R}_{p^r,2p^1-1}} \\ &= (c_r+1)\Theta(n-(c_r-1)p^r,p)|_{\mathrm{A}_{p^r,2p^1-1}} + c_r\Theta(n-(c_r-1)p^r,p)|_{\mathrm{V}_{p^r,2p^1-1}}.\end{aligned} \tag{8}$$

Similarly to the preceding proof, we have

$$\Theta(n-(c_r-1)p^r,p)|_{\mathrm{V}_{p^r,2p^1-1}} = \Theta(n-(c_r-1)p^r,p) - 2\Theta(n-c_rp^r,p)$$

and

$$\Theta(n-(c_r-1)p^r,p)|_{\mathrm{A}_{p^r,2p^1-1}} = \Theta(n-c_rp^r,p).$$

Substituting the expressions with unrestricted Θ in (8) and combining "like" terms, we obtain general recurrence relation (5). □

Proof of special recurrence relation (6). It is clear from the discussion in Section 2 that the nth row, $n \geq p^i - 1$, of the p_i-divisibility triangle contains only zeros if and only if $p^i \mid n+1$. It follows that for $n \geq p^{i+1}-1$, if the nth row of the p_i-divisibility triangle is not all zeros, then the nth row of the p_{i+1}-divisibility triangle is not all zeros. Because $\theta_r(n,p)$ is a count of the number of positions containing 1 in the nth rows of all the subprime-divisibility triangles for p_1 through p_r, $\theta_r(n,p) > 0$ if and only if $p \nmid n+1$. Similarly, $\theta_{r-1}(n,p) > 0$ if and only if $p^2 \nmid n+1$. And so on. It remains to show that if $\theta_r(n,p) = 0$, then $p \mid \theta_j(p,n)$ for $0 \leq j < r$, and this easily follows by induction on r. Noting that the applicability condition $n = p^k m - 1$ for special recurrence relation (6) is equivalent to $p^k \mid n+1$, we have proved relation (6). □

Remark. From the foregoing, it is clear that if $\theta_j(n,p) = 0$, then $\theta_k(n,p) = 0$ for all $k \geq j$. Moreover, $\theta_1(n,p) = 0$ if and only if n has the form $cp^s - 1$, where $0 < c < p$ and s is a nonnegative integer; $\theta_2(n,p) = 0$ if and only if n has the form $c_rp^r + cp^{r-1} - 1$, where $r > 1$; and so on.

5. Conclusion

We briefly consider the computational efficiency of different ways of computing $\Theta(n,p)$ by looking at the basic rate of increase in the number of computational steps as a function of increasing r. Traditionally, $\Theta(n,p)$ has been computed by counting the number of carries when adding $n-k$ and k in base p (Kummer's formula for the degree of the highest power of p that divides $\binom{n}{k}$ [7]). To compute $\Theta(n,p)$, we sum $n-1$ pairs (there is no point in summing for $\binom{n}{0}$ and $\binom{n}{n}$). This means summing $c_rp^r + \cdots + c_0 - 1$ pairs, and the basic rate of increase is as p^r.

General formula (1) for $\theta_j(n,p)$, which follows because the subprime factorization of Pascal's triangle of binomial coefficients gives essentially the same regular binary tiling for all powers of all primes and the tiles only differ in scale and resolution, reduces the basic rate of increase from p^r to $\sum_{i=0}^{r}\binom{r}{i} = 2^r$.

Using the general recurrence relations in Theorem 1 in the general case with $n = c_0+c_1p+\cdots+c_rp^r$, we calculate one value for $\Theta(c_0,p)$, two values for $\Theta(c_0+p,p)$, two values for $\Theta(c_0+c_1p,p)$, …, $r+1$ values for $\Theta(c_0+c_1+\cdots+p^r,p)$, and $r+1$ values for $\Theta(c_0+c_1p+\cdots+c_rp^r,p)$. In all, the number of intermediate values calculated before calculating the final $r+1$ values is r^2+2r-1. The number of computational steps for calculating each value is obviously bounded by a small positive constant independent of r. Therefore, the basic rate of increase in computational steps is as r^2, which is an improvement over 2^r.

References

[1] L. Carlitz. The number of binomial coefficients divisible by a fixed power of a prime. *Rend. Circ. Mat. Palermo (2)*, **16** (1967), 299–320.

[2] W. B. Everett. Number of binomial coefficients divisible by a fixed power of a prime. *Integers*, **8**, (2008), A11.

[3] N. J. Fine. Binomial coefficients modulo a prime. *Amer. Math. Monthly*, **54** (1947), 589–592.

[4] J. W. L. Glaisher. On the residue of a binomial-theorem coefficient with respect to a prime modulus. *Quart. J. Math.*, **30** (1899), 150–156.

[5] F. T. Howard. The number of binomial coefficients divisible by a fixed power of 2. *Proc. Amer. Math. Soc.*, **29** (1971), 236–242.

[6] F. T. Howard. Formulas for the number of binomial coefficients divisible by a fixed power of a prime. *Proc. Amer. Math. Soc.*, **37** (1973), 358–362.

[7] E. E. Kummer. Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen. *J. Reine Angew. Math.*, **44** (1852), 93–146.

[8] E. S. Rowland. The number of nonzero binomial coefficients modulo p^α. arXiv:1001.1783v2.

[9] V. Shevelev. Binomial coefficient predictors. *J. Integer Sequences*, **14** (2011), 11.2.8; Binomial predictors. arXiv:0907.3302v4.

[10] S. Wolfram. Geometry of binomial coefficients. *Amer. Math. Monthly*, **91** (1984), 566–571. http://www.stephenwolfram.com/publications/articles/ mathematics/84-geometry/1/text.html.