A NOVEL APPROACH TO THE DISCOVERY OF BINARY BBP-TYPE FORMULAS FOR POLYLOGARITHM CONSTANTS

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Abstract

Using a clear and straightforward approach, we discover and prove new binary digit extraction BBP-type formulas for polylogarithm constants. Some known results are also rediscovered in a more direct and elegant manner. Numerous experimentally discovered and hitherto unproved binary BBP-type formulas are also proved.

1. Introduction

This paper is concerned with proving each of a lengthy list of conjectured binary BBP-type formulas collected in the "Compendium of BBP-Type Formulas", an online collection of BBP-type formulas for various mathematical constants [1]. The formulas have, in most cases, been outstanding for upwards of fifteen years, in spite of many thousands of downloads of the BBP Compendium. New binary BBP-type formulas, together with their proofs will also be derived.

BBP-type formulas are formulas of the form

$$c = \sum_{k=0}^{\infty} 1/b^k \sum_{j=1}^{l} a_j/(kl+j)^s$$

where s, b, l and a_j are integers, and c is some constant. Formulas of this type were first introduced in a 1996 paper [2], where a formula of this type for π was given. Such formulas allow digit extraction — the i-th digit of a mathematical constant c in base b can be calculated directly, without needing to compute any of the previous $i - 1$ digits, by means of simple algorithms that do not require multiple-precision arithmetic [1].

Apart from digit extraction, another reason the study of BBP-Type formulas has continued to attract attention is that BBP-Type constants are conjectured to be

either rational or normal to base b [3, 4, 5]; that is, their base-b digits are randomly distributed.

BBP-Type formulas are usually discovered experimentally, through computer searches, by using Bailey and Ferguson's PSLQ (Partial Sum of Squares – Lower Quadrature) algorithm [6] or its variations. PSLQ and other integer relation finding schemes typically do not suggest proofs [4, 7]. Formal proofs must be developed after the formulas have been discovered.

Our approach in this paper is the systematic symbolic (that is, non-computer-search-based) discovery of BBP-type formulas. The methods used here aim to complement the experimental approaches that have dominated the area. Through fundamental methods, a wide range of interesting, mostly new, BBP-type formulas will be obtained, together with their proofs. It should be noted that in this paper, unlike in [2] or [8], for example, no evaluation of complicated integrals is necessary. The BBP-type formulas come as natural consequences of the corresponding polylogarithm identities.

2. Notation

Degree s ($\in \mathbb{Z}^+$ in this paper) polylogarithm functions Li_s are defined by

$$\text{Li}_s[z] = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \quad |z| \leq 1\,.$$

In particular, for $|z| = 1$ and $x \in \mathbb{R}$ we have

$$\begin{aligned}
\text{Li}_{2n}[e^{ix}] &= \text{Gl}_{2n}(x) + i\text{Cl}_{2n}(x) \\
\text{Li}_{2n+1}[e^{ix}] &= \text{Cl}_{2n+1}(x) + i\text{Gl}_{2n+1}(x)\,,
\end{aligned} \tag{1}$$

where Gl and Cl are Clausen sums [9] defined, for $n \in \mathbb{Z}^+$ by

$$\begin{aligned}
\text{Cl}_{2n}(x) &= \sum_{k=1}^{\infty} \frac{\sin kx}{k^{2n}}, \quad \text{Cl}_{2n+1}(x) = \sum_{k=1}^{\infty} \frac{\cos kx}{k^{2n+1}} \\
\text{Gl}_{2n}(x) &= \sum_{k=1}^{\infty} \frac{\cos kx}{k^{2n}}, \quad \text{Gl}_{2n+1}(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^{2n+1}}\,.
\end{aligned} \tag{2}$$

We shall find the following formulas useful:

$$\text{Gl}_{2n}(x) = (-1)^{1+[n/2]} 2^{n-1} \pi^n \text{B}_n(x/2\pi)/n!$$

$$\frac{1}{m^{n-1}} \text{Cl}_n(mx) = \sum_{r=0}^{m-1} \text{Cl}_n(x + 2\pi r/m). \tag{3}$$

Here $[n/2]$ denotes the integer part of $n/2$ and B_n are the Bernoulli polynomials defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{\text{B}_n(x)t^n}{n!}.$$

In order to save space, we will give the BBP-type formulas using the compact P-notation [1]:

$$\sum_{k=0}^{\infty} \frac{1}{b^k} \sum_{j=1}^{l} \frac{a_j}{(kl+j)^s} \equiv P(s, b, l, A),$$

where s, b and l are integers, and $A = (a_1, a_2, \ldots, a_l)$ is a vector of integers.

3. Scheme for Obtaining the BBP-Type Formulas

The derivation of a desired degree s BBP-type formula proceeds in two stages

1. An attempt is made to express the polylogarithm constant, c, of interest as a linear combination of the real or imaginary parts of polylogarithms:

$$c = \sum_j \{\alpha_j \, \text{Re} \, \text{Li}_s \, [p_j \exp(ix_j)]\} \tag{4}$$

or

$$c = \sum_j \{\beta_j \, \text{Im} \, \text{Li}_s \, [q_j \exp(iy_j)]\} \tag{5}$$

for $\alpha_j, \beta_j \in \mathbb{Q}$, $p_j, q_j \in (0, 1)$ and x_j, y_j rational multiples of π.

2. The identities

$$\text{Re} \, \text{Li}_s \, [pe^{ix}] = \sum_{k=1}^{\infty} \frac{p^k \cos kx}{k^s} \tag{6}$$

and

$$\operatorname{Im} \operatorname{Li}_s \left[p e^{ix} \right] = \sum_{k=1}^{\infty} \frac{p^k \sin kx}{k^s} , \tag{7}$$

for $p \in [0,1]$, $x \in [0, 2\pi]$ and $s \in \mathbb{Z}^+$, for the real and imaginary parts of a polylogarithm function are then employed to write each constituent term of the linear combination as a BBP-type formula and the indicated combination is then formed. In particular if a binary formula is sought, then p_j and q_j in the above formulas must be taken as positive integral powers of $1/2$ for $x = \pi$, $x = \pi/2$ or $x = \pi/3$ (odd positive integral powers of $1/\sqrt{2}$ for $x = \pi/4$ or $x = 3\pi/4$).

To accomplish the first stage, polylogarithm functional equations are evaluated at certain carefully chosen coordinates and real and imaginary parts are taken. Sometimes it will be necessary to simultaneously solve two or more equations that couple several polylogarithm constants.

As a concrete example of how Eqs. (6) and (7) give rise to BBP-type formulas, consider the choice $p = 1/\sqrt{2}^q$ and $x = \pi/4$ in Eq. (6) for $q \in \mathbb{Z}^+$ and $\mod (q,2) = 1$. This choice, together with the periodicity of $\cos(\pi/4)$ yield the very general BBP-type formula (with q a positive odd integer):

$$
\begin{aligned}
\operatorname{Re} \operatorname{Li}_s \left[\frac{1}{\sqrt{2}^q} \exp\left(\frac{i\pi}{4} \right) \right] = {} & \frac{1}{2^{12q-s}} P(s, 2^{12q}, 24, (2^{-\frac{1}{2}-\frac{q}{2}+12q}, 0, -2^{-\frac{1}{2}-\frac{q}{2}+11q}, \\
& -2^{10q}, -2^{-\frac{1}{2}-\frac{q}{2}+10q}, 0, 2^{-\frac{1}{2}-\frac{q}{2}+9q}, 2^{8q}, 2^{-\frac{1}{2}-\frac{q}{2}+8q}, 0, \\
& -2^{-\frac{1}{2}-\frac{q}{2}+7q}, -2^{6q}, -2^{-\frac{1}{2}-\frac{q}{2}+6q}, 0, 2^{-\frac{1}{2}-\frac{q}{2}+5q}, 2^{4q}, \\
& 2^{-\frac{1}{2}-\frac{q}{2}+4q}, 0, -2^{-\frac{1}{2}-\frac{q}{2}+3q}, -2^{2q}, -2^{-\frac{1}{2}-\frac{q}{2}+2q}, 0, \\
& 2^{-\frac{1}{2}+\frac{q}{2}}, 1)) .
\end{aligned}
\tag{8}
$$

We also have (with $x = 3\pi/4$)

$$
\begin{aligned}
\operatorname{Re} \operatorname{Li}_s \left[\frac{1}{\sqrt{2}^q} \exp\left(\frac{3i\pi}{4} \right) \right] = {} & \frac{1}{2^{12q-s}} P(s, 2^{12q}, 24, (-2^{-\frac{1}{2}-\frac{q}{2}+12q}, 0, 2^{-\frac{1}{2}-\frac{q}{2}+11q}, \\
& -2^{10q}, 2^{-\frac{1}{2}-\frac{q}{2}+10q}, 0, -2^{-\frac{1}{2}-\frac{q}{2}+9q}, 2^{8q}, -2^{-\frac{1}{2}-\frac{q}{2}+8q}, 0, \\
& 2^{-\frac{1}{2}-\frac{q}{2}+7q}, -2^{6q}, 2^{-\frac{1}{2}-\frac{q}{2}+6q}, 0, -2^{-\frac{1}{2}-\frac{q}{2}+5q}, 2^{4q}, \\
& -2^{-\frac{1}{2}-\frac{q}{2}+4q}, 0, 2^{-\frac{1}{2}-\frac{q}{2}+3q}, -2^{2q}, 2^{-\frac{1}{2}-\frac{q}{2}+2q}, 0, \\
& -2^{-\frac{1}{2}+\frac{q}{2}}, 1)) .
\end{aligned}
\tag{9}
$$

Another example is ($q \in \mathbb{Z}^+$)

$$\mathrm{Li}_s\left[-\frac{1}{2^q}\right] = \mathrm{Re}\,\mathrm{Li}_s\left[\frac{1}{2^q}\exp(i\pi)\right]$$

$$= \frac{1}{2^{12q-s}}P(s, 2^{12q}, 24, (0, -2^{11q}, 0, 2^{10q}, 0, -2^{9q}, 0, 2^{8q}, 0, -2^{7q},$$

$$0, 2^{6q}, 0, -2^{5q}, 0, 2^{4q}, 0, -2^{3q}, 0, 2^{2q}, 0, -2^q, 0, 1)). \tag{10}$$

The reader should note that the series given above are not the only possible ones for the indicated constants; in general, the series used will depend on the particular base and length that are targeted. For instance, a base 2^{4q}, length 8 version of (9), for q an odd positive integer, is

$$\mathrm{Re}\,\mathrm{Li}_s\left[\frac{1}{\sqrt{2}^q}\exp\left(\frac{3i\pi}{4}\right)\right] = \frac{1}{2^{4q}}P(s, 2^{4q}, 8, (-2^{-\frac{1}{2}-\frac{q}{2}+4q}, 0, 2^{-\frac{1}{2}-\frac{q}{2}+3q},$$

$$-2^{2q}, 2^{-\frac{1}{2}-\frac{q}{2}+2q}, 0, -2^{-\frac{1}{2}+\frac{q}{2}}, 1)), \tag{11}$$

while a base 2^{4q}, length 24 version of the same series is

$$\mathrm{Re}\,\mathrm{Li}_s\left[\frac{1}{\sqrt{2}^q}\exp\left(\frac{3i\pi}{4}\right)\right] = \frac{3^s}{2^{4q}}P(s, 2^{4q}, 24, (0, 0, -2^{-\frac{1}{2}-\frac{q}{2}+4q}, 0, 0, 0, 0, 0,$$

$$2^{-\frac{1}{2}-\frac{q}{2}+3q}, 0, 0, -2^{2q}, 0, 0, 2^{-\frac{1}{2}-\frac{q}{2}+2q}, 0, 0, 0,$$

$$0, 0, -2^{-\frac{1}{2}+\frac{q}{2}}, 0, 0, 1)). \tag{12}$$

It is of course possible to give BBP-type formulas in general bases for other classes of polylogarithm constants. This is however not the subject matter of this paper.

By way of a specific illustration of how to derive a BBP-type formula for a polylogarithm constant, let us apply the above procedure to obtain a base 2^{12} length 24 formula for $\log^2 2$. The first step is to express this constant as a linear combination of polylogarithms. This is accomplished through the identity (Eq. (17), Section 5.1)

$$\log^2 2 = 2\,\mathrm{Li}_2\left[-\frac{1}{4}\right] - 4\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right] - 4\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{(\sqrt{2})^3}\exp\left(\frac{\pi i}{4}\right)\right].$$

The next step is to now write each of the three constituent members on the Right Hand Side as a BBP-type formula and then form the indicated combination. This

is easily done by choosing $q = 2 = s$ in Eq. (10), $q = 1, s = 2$ in Eq. (8) and $q = 3$, $s = 2$ in Eq. (11) and the result is

$$\log^2 2 = \frac{1}{2^{10}} P(2, 2^{12}, 24, (2^{11}, 0, -5 \cdot 2^{11}, -7 \cdot 2^{10}, -2^9, 0, 2^8, 7 \cdot 2^8, 5 \cdot 2^8, 0, -2^6, 2^7,$$
$$- 2^5, 0, 5 \cdot 2^5, 7 \cdot 2^4, 2^3, 0, -2^2, -7 \cdot 2^2, -5 \cdot 2^2, 0, 1, -2)).$$

4. Degree 1 Formulas

Degree 1 BBP-type formulas in general bases are discussed in [10]. Binary formulas are easily obtained by choosing bases that are powers of 2. Degree 1 formulas will not be discussed further in this paper.

5. Degree 2 Formulas

5.1. Generators of Degree 2 BBP-Type Formulas

The dilogarithm reflection formula (Eq. A.2.1.7 of [9]) is

$$\frac{\pi^2}{6} - \log x \log(1 - x) = \text{Li}_2[x] + \text{Li}_2[1 - x].$$

Putting $x = 1/2$ in the above formula gives the well-known result:

$$\frac{\pi^2}{12} - \frac{\log^2 2}{2} = \text{Li}_2 \left[\frac{1}{2} \right] \tag{13}$$

A two-variable functional equation for dilogarithms, due to Kummer (Eq. A.2.1.19 of [9]) is

$$\text{Li}_2 \left[\frac{x(1-y)^2}{y(1-x)^2} \right] = \text{Li}_2 \left[-\frac{x(1-y)}{(1-x)} \right] + \text{Li}_2 \left[-\frac{(1-y)}{y(1-x)} \right]$$
$$+ \text{Li}_2 \left[\frac{x}{y} \frac{(1-y)}{(1-x)} \right] + \text{Li}_2 \left[\frac{1-y}{1-x} \right] + \frac{1}{2} \log^2 y. \tag{14}$$

Putting $x = -1$ and $y = 1/2$ in Kummer's formula gives

$$\log^2 2 = 2 \text{Li}_2 \left[-\frac{1}{8} \right] - 4 \text{Li}_2 \left[\frac{1}{4} \right] - 4 \text{Li}_2 \left[-\frac{1}{2} \right], \tag{15}$$

while using $x = \exp(i\pi/3)$ and $y = 1/2$ yields

$$\pi^2 = 72 \, \text{Re} \, \text{Li}_2 \left[\frac{1}{2} \exp \left(\frac{\pi i}{3} \right) \right] - 18 \text{Li}_2 \left[\frac{1}{4} \right]. \tag{16}$$

Putting $x = 1/2$ and $y = \exp(i\pi/2)$ in Kummer's formula gives

$$\log^2 2 = 2\mathrm{Li}_2\left[-\frac{1}{4}\right] - 4\mathrm{Re}\ \mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right] - 4\mathrm{Re}\ \mathrm{Li}_2\left[\frac{1}{\left(\sqrt{2}\right)^3}\exp\left(\frac{\pi i}{4}\right)\right]. \quad (17)$$

Putting $x = -1$ and $y = (1+i)/2$ in Kummer's formula and taking real and imaginary parts give

$$\frac{\pi\log 2}{8} = 2\,\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] - 2\,\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]$$
$$- \mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \quad (18)$$

and

$$\frac{\pi^2}{32} - \frac{1}{8}\log^2 2 = 2\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] + 2\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]$$
$$- \mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]. \quad (19)$$

Another two-variable functional equation for dilogarithms, due to Abel (Eq. A.2.1.16 of [9]) is

$$\mathrm{Li}_2\left[\frac{x}{1-x}\cdot\frac{y}{1-y}\right] = \mathrm{Li}_2\left[\frac{x}{(1-y)}\right] + \mathrm{Li}_2\left[\frac{y}{(1-x)}\right]$$
$$- \mathrm{Li}_2\left[x\right] - \mathrm{Li}_2\left[y\right] - \log(1-x)\log(1-y). \quad (20)$$

Putting $x = i$ and $y = -1$ in Abel's formula and taking real and imaginary parts, gives

$$\frac{5\pi^2}{48} - \frac{1}{2}\log^2 2 = \mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{\left(\sqrt{2}\right)^3}\exp\left(\frac{\pi i}{4}\right)\right]$$
$$- \mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{\pi i}{2}\right)\right] - \mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right] \quad (21)$$

and

$$\mathrm{G} - \frac{\pi\log 2}{4} = \mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{\pi i}{2}\right)\right]$$
$$+ \mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{\left(\sqrt{2}\right)^3}\exp\left(\frac{\pi i}{4}\right)\right] - \mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right]. \quad (22)$$

Putting $x = 1/2$, $y = \exp(i\pi/3)$ in Abel's formula and taking the imaginary part, gives

$$5\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \pi \log 2 = 6\,\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{\pi i}{3}\right)\right].\tag{23}$$

Putting $x = i = y$ in Abel's formula and taking real and imaginary parts, gives

$$\frac{5\pi^2}{48} - \frac{\log^2 2}{4} = \mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{\pi i}{2}\right)\right] - 2\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right]\tag{24}$$

and

$$2\,\mathrm{G} - \frac{\pi \log 2}{4} = 2\,\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right] + \mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{\pi i}{2}\right)\right].\tag{25}$$

5.2. Base 2^{12} Binary BBP-Type Formulas

Solving Equations (21) and (24) simultaneously, we find

$$\begin{aligned}
\log^2 2 = {}& 8\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{\pi i}{2}\right)\right] - 4\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right] \\
& - 4\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{(\sqrt{2})^3}\exp\left(\frac{\pi i}{4}\right)\right]
\end{aligned}\tag{26}$$

and

$$\begin{aligned}
\pi^2 = {}& \frac{144}{5}\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{\pi i}{2}\right)\right] - \frac{144}{5}\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right] \\
& - \frac{48}{5}\,\mathrm{Re}\,\mathrm{Li}_2\left[\frac{1}{(\sqrt{2})^3}\exp\left(\frac{\pi i}{4}\right)\right].
\end{aligned}\tag{27}$$

Solving Equations (22) and (25) simultaneously, we find

$$\mathrm{G} = 3\,\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right] - \mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{(\sqrt{2})^3}\exp\left(\frac{\pi i}{4}\right)\right]\tag{28}$$

and

$$\begin{aligned}
\pi \log 2 = {}& 16\,\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right] - 4\,\mathrm{Im}\,\mathrm{Li}_2\left[\frac{1}{2}\exp\left(\frac{\pi i}{2}\right)\right] \\
& - 8\,\mathrm{Im}\mathrm{Li}_2\left[\frac{1}{(\sqrt{2})^3}\exp\left(\frac{\pi i}{4}\right)\right].
\end{aligned}\tag{29}$$

Identities (26), (27), (28) and (29) facilitate the derivation of base 2^{12}, length 24 BBP-type formulas for the respective polylogarithm constants through the prescription of Section 3. The explicit formulas or their variants are listed in the BBP Compendium.

5.3. Base 2^{60} length 120 Formulas

Solving Equations (13) and (19) simultaneously, we have

$$
\begin{aligned}
\pi^2 = {}& 192\operatorname{Re}\operatorname{Li}_2\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] + 192\operatorname{Re}\operatorname{Li}_2\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\
& -96\operatorname{Re}\operatorname{Li}_2\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - 24\operatorname{Li}_2\left[\frac{1}{2}\right]
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
\log^2 2 = {}& 32\operatorname{Re}\operatorname{Li}_2\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] + 32\operatorname{Re}\operatorname{Li}_2\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\
& -16\operatorname{Re}\operatorname{Li}_2\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - 6\operatorname{Li}_2\left[\frac{1}{2}\right].
\end{aligned}
\tag{31}
$$

Using Equations (18) in Equations (22), we have

$$
\begin{aligned}
G = {}& 5\operatorname{Im}\operatorname{Li}_2\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] - \operatorname{Im}\operatorname{Li}_2\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i3\pi}{4}\right)\right] \\
& -3\operatorname{Im}\operatorname{Li}_2\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - 2\operatorname{Im}\operatorname{Li}_2\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]
\end{aligned}
\tag{32}
$$

Applying the prescriptions of Section 3 to Equations (30), (31), (18), and (32), respectively, we obtain the following base 2^{60}, length 120 binary BBP-type formulas:

$$
\begin{aligned}
\pi^2 = {}& \frac{3}{2^{54}}P(2, 2^{60}, 120, (0, -2^{58}, 3^2\cdot 2^{58}, -3^2\cdot 2^{57}, -5^2\cdot 2^{56}, -2^{56}, 0, \\
& 7\cdot 2^{55}, -3^2\cdot 2^{55}, -2^{54}, 0, -3^3\cdot 2^{53}, 0, -2^{52}, 7\cdot 2^{51}, 7\cdot 2^{51}, 0, -2^{50}, 0, 2^{53}, \\
& 3^2\cdot 2^{49}, -2^{48}, 0, 5^2\cdot 2^{47}, 5^2\cdot 2^{46}, -2^{46}, 3^2\cdot 2^{46}, -3^2\cdot 2^{45}, 0, -2^{44}, 0, 7\cdot 2^{43}, \\
& -3^2\cdot 2^{43}, -2^{42}, -5^2\cdot 2^{41}, -3^3\cdot 2^{41}, 0, -2^{40}, -3^2\cdot 2^{40}, -3^2\cdot 2^{40}, 0, -2^{38}, 0, \\
& -3^2\cdot 2^{37}, -7\cdot 2^{36}, -2^{36}, 0, 5^2\cdot 2^{35}, 0, -2^{34}, 3^2\cdot 2^{34}, -3^2\cdot 2^{33}, 0, -2^{32}, 5^2\cdot 2^{31}, \\
& 7\cdot 2^{31}, -3^2\cdot 2^{31}, -2^{30}, 0, -2^{30}, 0, -2^{28}, -3^2\cdot 2^{28}, 7\cdot 2^{27}, 5^2\cdot 2^{26}, -2^{26}, 0, \\
& -3^2\cdot 2^{25}, 3^2\cdot 2^{25}, -2^{24}, 0, 5^2\cdot 2^{23}, 0, -2^{22}, -7\cdot 2^{21}, -3^2\cdot 2^{21}, 0, -2^{20}, 0, \\
& -3^2\cdot 2^{20}, -3^2\cdot 2^{19}, -2^{18}, 0, -3^3\cdot 2^{17}, -5^2\cdot 2^{16}, -2^{16}, -3^2\cdot 2^{16}, 7\cdot 2^{15}, 0, \\
& -2^{14}, 0, -3^2\cdot 2^{13}, 3^2\cdot 2^{13}, -2^{12}, 5^2\cdot 2^{11}, 5^2\cdot 2^{11}, 0, -2^{10}, 3^2\cdot 2^{10}, 2^{13}, \\
& 0, -2^8, 0, 7\cdot 2^7, 7\cdot 2^6, -2^6, 0, -3^3\cdot 2^5, 0, -2^4, -3^2\cdot 2^4, 7\cdot 2^3, 0, -2^2, \\
& -5^2\cdot 2, -3^2\cdot 2, 3^2\cdot 2, -1, 0, 0)),
\end{aligned}
\tag{33}
$$

$$\log^2 2 \;=\; \frac{1}{2^{57}} P(2, 2^{60}, 120, (0, -3\cdot 2^{59}, 3^2\cdot 2^{60}, -19\cdot 2^{58}, -5^2\cdot 2^{58}, -3\cdot 2^{57}, 0, 13\cdot 2^{56}, 0,$$
$$-3^2\cdot 2^{57}, -3\cdot 2^{55}, 0, -5\cdot 11\cdot 2^{54}, 0, -3\cdot 2^{53}, 7\cdot 2^{53}, 13\cdot 2^{52}, 0, -3\cdot 2^{51}, 31\cdot 2^{50},$$
$$3^2\cdot 2^{51}, -3\cdot 2^{49}, 0, 7^2\cdot 2^{48}, 5^2\cdot 2^{48}, -3\cdot 2^{47}, 3^2\cdot 2^{48}, -19\cdot 2^{46}, 0, -3\cdot 2^{45}, 0,$$
$$13\cdot 2^{44}, -3^2\cdot 2^{45}, -3\cdot 2^{43}, -5^2\cdot 2^{43}, -5\cdot 11\cdot 2^{42}, 0, -3\cdot 2^{41}, -3^2\cdot 2^{42}, -37\cdot 2^{40},$$
$$0, -3\cdot 2^{39}, 0, -19\cdot 2^{38}, -7\cdot 2^{38}, -3\cdot 2^{37}, 0, 7^2\cdot 2^{36}, 0, -3\cdot 2^{35}, 3^2\cdot 2^{36},$$
$$-19\cdot 2^{34}, 0, -3\cdot 2^{33}, 5^2\cdot 2^{33}, 13\cdot 2^{32}, -3^2\cdot 2^{33}, -3\cdot 2^{31}, 0, -5\cdot 2^{30}, 0, -3\cdot 2^{29},$$
$$-3^2\cdot 2^{30}, 13\cdot 2^{28}, 5^2\cdot 2^{28}, -3\cdot 2^{27}, 0, -19\cdot 2^{26}, 3^2\cdot 2^{27}, -3\cdot 2^{25}, 0, 7^2\cdot 2^{24}, 0,$$
$$-3\cdot 2^{23}, -7\cdot 2^{23}, -19\cdot 2^{22}, 0, -3\cdot 2^{21}, 0, -37\cdot 2^{20}, -3^2\cdot 2^{21}, -3\cdot 2^{19}, 0,$$
$$-5\cdot 11\cdot 2^{18}, -5^2\cdot 2^{18}, -3\cdot 2^{17}, -3^2\cdot 2^{18}, 13\cdot 2^{16}, 0, -3\cdot 2^{15}, 0, -19\cdot 2^{14}, 3^2\cdot 2^{15},$$
$$-3\cdot 2^{13}, 5^2\cdot 2^{13}, 7^2\cdot 2^{12}, 0, -3\cdot 2^{11}, 3^2\cdot 2^{12}, 31\cdot 2^{10}, 0, -3\cdot 2^{9}, 0, 13\cdot 2^{8}, 7\cdot 2^{8},$$
$$-3\cdot 2^{7}, 0, -5\cdot 11\cdot 2^{6}, 0, -3\cdot 2^{5}, -3^2\cdot 2^{6}, 13\cdot 2^{4}, 0, -3\cdot 2^{3}, -5^2\cdot 2^{3}, -19\cdot 2^{2},$$
$$3^2\cdot 2^{3}, -3\cdot 2, 0, -1)), \tag{34}$$

$$\pi\log 2 \;=\; \frac{1}{2^{55}} P(2, 2^{60}, 120, (0, 2^{60}, -3^2\cdot 2^{57}, 0, -5^2\cdot 2^{55}, -13\cdot 2^{56}, 0, 0, -3^2\cdot 2^{54},$$
$$-17\cdot 2^{53}, 0, 0, 0, -2^{54}, -7\cdot 2^{50}, 0, 0, 13\cdot 2^{50}, 0, 0, 3^2\cdot 2^{48}, -2^{50}, 0, 0, 5^2\cdot 2^{45}, 2^{48},$$
$$-3^2\cdot 2^{45}, 0, 0, -2^{43}, 0, 0, -3^2\cdot 2^{42}, 2^{44}, 5^2\cdot 2^{40}, 0, 0, -2^{42}, 3^2\cdot 2^{39}, 0, 0, 13\cdot 2^{38}, 0,$$
$$0, -7\cdot 2^{35}, -2^{38}, 0, 0, 0, -17\cdot 2^{33}, -3^2\cdot 2^{33}, 0, 0, -13\cdot 2^{32}, -5^2\cdot 2^{30}, 0, -3^2\cdot 2^{30},$$
$$2^{32}, 0, 0, 0, -2^{30}, 3^2\cdot 2^{27}, 0, 5^2\cdot 2^{25}, 13\cdot 2^{26}, 0, 0, 3^2\cdot 2^{24}, 17\cdot 2^{23}, 0, 0, 0, 2^{24},$$
$$7\cdot 2^{20}, 0, 0, -13\cdot 2^{20}, 0, 0, -3^2\cdot 2^{18}, 2^{20}, 0, 0, -5^2\cdot 2^{15}, -2^{18}, 3^2\cdot 2^{15}, 0, 0, 2^{13}, 0,$$
$$0, 3^2\cdot 2^{12}, -2^{14}, -5^2\cdot 2^{10}, 0, 0, 2^{12}, -3^2\cdot 2^{9}, 0, 0, -13\cdot 2^{8}, 0, 0, 7\cdot 2^{5}, 2^{8}, 0, 0, 0,$$
$$17\cdot 2^{3}, 3^2\cdot 2^{3}, 0, 0, 13\cdot 2^{2}, 5^2, 0, 3^2, -2^{2}, 0, 0)) \tag{35}$$

and

$$G \;=\; \frac{1}{2^{60}} P(2, 2^{60}, 120, (-2^{59}, 3\cdot 7\cdot 2^{59}, -7\cdot 2^{60}, 0, -7^2\cdot 2^{57}, -3\cdot 2^{61}, 2^{56}, 0,$$
$$-7\cdot 2^{57}, -29\cdot 2^{55}, -2^{54}, 0, 2^{53}, -3\cdot 7\cdot 2^{53}, -11\cdot 2^{53}, 0, -2^{51}, 3\cdot 2^{55}, -2^{50}, 0,$$
$$7\cdot 2^{51}, -3\cdot 7\cdot 2^{49}, 2^{48}, 0, 7^2\cdot 2^{47}, 3\cdot 7\cdot 2^{47}, -7\cdot 2^{48}, 0, 2^{45}, 2^{46}, 2^{44}, 0, -7\cdot 2^{45},$$
$$3\cdot 7\cdot 2^{43}, 7^2\cdot 2^{42}, 0, 2^{41}, -3\cdot 7\cdot 2^{41}, 7\cdot 2^{42}, 0, -2^{39}, 3\cdot 2^{43}, -2^{38}, 0,$$
$$-11\cdot 2^{38}, -3\cdot 7\cdot 2^{37}, 2^{36}, 0, -2^{35}, -29\cdot 2^{35}, -7\cdot 2^{36}, 0, 2^{33}, -3\cdot 2^{37}, -7^2\cdot 2^{32},$$
$$0, -7\cdot 2^{33}, 3\cdot 7\cdot 2^{31}, -2^{30}, 0, 2^{29}, -3\cdot 7\cdot 2^{29}, 7\cdot 2^{30}, 0, 7^2\cdot 2^{27}, 3\cdot 2^{31}, -2^{26}, 0,$$
$$7\cdot 2^{27}, 29\cdot 2^{25}, 2^{24}, 0, -2^{23}, 3\cdot 7\cdot 2^{23}, 11\cdot 2^{23}, 0, 2^{21}, -3\cdot 2^{25}, 2^{20}, 0, -7\cdot 2^{21},$$
$$3\cdot 7\cdot 2^{19}, -2^{18}, 0, -7^2\cdot 2^{17}, -3\cdot 7\cdot 2^{17}, 7\cdot 2^{18}, 0, -2^{15}, -2^{16}, -2^{14}, 0, 7\cdot 2^{15},$$
$$-3\cdot 7\cdot 2^{13}, -7^2\cdot 2^{12}, 0, -2^{11}, 3\cdot 7\cdot 2^{11}, -7\cdot 2^{12}, 0, 2^{9}, -3\cdot 2^{13}, 2^{8}, 0, 11\cdot 2^{8},$$
$$3\cdot 7\cdot 2^{7}, -2^{6}, 0, 2^{5}, 29\cdot 2^{5}, 7\cdot 2^{6}, 0, -2^{3}, 3\cdot 2^{7}, 7^2\cdot 2^{2}, 0,$$
$$7\cdot 2^{3}, -3\cdot 7\cdot 2, 1, 0)). \tag{36}$$

6. Degree 3 Formulas

No *proved* explicit Digit Extraction BBP-type formulas are known for π^3 and $\pi \log^2 2$. In what follows, we now present, together with their proofs, new binary degree 3 BBP-type formulas for these and the remaining three trilogarithm constants.

6.1. Generators of Degree 3 BBP-Type Formulas

A functional equation for trilogarithms (Eq. A.2.6.10 of [9]) reads

$$\text{Li}_3 \left[\frac{1-x}{1+x} \right] - \text{Li}_3 \left[\frac{x-1}{x+1} \right] = 2 \text{Li}_3 \left[1-x \right] + 2 \text{Li}_3 \left[\frac{1}{1+x} \right] - \frac{1}{2} \text{Li}_3 \left[1-x^2 \right] - \frac{7}{4} \zeta(3)$$
$$+ \frac{\pi^2}{6} \log(1+x) - \frac{1}{3} \log^3(1+x). \tag{37}$$

The use of $x = 1$ in the functional equation (37) gives the well-known formula

$$\frac{7}{8}\zeta(3) - \frac{\pi^2 \log 2}{12} + \frac{\log^3 2}{6} = \text{Li}_3 \left[\frac{1}{2} \right]. \tag{38}$$

Another functional identity for trilogarithms (Eq. A.2.6.11 of [9]) is

$$\text{Li}_3 \left[\frac{x(1-y)^2}{y(1-x)^2} \right] + \text{Li}_3 \left[xy \right] + \text{Li}_3 \left[\frac{x}{y} \right] - 2\text{Li}_3 \left[\frac{x(1-y)}{y(1-x)} \right] - 2\text{Li}_3 \left[\frac{x(1-y)}{(x-1)} \right]$$
$$- 2\text{Li}_3 \left[\frac{1-y}{1-x} \right] - 2\text{Li}_3 \left[\frac{(1-y)}{y(x-1)} \right] - 2\text{Li}_3 \left[x \right] - 2\text{Li}_3 \left[y \right] + 2\zeta(3) \tag{39}$$
$$= \log^2 y \log \left(\frac{1-y}{1-x} \right) - \frac{1}{3}\pi^2 \log y - \frac{1}{3} \log^3 y.$$

The use of $x = -1$, $y = i$ in the above equation gives

$$\frac{35}{8}\zeta(3) - \frac{5\pi^2 \log 2}{24} + \frac{1}{6} \log^3 2 = 8 \, \text{Re} \, \text{Li}_3 \left[\frac{1}{\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right]. \tag{40}$$

Plugging $x = -i$, $y = 1-i$ in Equation (39) and taking real and imaginary parts gives

$$\frac{7}{16}\zeta(3) + \frac{5\pi^2 \log 2}{192} - \frac{7 \log^3 2}{48} = 2 \, \text{Re} \, \text{Li}_3 \left[\frac{1}{\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right] + \frac{9}{4} \, \text{Re} \, \text{Li}_3 \left[\frac{1}{2} \exp \left(\frac{i\pi}{2} \right) \right]$$
$$- \text{Re} \, \text{Li}_3 \left[\frac{1}{2\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right] \tag{41}$$

and

$$\frac{3\pi \log^2 2}{32} - \frac{9\pi^3}{128} = \text{Im Li}_3 \left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) \right]$$
$$+ \frac{9}{4} \text{Im Li}_3 \left[\frac{1}{2} \exp\left(\frac{i\pi}{2}\right) \right] - 6 \text{Im Li}_3 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) \right]. \tag{42}$$

Yet another functional equation for trilogarithms (Eq. 6.96 p 174 of [9]) is

$$\text{Li}_3 \left[\frac{x(1-y)^2}{y(1-x)^2} \right] = 2\text{Li}_3 \left[\frac{-x(1-y)}{(1-x)} \right] + 2\text{Li}_3 \left[\frac{x(1-y)}{y(1-x)} \right]$$
$$+ 2\text{Li}_3 \left[\frac{-y(1-x)}{(1-y)} \right] + 2\text{Li}_3 \left[\frac{1-x}{1-y} \right] + 2\text{Li}_3 [x]$$
$$+ 2\text{Li}_3 [y] - 2\text{Li}_3 [xy] - \text{Li}_3 \left[\frac{x}{y} \right] - 2\zeta(3)$$
$$+ \frac{\pi^2}{3} \log\left(\frac{1-y}{1-x}\right) + \log(-y) \log^2(1-y) \tag{43}$$
$$- \log^2(-y) \log(1-x) - \frac{1}{3} \log^3 \left(\frac{1-y}{1-x}\right)$$
$$+ \frac{1}{3} \log^3 \left(\frac{-y(1-x)}{1-y}\right) + \frac{1}{3} \log^3 \left(-\frac{1-y}{y}\right)$$
$$- \frac{1}{3} \log^3(1-y).$$

Using $x = -1$, $y = 1 + i$ in Eq. (43), simplifying and taking real and imaginary parts gives

$$\frac{7}{2}\zeta(3) - \frac{15\pi^2 \log 2}{64} + \frac{5 \log^3 2}{16}$$
$$= 4 \text{Re Li}_3 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) \right] + 4 \text{Re Li}_3 \left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) \right] \tag{44}$$
$$+ \frac{7}{2} \text{Re Li}_3 \left[\frac{1}{2} \exp\left(\frac{i\pi}{2}\right) \right] - \text{Re Li}_3 \left[\frac{1}{4\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) \right]$$

and

$$\frac{13\pi^3}{128} - \frac{7\pi \log^2 2}{32} = 4 \text{Im Li}_3 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) \right] - 4 \text{Im Li}_3 \left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) \right]$$
$$+ \frac{7}{2} \text{Im Li}_3 \left[\frac{1}{2} \exp\left(\frac{i\pi}{2}\right) \right] - \text{Im Li}_3 \left[\frac{1}{4\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) \right]. \tag{45}$$

Using $x = -1$, $y = 1/2$ in Equation (43) gives the identity

$$7\zeta(3) - \pi^2 \log 2 + 3 \log^3 2 = 9 \text{L}_3 \left[\frac{1}{4} \right] - 2 \text{L}_3 \left[-\frac{1}{8} \right]. \tag{46}$$

We are now ready to derive formulas for π^3, $\pi \log^2 2$, $\zeta(3)$, $\pi^2 \log 2$ and $\log^3 2$.

6.2. Base 2^{12} Formulas

Solving Equations (38), (40) and (46) simultaneously, we find the BBP-ready identities:

$$\zeta(3) = \frac{128}{21} \operatorname{Re} \operatorname{Li}_3\left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] - \frac{88}{21} \operatorname{Li}_3\left[\frac{1}{2}\right] + \frac{12}{7} \operatorname{Li}_3\left[\frac{1}{4}\right]$$
$$- \frac{8}{21} \operatorname{Li}_3\left[-\frac{1}{8}\right] \tag{47}$$

$$\pi^2 \log 2 = \frac{320}{3} \operatorname{Re} \operatorname{Li}_3\left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] - \frac{328}{3} \operatorname{Li}_3\left[\frac{1}{2}\right] + 48 \operatorname{Li}_3\left[\frac{1}{4}\right]$$
$$- \frac{32}{3} \operatorname{Li}_3\left[-\frac{1}{8}\right] \tag{48}$$

and

$$\log^3 2 = \frac{64}{3} \operatorname{Re} \operatorname{Li}_3\left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] - \frac{80}{3} \operatorname{Li}_3\left[\frac{1}{2}\right] + 15 \operatorname{Li}_3\left[\frac{1}{4}\right]$$
$$- \frac{10}{3} \operatorname{Li}_3\left[-\frac{1}{8}\right] . \tag{49}$$

Application of the prescriptions of section 3 to the identities (47), (48) and (49) facilitates the derivation of base 2^{12}, length 24 BBP-type formulas for the respective polylogarithm constants. The explicit formulas or their variants are as listed in the BBP Compendium.

6.3. Base 2^{60} Formulas

Eliminating $\pi \log^2 2$ between Equations (42) and (45), we have the following BBP-ready formula for π^3:

$$\pi^3 = 16 \operatorname{Im} \operatorname{Li}_3\left[\frac{1}{4\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] + 160 \operatorname{Im} \operatorname{Li}_3\left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right]$$
$$+ \frac{80}{3} \operatorname{Im} \operatorname{Li}_3\left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4}\right)\right] - 140 \operatorname{Im} \operatorname{Li}_3\left[\frac{1}{2} \exp\left(\frac{i\pi}{2}\right)\right] . \tag{50}$$

Eliminating π^3 between Equations (42) and (45), we have the following BBP-ready formula for $\pi \log^2 2$:

$$
\begin{aligned}
\pi \log^2 2 = {} & 12 \operatorname{Im} \operatorname{Li}_3 \left[\frac{1}{4\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right] + 56 \operatorname{Im} \operatorname{Li}_3 \left[\frac{1}{\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right] \\
& + \frac{92}{3} \operatorname{Im} \operatorname{Li}_3 \left[\frac{1}{2\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right] - 81 \operatorname{Im} \operatorname{Li}_3 \left[\frac{1}{2} \exp \left(\frac{i\pi}{2} \right) \right] .
\end{aligned}
\tag{51}
$$

Eliminating $\pi^2 \log 2$ and $\log^3 2$ between Eqs. (40), (41) and (44) we find

$$
\begin{aligned}
\zeta(3) = {} & \frac{16}{7} \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{4\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right] + \frac{32}{7} \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right] \\
& - \frac{48}{7} \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{2\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right] - \frac{92}{7} \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{2} \exp \left(\frac{i\pi}{2} \right) \right] .
\end{aligned}
\tag{52}
$$

Eliminating $\zeta(3)$ and $\log^3 2$ between Eqs. (40), (41) and (44) we find

$$
\begin{aligned}
\pi^2 \log 2 = {} & \frac{312}{5} \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{4\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right] + \frac{336}{5} \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right] \\
& - \frac{904}{5} \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{2\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right] - \frac{1866}{5} \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{2} \exp \left(\frac{i\pi}{2} \right) \right] .
\end{aligned}
\tag{53}
$$

Elimination of $\pi^2 \log 2$ and $\zeta(3)$ between Eqs. (40), (41) and (44) gives

$$
\begin{aligned}
\log^3 2 = {} & 18 \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{4\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right] + 12 \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right] \\
& - 46 \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{2\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right] - \frac{243}{2} \operatorname{Re} \operatorname{Li}_3 \left[\frac{1}{2} \exp \left(\frac{i\pi}{2} \right) \right] .
\end{aligned}
\tag{54}
$$

As in the previous cases, the application of the prescriptions of section 3 to the identities (50), (51), (52), (53) and (54) facilitates the derivation of base 2^{60}, length 120 BBP-type formulas for the respective polylogarithm constants. The explicit formulas or their variants are as listed in the BBP Compendium.

7. Degree 4 Formulas

Next we derive binary BBP-type formulas for π^4, $\pi^2 \log^2 2$, $\log^4 2$ and two linear combinations of $\operatorname{Cl}_4(\pi/2)$, $\pi \log^3 2$ and $\pi^3 \log 2$. We will also give formal proofs of the known but hitherto unproved formulas for these polylogarithm constants.

7.1. Generators of Degree 4 Binary BBP-Type Formulas

The two-variable degree 4 polylogarithm functional equation (Eq. 7.90, p. 211 of [9]) reads

$$
\mathrm{Li}_4\left[-\frac{x^2y\eta}{\xi}\right] + \mathrm{Li}_4\left[-\frac{y^2x\xi}{\eta}\right] + \mathrm{Li}_4\left[\frac{x^2y}{\eta^2\xi}\right] + \mathrm{Li}_4\left[\frac{y^2x}{\xi^2\eta}\right]
$$

$$
= 6\,\mathrm{Li}_4\left[xy\right] + 6\,\mathrm{Li}_4\left[\frac{xy}{\eta\,\xi}\right] + 6\,\mathrm{Li}_4\left[-\frac{xy}{\eta}\right] + 6\,\mathrm{Li}_4\left[-\frac{xy}{\xi}\right]
$$

$$
+ 3\,\mathrm{Li}_4\left[x\eta\right] + 3\,\mathrm{Li}_4\left[y\xi\right] + 3\,\mathrm{Li}_4\left[\frac{x}{\eta}\right] + 3\,\mathrm{Li}_4\left[\frac{y}{\xi}\right] + 3\,\mathrm{Li}_4\left[-\frac{x\eta}{\xi}\right] \qquad (55)
$$

$$
+ 3\,\mathrm{Li}_4\left[-\frac{y\xi}{\eta}\right] + 3\,\mathrm{Li}_4\left[-\frac{x}{\eta\,\xi}\right] + 3\,\mathrm{Li}_4\left[-\frac{y}{\eta\,\xi}\right] - 6\,\mathrm{Li}_4\left[x\right]
$$

$$
- 6\,\mathrm{Li}_4\left[y\right] - 6\,\mathrm{Li}_4\left[-\frac{x}{\xi}\right] - 6\,\mathrm{Li}_4\left[-\frac{y}{\eta}\right] + 3/2\,\log^2\xi\log^2\eta\,,
$$

where $\xi = 1 - x$, $\eta = 1 - y$.

Evaluating Equation (55) at coordinates $(1/2, 1/2)$ gives

$$
\frac{\pi^4}{9} - \frac{1}{2}\,\pi^2\log^2 2 + \frac{5}{4}\,\log^4 2 = 24\,\mathrm{Li}_4\left[\frac{1}{2}\right] + 2\,\mathrm{Li}_4\left[-\frac{1}{8}\right] - \frac{27}{2}\,\mathrm{Li}_4\left[\frac{1}{4}\right]. \qquad (56)
$$

Evaluating the functional equation Eq. (55) at coordinates (i, i), simplifying and taking real and imaginary parts, we obtain

$$
\frac{349\pi^4}{9216} - \frac{7\pi^2\log^2 2}{128} + \frac{5}{64}\,\log^4 2
$$

$$
= 2\,\mathrm{Re}\,\mathrm{Li}_4\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] + 12\,\mathrm{Re}\,\mathrm{Li}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \qquad (57)
$$

$$
- \frac{27}{4}\,\mathrm{Re}\,\mathrm{Li}_4\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] - 6\,\mathrm{Li}_4\left[\frac{1}{2}\right]
$$

and

$$
20\mathrm{Cl}_4\left(\frac{\pi}{2}\right) + \frac{3\pi\log^3 2}{32} - \frac{27\pi^3\log 2}{128}
$$

$$
= -2\,\mathrm{Im}\,\mathrm{Li}_4\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - \frac{27}{4}\,\mathrm{Im}\,\mathrm{Li}_4\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] \qquad (58)
$$

$$
+ 36\,\mathrm{Im}\,\mathrm{Li}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right].
$$

Evaluating Equation (55) at $(1 + i, 1/2)$, simplifying and taking the real part

yields

$$
\begin{aligned}
\frac{1697\pi^4}{9216} &- \frac{137\pi^2 \log^2 2}{384} + \frac{115}{192}\log^4 2 \\
&= -14\operatorname{Re}\operatorname{Li}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right] + 30\operatorname{Re}\operatorname{Li}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\
&\quad - 6\operatorname{Re}\operatorname{Li}_4\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{2}\right)\right] - 7\operatorname{Li}_4\left[\frac{1}{2}\right] + \frac{11}{8}\operatorname{Li}_4\left[-\frac{1}{4}\right].
\end{aligned} \tag{59}
$$

Finally, evaluating the identity at $((1-i)/2, 1/2)$ and taking real and imaginary parts, we find

$$
\begin{aligned}
\frac{1265\pi^4}{18432} &- \frac{113\pi^2 \log^2 2}{768} + \frac{91}{384}\log^4 2 \\
&= \operatorname{Re}\operatorname{Li}_4\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{\pi i}{4}\right)\right] - 12\operatorname{Re}\operatorname{Li}_4\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\
&\quad + 5\operatorname{Re}\operatorname{Li}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - 7\operatorname{Re}\operatorname{Li}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right] \\
&\quad - \frac{95}{8}\operatorname{Re}\operatorname{Li}_4\left[\frac{1}{2}\exp\left(\frac{\pi i}{2}\right)\right] + \frac{11}{8}\operatorname{Li}_4\left[-\frac{1}{4}\right] + 6\operatorname{Li}_4\left[\frac{1}{2}\right]
\end{aligned} \tag{60}
$$

and

$$
\begin{aligned}
12\operatorname{Cl}_4\left(\frac{\pi}{2}\right) &+ \frac{29\pi \log^3 2}{192} - \frac{47\pi^3 \log 2}{256} \\
&= -6\operatorname{Im}\operatorname{Li}_4\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] + \frac{95}{8}\operatorname{Im}\operatorname{Li}_4\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] \\
&\quad + \operatorname{Im}\operatorname{Li}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] + 7\operatorname{Im}\operatorname{Li}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3\pi i}{4}\right)\right] \\
&\quad - \operatorname{Im}\operatorname{Li}_4\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right].
\end{aligned} \tag{61}
$$

We are now ready to derive formulas for π^4, $\pi^2 \log^2 2$, $\log^4 2$, and the two linear combinations of $\operatorname{Cl}_4(\pi/2)$, $\pi \log^3 2$ and $\pi^3 \log 2$.

7.2. Base 2^{12} Binary BBP-Type Formulas

Solving Equations (56), (57) and (59) simultaneously, we find

$$
\begin{aligned}
\pi^4 = &\frac{27648}{41}\operatorname{Re}\operatorname{Li}_4\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] + \frac{51840}{41}\operatorname{Re}\operatorname{Li}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\
&+ \frac{24192}{41}\operatorname{Re}\operatorname{Li}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right] - \frac{58320}{41}\operatorname{Re}\operatorname{Li}_4\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] \\
&- \frac{32832}{41}\operatorname{Li}_4\left[\frac{1}{2}\right] - \frac{3888}{41}\operatorname{Li}_4\left[\frac{1}{4}\right] + \frac{576}{41}\operatorname{Li}_4\left[-\frac{1}{8}\right] - \frac{2376}{41}\operatorname{Li}_4\left[-\frac{1}{4}\right],
\end{aligned} \tag{62}
$$

$$\pi^2 \log^2 2 = \frac{98944}{123} \operatorname{Re} \operatorname{Li}_4 \left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) \right] + \frac{47408}{41} \operatorname{Re} \operatorname{Li}_4 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) \right]$$

$$+ \frac{31920}{41} \operatorname{Re} \operatorname{Li}_4 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{3i\pi}{4}\right) \right] - \frac{65142}{41} \operatorname{Re} \operatorname{Li}_4 \left[\frac{1}{2} \exp\left(\frac{i\pi}{2}\right) \right]$$

$$- \frac{30200}{41} \operatorname{Li}_4 \left[\frac{1}{2} \right] - \frac{6606}{41} \operatorname{Li}_4 \left[\frac{1}{4} \right] + \frac{2936}{123} \operatorname{Li}_4 \left[-\frac{1}{8} \right]$$

$$- \frac{3135}{41} \operatorname{Li}_4 \left[-\frac{1}{4} \right]$$

(63)

and

$$\log^2 4 = \frac{161024}{615} \operatorname{Re} \operatorname{Li}_4 \left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) \right] + \frac{71776}{205} \operatorname{Re} \operatorname{Li}_4 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) \right]$$

$$+ \frac{53088}{205} \operatorname{Re} \operatorname{Li}_4 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{3i\pi}{4}\right) \right] - \frac{104364}{205} \operatorname{Re} \operatorname{Li}_4 \left[\frac{1}{2} \exp\left(\frac{i\pi}{2}\right) \right]$$

$$- \frac{41872}{205} \operatorname{Li}_4 \left[\frac{1}{2} \right] - \frac{13698}{205} \operatorname{Li}_4 \left[\frac{1}{4} \right] + \frac{6088}{615} \operatorname{Li}_4 \left[-\frac{1}{8} \right]$$

$$- \frac{5214}{205} \operatorname{Li}_4 \left[-\frac{1}{4} \right] .$$

(64)

The identities (58), (62), (63) and (64) facilitate the derivation of base 2^{12}, length 24 BBP-type formulas for the respective polylogarithm constants. The explicit formulas are listed in the BBP Compendium.

7.3. Base 2^{60} Binary BBP-Type Formulas

Solving Eqs. (57), (59) and (60) simultaneously, we find

$$\pi^4 = \frac{57600}{71} \operatorname{ReLi}_4 \left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) \right] + \frac{442368}{497} \operatorname{ReLi}_4 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) \right]$$

$$+ \frac{13824}{71} \operatorname{ReLi}_4 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{3i\pi}{4}\right) \right] + \frac{239328}{497} \operatorname{ReLi}_4 \left[\frac{1}{2} \exp\left(\frac{i\pi}{2}\right) \right]$$

(65)

$$- \frac{34560}{497} \operatorname{ReLi}_4 \left[\frac{1}{4\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) \right] - \frac{432000}{497} \operatorname{Li}_4 \left[\frac{1}{2} \right] - \frac{4752}{71} \operatorname{Li}_4 \left[-\frac{1}{4} \right] ,$$

$$\pi^2 \log^2 2 = \frac{73632}{71} \text{ReLi}_4 \left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4} \right) \right] + \frac{258592}{497} \text{ReLi}_4 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4} \right) \right]$$

$$+ \frac{7584}{71} \text{ReLi}_4 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{3i\pi}{4} \right) \right] + \frac{818152}{497} \text{ReLi}_4 \left[\frac{1}{2} \exp\left(\frac{i\pi}{2} \right) \right]$$

$$- \frac{58720}{497} \text{ReLi}_4 \left[\frac{1}{4\sqrt{2}} \exp\left(\frac{i\pi}{4} \right) \right] - \frac{423872}{497} \text{Li}_4 \left[\frac{1}{2} \right]$$

$$- \frac{6512}{71} \text{Li}_4 \left[-\frac{1}{4} \right]$$

(66)

and

$$\log^4 2 = \frac{25440}{71} \text{ReLi}_4 \left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4} \right) \right] + \frac{42928}{497} \text{ReLi}_4 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4} \right) \right]$$

$$- \frac{1392}{71} \text{ReLi}_4 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{3i\pi}{4} \right) \right] + \frac{413758}{497} \text{ReLi}_4 \left[\frac{1}{2} \exp\left(\frac{i\pi}{2} \right) \right]$$

$$- \frac{24352}{497} \text{ReLi}_4 \left[\frac{1}{4\sqrt{2}} \exp\left(\frac{i\pi}{4} \right) \right] - \frac{125480}{497} \text{Li}_4 \left[\frac{1}{2} \right]$$

$$- \frac{2255}{71} \text{Li}_4 \left[-\frac{1}{4} \right] .$$

(67)

The identities (61), (65), (66) and (67) facilitate the derivation of base 2^{60}, length 120 BBP-type formulas for the respective polylogarithm constants. The explicit formulas are listed in the BBP Compendium.

8. Degree 5 Formulas

Next we give formal proofs for the binary BBP-type formulas for $\zeta(5)$, $\pi^4 \log 2$, $\pi^2 \log^3 2$ and $\log^5 2$. These formulas were found experimentally by David H. Bailey, using his PSLQ algorithm. This section provides the first avenue where the hitherto unproved formulas are formally proved.

8.1. Generators of Degree 5 Binary BBP-Type Formulas

The following two-variable degree 5 polylogarithm functional equation was derived by Broadhurst [11] (Eq. 63 pg. 5)

$$
\begin{aligned}
&\operatorname{Li}_5\left[\frac{x\alpha}{y\beta}\right] + \operatorname{Li}_5\left[x\alpha\, y\eta\right] + \operatorname{Li}_5\left[\frac{x\alpha\,\beta}{\eta}\right] + \operatorname{Li}_5\left[x\xi\, y\beta\right] + \operatorname{Li}_5\left[\frac{x\xi}{y\eta}\right] \\
&\quad + \operatorname{Li}_5\left[\frac{x\xi\,\eta}{\beta}\right] + \operatorname{Li}_5\left[\frac{\alpha\, y\beta}{\xi}\right] + \operatorname{Li}_5\left[\frac{\alpha}{\xi\, y\eta}\right] + \operatorname{Li}_5\left[\frac{\alpha\,\eta}{\xi\,\beta}\right] \\
&\quad - 9\operatorname{Li}_5\left[xy\right] - 9\operatorname{Li}_5\left[x\beta\right] - 9\operatorname{Li}_5\left[x\eta\right] - 9\operatorname{Li}_5\left[\frac{x}{y}\right] - 9\operatorname{Li}_5\left[\frac{x}{\beta}\right] \\
&\quad - 9\operatorname{Li}_5\left[\frac{x}{\eta}\right] - 9\operatorname{Li}_5\left[\alpha\, y\right] - 9\operatorname{Li}_5\left[\alpha\,\beta\right] - 9\operatorname{Li}_5\left[\alpha\,\eta\right] \\
&\quad - 9\operatorname{Li}_5\left[\frac{\alpha}{y}\right] - 9\operatorname{Li}_5\left[\frac{\alpha}{\beta}\right] - 9\operatorname{Li}_5\left[\frac{\alpha}{\eta}\right] - 9\operatorname{Li}_5\left[\xi\, y\right] - 9\operatorname{Li}_5\left[\xi\,\beta\right] \\
&\quad - 9\operatorname{Li}_5\left[\xi\,\eta\right] - 9\operatorname{Li}_5\left[\frac{y}{\xi}\right] - 9\operatorname{Li}_5\left[\frac{\beta}{\xi}\right] - 9\operatorname{Li}_5\left[\frac{\eta}{\xi}\right] \\
&\quad + 18\operatorname{Li}_5\left[x\right] + 18\operatorname{Li}_5\left[\alpha\right] + 18\operatorname{Li}_5\left[\xi\right] + 18\operatorname{Li}_5\left[y\right] + 18\operatorname{Li}_5\left[\beta\right] \\
&\quad + 18\operatorname{Li}_5\left[\eta\right] - 18\,\zeta(5) = 3/10\,(\log\xi)^5 + 3/4\,(\log y - \log x)\,(\log\xi)^4 \\
&\quad + 3/2\,(3\log y - \log\eta)\,(\log\eta)^2\,(\log\xi)^2 + 1/2\,\pi^2\,(\log\xi - 3\log\eta)\,(\log\xi)^2 \\
&\quad + 1/5\,\pi^4\log\xi\,.
\end{aligned}
\tag{68}
$$

In the above formula $\xi = 1 - x$, $\eta = 1 - y$, $\alpha = -x/\xi$ and $\beta = -y/\eta$.

Evaluating Eq. (68) at coordinates $(1/2, 1/2)$ gives

$$
\begin{aligned}
\frac{403}{4}\,\zeta(5) - \frac{2}{3}\,\pi^4\log 2 + \pi^2\log^3 2 - \frac{3}{2}\,\log^5 2 &= 144\operatorname{Li}_5\left[\frac{1}{2}\right] - \frac{81}{2}\operatorname{Li}_5\left[\frac{1}{4}\right] \\
&\quad + 4\operatorname{Li}_5\left[-\frac{1}{8}\right].
\end{aligned}
\tag{69}
$$

Evaluating at $(-i, i)$ and taking the real part gives

$$
\begin{aligned}
\frac{4371}{128}\,\zeta(5) - \frac{349}{3072}\,\pi^4\log 2 &+ \frac{7}{128}\,\pi^2\log^3 2 - \frac{3}{64}\,\log^5 2 \\
&= 36\operatorname{Re}\operatorname{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - 36\operatorname{Re}\operatorname{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right] \\
&\quad + 4\operatorname{Re}\operatorname{Li}_5\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - 18\operatorname{Li}_5\left[\frac{1}{2}\right] - \frac{9}{16}\operatorname{Li}_5\left[-\frac{1}{4}\right].
\end{aligned}
\tag{70}
$$

Evaluating the identity at $(-1, i)$ and taking the real part gives

$$
\frac{279}{8}\,\zeta(5) - \frac{977}{6144}\,\pi^4 \log 2 + \frac{97}{768}\,\pi^2 \log^3 2 - \frac{15}{128}\,\log^5 2
$$
$$
= 2\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - 36\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \tag{71}
$$
$$
- 32\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right] + 37\,\mathrm{Li}_5\left[\frac{1}{2}\right] + \frac{7}{8}\,\mathrm{Li}_5\left[-\frac{1}{4}\right].
$$

Broadhurst proved (Eq. 68 of [11] written out) that

$$
\frac{31}{32}\,\zeta(5) - \frac{343}{99360}\,\pi^4 \log 2 + \frac{5}{2484}\,\pi^2 \log^3 2 - \frac{2}{1035}\,\log^5 2
$$
$$
= \frac{128}{69}\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - \frac{20}{69}\,\mathrm{Li}_5\left[\frac{1}{2}\right]. \tag{72}
$$

8.2. Base 2^{60} Binary BBP-Type Formulas

Solving Eqs. (69), (70), (71) and (72) simultaneously for $\zeta(5)$, $\pi^4 \log 2$, $\pi^2 \log^3 2$ and $\log^5 2$ we find

$$
\zeta(5) = \frac{1317888}{1457}\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - \frac{377856}{62651}\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]
$$
$$
+ \frac{56097792}{62651}\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right] + \frac{1240064}{62651}\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]
$$
$$
- \frac{929664}{62651}\,\mathrm{Li}_5\left[\frac{1}{2}\right] + \frac{644112}{62651}\,\mathrm{Li}_5\left[\frac{1}{4}\right] \tag{73}
$$
$$
- \frac{63616}{62651}\,\mathrm{Li}_5\left[-\frac{1}{8}\right] + \frac{616752}{62651}\,\mathrm{Li}_5\left[-\frac{1}{4}\right],
$$

$$
\pi^4 \log 2 = \frac{18593280}{47}\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - \frac{5294592}{2021}\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]
$$
$$
+ \frac{794230272}{2021}\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right]
$$
$$
+ \frac{16467456}{2021}\,\mathrm{Re}\,\mathrm{Li}_5\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \tag{74}
$$
$$
- \frac{11408256}{2021}\,\mathrm{Li}_5\left[\frac{1}{2}\right] + \frac{9121248}{2021}\,\mathrm{Li}_5\left[\frac{1}{4}\right]
$$
$$
- \frac{900864}{2021}\,\mathrm{Li}_5\left[-\frac{1}{8}\right] + \frac{8769816}{2021}\,\mathrm{Li}_5\left[-\frac{1}{4}\right],
$$

$$\pi^2 \log^3 2 = \frac{17440704}{47} \operatorname{Re} \operatorname{Li}_5 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4} \right) \right] - \frac{4835904}{2021} \operatorname{Re} \operatorname{Li}_5 \left[\frac{1}{4\sqrt{2}} \exp\left(\frac{i\pi}{4} \right) \right]$$
$$+ \frac{747218112}{2021} \operatorname{Re} \operatorname{Li}_5 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{3i\pi}{4} \right) \right]$$
$$+ \frac{12619200}{2021} \operatorname{Re} \operatorname{Li}_5 \left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4} \right) \right] \tag{75}$$
$$- \frac{7432176}{2021} \operatorname{Li}_5 \left[\frac{1}{2} \right] + \frac{8731962}{2021} \operatorname{Li}_5 \left[\frac{1}{4} \right]$$
$$- \frac{862416}{2021} \operatorname{Li}_5 \left[-\frac{1}{8} \right] + \frac{8350599}{2021} \operatorname{Li}_5 \left[-\frac{1}{4} \right]$$

and

$$\log^5 2 = \frac{6218880}{47} \operatorname{Re} \operatorname{Li}_5 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4} \right) \right] - \frac{1689472}{2021} \operatorname{Re} \operatorname{Li}_5 \left[\frac{1}{4\sqrt{2}} \exp\left(\frac{i\pi}{4} \right) \right]$$
$$+ \frac{266699392}{2021} \operatorname{Re} \operatorname{Li}_5 \left[\frac{1}{\sqrt{2}} \exp\left(\frac{3i\pi}{4} \right) \right]$$
$$+ \frac{3780736}{2021} \operatorname{Re} \operatorname{Li}_5 \left[\frac{1}{2\sqrt{2}} \exp\left(\frac{i\pi}{4} \right) \right] \tag{76}$$
$$- \frac{2092736}{2021} \operatorname{Li}_5 \left[\frac{1}{2} \right] + \frac{3217563}{2021} \operatorname{Li}_5 \left[\frac{1}{4} \right]$$
$$- \frac{317784}{2021} \operatorname{Li}_5 \left[-\frac{1}{8} \right] + \frac{3005666}{2021} \operatorname{Li}_5 \left[-\frac{1}{4} \right].$$

Identities (73), (74), (75) and (76) facilitate the derivation of base 2^{60}, length 120 BBP-type formulas for the respective polylogarithm constants. The explicit formulas are as listed in the BBP Compendium.

9. Zero Relations

BBP zero relations are BBP-type formulas that evaluate to zero. Considering that BBP-type formulas are usually discovered through computer searches, the need to study zero relations is aptly set forth in the BBP Compendium [1]:

> Knowledge of these zero relations is essential for finding formulas using integer relation programs (such as PSLQ). This is because unless these zero relations are excluded from the search for a conjectured BBP-type formula, the search may only recover a zero relation.

9.1. Degree 2 Zero Relations

9.1.1. Base 2^{12} Relations

Eliminating π^2 between Eqs. (16) and (27), we find the identity

$$
\begin{aligned}
0 = \ & 72 \operatorname{Re} \operatorname{Li}_2 \left[\frac{1}{2} \exp \left(\frac{i\pi}{2} \right) \right] - 72 \operatorname{Re} \operatorname{Li}_2 \left[\frac{1}{\sqrt{2}} \exp \left(\frac{3i\pi}{4} \right) \right] \\
& -24 \operatorname{Re} \operatorname{Li}_2 \left[\frac{1}{2\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right] - 180 \operatorname{Re} \operatorname{Li}_2 \left[\frac{1}{2} \exp \left(\frac{i\pi}{3} \right) \right] \\
& +45 \operatorname{Li}_2 \left[\frac{1}{4} \right],
\end{aligned}
\tag{77}
$$

which leads immediately to the zero relation

$$
\begin{aligned}
0 = \ & P(2, 2^{12}, 24, (2^{11}, -5 \cdot 2^{11}, -2^{12}, 3 \cdot 2^{12}, -2^9, 5 \cdot 2^{10}, 2^8, \\
& 3^2 \cdot 2^9, 2^9, -5 \cdot 2^7, -2^6, 0, -2^5, -5 \cdot 2^5, 2^6, 3^2 \cdot 2^5, 2^3, \\
& 5 \cdot 2^4, -2^2, 3 \cdot 2^4, -2^3, -5 \cdot 2, 1, 0)).
\end{aligned}
\tag{78}
$$

Eliminating $\log^2 2$ between Eq. (15) and (17) gives the identity

$$
\begin{aligned}
0 = \ & 2 \operatorname{Re} \operatorname{Li}_2 \left[\frac{1}{\sqrt{2}} \exp \left(\frac{3i\pi}{4} \right) \right] + 2 \operatorname{Re} \operatorname{Li}_2 \left[\frac{1}{2\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right] \\
& +\operatorname{Li}_2 \left[-\frac{1}{8} \right] - 2 \operatorname{Li}_2 \left[-\frac{1}{2} \right] - 2 \operatorname{Li}_2 \left[\frac{1}{4} \right] - \operatorname{Li}_2 \left[-\frac{1}{4} \right],
\end{aligned}
\tag{79}
$$

which produces the zero relation

$$
\begin{aligned}
0 = \ & P(2, 2^{12}, 24, (2^{11}, -2^{13}, -5 \cdot 2^{11}, 13 \cdot 2^{10}, -2^9, 7 \cdot 2^{10}, 2^8, 3^3 \cdot 2^8, \\
& 5 \cdot 2^8, -2^9, -2^6, 2^8, -2^5, -2^7, 5 \cdot 2^5, 3^3 \cdot 2^4, 2^3, 7 \cdot 2^4, \\
& -2^2, 13 \cdot 2^2, -5 \cdot 2^2, -2^3, 1, 0)).
\end{aligned}
\tag{80}
$$

9.1.2. A Base 2^{60} Relation

Eliminating $\pi \log 2$ between Eq. (18) and Eq. (29) gives the identity

$$
\begin{aligned}
0 = \ & 5 \operatorname{Im} \operatorname{Li}_2 \left[\frac{1}{2} \exp \left(\frac{i\pi}{2} \right) \right] - 4 \operatorname{Im} \operatorname{Li}_2 \left[\frac{1}{\sqrt{2}} \exp \left(\frac{i3\pi}{4} \right) \right] \\
& -2 \operatorname{Im} \operatorname{Li}_2 \left[\frac{1}{2\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right] - 2 \operatorname{Im} \operatorname{Li}_2 \left[\frac{1}{4\sqrt{2}} \exp \left(\frac{i\pi}{4} \right) \right],
\end{aligned}
\tag{81}
$$

from which we get, immediately, the binary BBP-type zero relation

$$
\begin{aligned}
0 = {} & P(2, 2^{60}, 120, (-2^{59}, 3 \cdot 2^{60}, -11 \cdot 2^{57}, 0, -23 \cdot 2^{56}, -3 \cdot 7 \cdot 2^{56}, 2^{56}, 0, -11 \cdot 2^{54}, \\
& -13 \cdot 2^{54}, -2^{54}, 0, 2^{53}, -3 \cdot 2^{54}, -7 \cdot 2^{52}, 0, -2^{51}, 3 \cdot 7 \cdot 2^{50}, -2^{50}, 0, 11 \cdot 2^{48}, -3 \cdot 2^{50}, \\
& 2^{48}, 0, 23 \cdot 2^{46}, 3 \cdot 2^{48}, -11 \cdot 2^{45}, 0, 2^{45}, 2^{46}, 2^{44}, 0, -11 \cdot 2^{42}, 3 \cdot 2^{44}, 23 \cdot 2^{41}, 0, 2^{41}, \\
& -3 \cdot 2^{42}, 11 \cdot 2^{39}, 0, -2^{39}, 3 \cdot 7 \cdot 2^{38}, -2^{38}, 0, -7 \cdot 2^{37}, -3 \cdot 2^{38}, 2^{36}, 0, -2^{35}, -13 \cdot 2^{34}, \\
& -11 \cdot 2^{33}, 0, 2^{33}, -3 \cdot 7 \cdot 2^{32}, -23 \cdot 2^{31}, 0, -11 \cdot 2^{30}, 3 \cdot 2^{32}, -2^{30}, 0, 2^{29}, -3 \cdot 2^{30}, \\
& 11 \cdot 2^{27}, 0, 23 \cdot 2^{26}, 3 \cdot 7 \cdot 2^{26}, -2^{26}, 0, 11 \cdot 2^{24}, 13 \cdot 2^{24}, 2^{24}, 0, -2^{23}, 3 \cdot 2^{24}, 7 \cdot 2^{22}, 0, \\
& 2^{21}, -3 \cdot 7 \cdot 2^{20}, 2^{20}, 0, -11 \cdot 2^{18}, 3 \cdot 2^{20}, -2^{18}, 0, -23 \cdot 2^{16}, -3 \cdot 2^{18}, 11 \cdot 2^{15}, 0, -2^{15}, \\
& -2^{16}, -2^{14}, 0, 11 \cdot 2^{12}, -3 \cdot 2^{14}, -23 \cdot 2^{11}, 0, -2^{11}, 3 \cdot 2^{12}, -11 \cdot 2^{9}, 0, 2^{9}, -3 \cdot 7 \cdot 2^{8}, 2^{8}, \\
& 0, 7 \cdot 2^{7}, 3 \cdot 2^{8}, -2^{6}, 0, 2^{5}, 13 \cdot 2^{4}, 11 \cdot 2^{3}, 0, -2^{3}, 3 \cdot 7 \cdot 2^{2}, 23 \cdot 2, \\
& 0, 11, -3 \cdot 2^{2}, 1, 0)). \qquad (82)
\end{aligned}
$$

9.2. Degree 3 Zero Relations

9.2.1. Base 2^{12} Zero Relation

Eliminating $\pi^2 \log 2$ and $\log^3 2$ between Equations (40), (41) and (46), and subtracting the resulting identity from Eq. (47) gives the BBP-ready identity

$$
\begin{aligned}
0 = {} & -\frac{88}{21} \operatorname{Li}_3\left[\frac{1}{2}\right] - \frac{132}{133} \operatorname{Li}_3\left[\frac{1}{4}\right] + \frac{88}{399} \operatorname{Li}_3\left[-\frac{1}{8}\right] - \frac{1584}{133} \operatorname{Re}\operatorname{Li}_3\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] \\
& + \frac{704}{133} \operatorname{Re}\operatorname{Li}_3\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] + \frac{704}{399} \operatorname{Re}\operatorname{Li}_3\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right], \qquad (83)
\end{aligned}
$$

which yields the only *expected* degree 3 base 2^{12} length 24 zero relation listed in the BBP Compendium and which was originally discovered by Bailey [1], using his PSLQ program.

9.2.2. Base 2^{60} Zero Relations

Solving Equations (38), (44) and (46) for $\zeta(3)$ and subtracting Eq. (52) gives the identity

$$
\begin{aligned}
0 = {} & \frac{50}{21} \operatorname{Li}_3\left[\frac{1}{2}\right] - \frac{5}{7} \operatorname{Li}_3\left[\frac{1}{4}\right] + \frac{10}{63} \operatorname{Li}_3\left[-\frac{1}{8}\right] - \frac{319}{63} \operatorname{Re}\operatorname{Li}_3\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] \\
& - \frac{236}{63} \operatorname{Re}\operatorname{Li}_3\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] - \frac{8}{9} \operatorname{Re}\operatorname{Li}_3\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\
& + \frac{68}{63} \operatorname{Re}\operatorname{Li}_3\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right]. \qquad (84)
\end{aligned}
$$

Solving Equations (41), (44) and (46) for $\zeta(3)$ and subtracting Equation (52) gives the identity

$$
\begin{aligned}
0 \;=\; & -\frac{30}{119}\,\mathrm{Li}_3\left[\frac{1}{4}\right] + \frac{20}{357}\,\mathrm{Li}_3\left[-\frac{1}{8}\right] - \frac{7}{3}\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] \\
& -\frac{52}{357}\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] + \frac{8}{357}\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\
& +\frac{76}{357}\,\mathrm{Re}\,\mathrm{Li}_3\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right].
\end{aligned}
\tag{85}
$$

The explicit base 2^{60} zero relations resulting from the identities (84) and (85) are as listed in the BBP Compendium.

9.3. Degree 4 Zero Relation

9.3.1. A Base 2^{60} Relation

Subtracting Eq. (65) from Eq. (62), we obtain the identity

$$
\begin{aligned}
0 \;=\; & -\frac{398592}{2911}\mathrm{Re}\mathrm{Li}_4\left[\frac{1}{2\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] + \frac{7627392}{20377}\mathrm{Re}\mathrm{Li}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\
& +\frac{1150848}{2911}\mathrm{Re}\mathrm{Li}_4\left[\frac{1}{\sqrt{2}}\exp\left(\frac{3i\pi}{4}\right)\right] + \frac{34560}{497}\mathrm{Re}\mathrm{Li}_4\left[\frac{1}{4\sqrt{2}}\exp\left(\frac{i\pi}{4}\right)\right] \\
& -\frac{38797488}{20377}\mathrm{Re}\mathrm{Li}_4\left[\frac{1}{2}\exp\left(\frac{i\pi}{2}\right)\right] + \frac{1394496}{20377}\mathrm{Li}_4\left[\frac{1}{2}\right] \\
& -\frac{3888}{41}\mathrm{Li}_4\left[\frac{1}{4}\right] + \frac{576}{41}\mathrm{Li}_4\left[-\frac{1}{8}\right] + \frac{26136}{2911}\mathrm{Li}_4\left[-\frac{1}{4}\right].
\end{aligned}
\tag{86}
$$

As usual, writing each component of Eq. (86) as a base 2^{60}, length 120 binary BBP-type formula and forming the indicated combination, we obtain the only degree 4 binary zero relation, given explicitly in the BBP Compendium.

10. Conclusion

Using a clear and straightforward approach, we have obtained and proved interesting new binary digit extraction BBP-type formulas for polylogarithm constants. Some known results were also rediscovered in a clearer and more elegant manner. Experimentally discovered binary BBP-type formulas are also proved.

References

[1] D. H. Bailey. A compendium of bbp-type formulas for mathematical constants
`http://crd.lbl.gov/~dhbailey/dhbpapers/bbp-formulas.pdf`. 2011.

[2] David H. Bailey, Peter B. Borwein, and Simon Plouffe. On the rapid computation of various polylogarithmic constants. *Mathematics of Computation*, 66(218):903–913, 1997.

[3] David H. Bailey and Richard E. Crandall. On the random character of fundamental constant expansions. *Experimental Mathematics*, 10:175, 2001.

[4] Jonathan M. Borwein, William F. Galway, and David Borwein. Finding and excluding b-ary Machin-type BBP formulae. *Canadian Journal of Mathematics*, 56:1339–1342, 2004.

[5] Marc Chamberland. Binary bbp-formulae for logarithms and generalized gaussian-mersenne primes. *Journal of Integer Sequences*, 6, 2003.

[6] H. R. P. Ferguson, D. H. Bailey, and S. Arno. Analysis of pslq, an integer relation finding algorithm. *Math. Comput.*, 68:351–369, 1999.

[7] David H Bailey. Algorithms for Experimental Mathematics I
`http://crd.lbl.gov/~dhbailey/dhbpapers`. 2006.

[8] G. Huvent. Formules BBP
`http://math.univ-lille1.fr/~bbecker/ano/seminaries/expo_huvent01.pdf`. 2001.

[9] Leonard Lewin. *Polylogarithms and associated functions*. Elsevier North Holland Inc., 1981.

[10] Kunle Adegoke. A non-pslq route to bbp-type formulas. *Journ. Maths. Res.*, 2:56–64, 2010.

[11] D. J. Broadhurst. Polylogarithmic ladders, hypergeometric series and the ten millionth digits of $\zeta(3)$ and $\zeta(5)$. `arXiv:math/9803067v1 [math.CA]`, 1998.