ANALYZING TWO-COLOR BABYLON

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Abstract

We examine the impartial combinatorial game Babylon. We abstract the game so
that it is suitable to combinatorial analysis, and present a full characterization and
strategies for the cases with odd number of tokens and only two colors. We also
demonstrate partial results on games with even tokens with two colors, an initial
extension to three colors, and offer directions for future work.

1. Introduction

The impartial combinatorial game Babylon was invented by French designer Bruno
Faidutti in 2003 [1]. The game begins with 12 tokens, 3 in each of 4 colors. In
Babylon, two players take turns combining two piles of tokens by placing one on
top of the other, where piles can be combined if they have the same color token on
top or the same height. A player wins if they are the last player able to combine
two piles.

We abstract the game of Babylon for our analysis. A game B is defined by the
triple

$$B = (C, T, R)$$

where C is a natural number to denote the number of colors, T is a natural number
to denote the number of tokens, and R is a monotonically decreasing C-tuple of
whole numbers to denote the count of tokens in each color and sums to T. The
original game of Babylon would be denoted as

$$B = (4, 12, \{3, 3, 3, 3\})$$

It is important to note that after each move, the number of piles decreases by one. As a result, the parity of the number of piles a player sees before their move is always the same.

2. Background

In earlier investigations of $(4, 12, \{3, 3, 3, 3\})$ we and others found it is a win for the second player [2, 3, 4]. A simulation of the game in Java (with reductions on the number of nodes based on symmetric game positions) provided the full game tree of 688 states, and allowed for backward propagation of the end-game states to the initial move.

While this analysis proved fruitful, it did not lead to insight into strategies for the second player to follow nor a general solution for the game. To facilitate this research, we restrict the original game to smaller cases, shrinking both the number of colors and the number of tokens.

3. Analyzing Simpler Cases

Our analysis for this game is with two colors, letting T and R vary. Formally, our games are of the type $(2, T, R)$.

In these games, we use the convention of the most common color in R being denoted as blue, and the least common color denoted as red. If the ratio is exactly even, then this distinction will be irrelevant.

An individual state of the game Babylon can be represented as a list of the individual blue pile heights followed by a list of the individual red pile heights. A sample starting position would be $(1, 1, 1, 1 : 1, 1, 1)$, and a sample midgame position would be $(3, 2 : 1, 1)$. A full game tree for $(2, 7, \{4, 3\})$ can be see in Figure 1.

Unlike other combinatorial games such as Domineering, the game is not decomposable. Piles are limited in their interaction by their height and top color, and both of these can change throughout the game to open up possible moves. We will therefore avoid a discussion of Grundy numbers and instead analyze this game with respect to P and N. Table 1 shows the results of game tree analysis on the first few $(2, T, R)$ games, conducted by computer simulation.

We immediately notice a pattern in the first two rows, where if there are only one or two red tokens, the first player will always win the game. Also, there appears to be a checkerboard pattern of P and N when there are greater than 2 red stones. Based on these initial observations, we next formulate proofs to capture our observations.
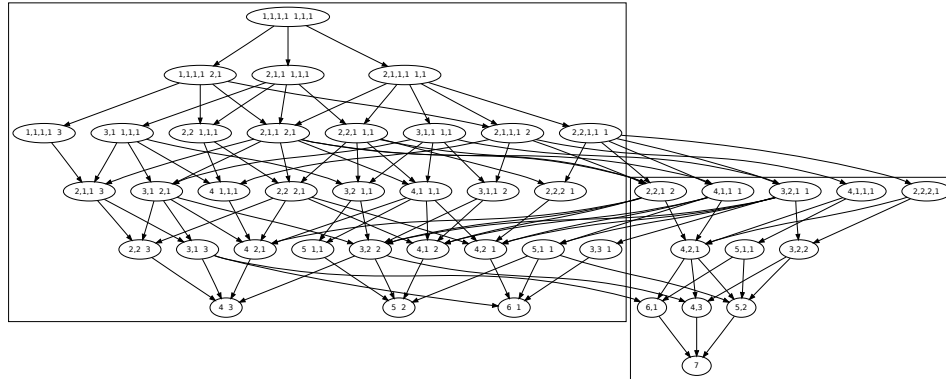
Figure 1: Game Tree for $(2, 7, \{4, 3\})$

4. Even Number of Tokens, One or Two Red Tokens

Theorem 1 $(2, 2m, \{2m - 1, 1\})$ *is a win for the first player.*

Proof. Assume we start Babylon with the starting configuration $(2, 2m, \{2m-1, 1\})$. The first player wins if the game results in one pile after an odd number of moves. Thus the winning move for the first player is to cover the one red token with a blue token. Since all the piles now have the same color on top and there is an odd number of piles, the game will now end after an odd number of moves.   □

Theorem 2 $(2, 2m, \{2m - 2, 2\})$ *is a win for the first player.*

Proof. Assume we start with the configuration $(2, 2m, \{2m-2, 2\})$. Again, the first player will win if the game results in one pile. If $2m = 4$, the first player should place a token on top of a similar token. The second player's only possible move is to combine the other two tokens, and the first player wins by combining the two piles of size two. If $2m > 4$, the first player's move should be to cover a red token with a blue token. The strategy then breaks into cases:

- If the second player created another pile of size 2 containing the other red token, place the first player's pile on top of the second player's pile. All piles have a blue token on top, and the first player will win.

- If the second player created another pile of size 2 not containing the other red token, cover that last red token with a blue token. (Since there are an even

		Blue							
		1	2	3	4	5	6	7	8
	1	N	N	N	N	N	N	N	N
Red	2		N	N	N	N	N	N	N
	3			P	N	P	N	P	N
	4				P	N	P	N	P
	5					P	N	P	N
	6						P	N	P
	7							P	N
	8								P

Table 1: N and P classifications for Babylon games $(2, b + r, \{b, r\})$

number of tokens and at least six tokens, this blue token exists.) All piles have a blue token on top, and the first player will win.

- If the second player placed a blue token on top of the first player's pile, there is still one red token and at least one blue token (both in piles of height 1). Place a blue token on top of the only red token. All piles have blue on top, and the first player will win. □

5. Odd Number of Tokens

Theorem 3 $(2, m + n, \{m, n\})$ *is a win for the first player if* $m + n$ *is odd.*

Proof. Assume we have the starting configuration $(2, m + n, \{m, n\})$ where $n < m$. Assume we will be the starting player, so when we end our turn, there will be an even number of piles. We will make the following definitions for game positions after our turn. Note that the color of a pile will refer to the color of the token on top of that pile. The colors of the tokens under that top token are irrelevant.

Definition A position is *single minority pile* (or *SMP*) if there exists one pile of one color with all other piles having the other color and none of these piles equal in height to the first pile.

Definition A position is *preferred pile determined* (or *PPD*) if

- there exists a pile of height 4 or greater,

- all other piles have height 1, and

- the color of the minority of the height 1 piles matches the color of the pile of height 4 or greater.

We will refer to the pile of height 4 or greater as the *preferred pile*.

We play using the following algorithm, moving through at most four stages. Our goal is to reach an SMP position, but we may reach a PPD position before the SMP position.

Stage 1: As our first move, place a red token on a blue token.

- If $n = 1$ and $m = 2$, we have won the game.

- If $n = 1$ and $m > 2$, the game position is SMP — move to stage 4.

- Otherwise, move to stage 2.

Stage 2: After our opponent's play, we will make a move that will create a PPD or SMP position. In either case, we will create a preferred pile and a minority color (the color on top of the preferred pile.)

- If $n = 2$ and our opponent's move was to place the remaining red token on top of the pile of height 2, make any other move. The game position is now SMP — move to stage 4.

- If $n = 2$ and our opponent's move was to place a red on a blue token, place this new pile on top of the first pile of height 2. The game position is now SMP — move to stage 4.

- If $n > 2$ and our opponent's move was to place a red token on top of the pile of height 2, we place another red token on the pile of height 3. This pile is our preferred pile, the minority color is red, and our position is PPD (note that we have fewer red piles of height 1 than we do blue) — move to stage 3.

- If $n > 2$ and our opponent's move was to place a red token on top of either a red or blue token, place the first pile of height 2 on top of this new pile. The game position is now PPD with minority color red — move to stage 3.

- If $n \geq 2$ and our opponent's move was to place a blue token on top of either a blue or red token, consider the remaining piles of height 1. Since there are an odd number of these piles, one type of token is the minority. Place the pile of height 2 with the minority color on top on top of the other pile of height 2. The game position is now PPD — move to stage 3.

Stage 3: After our last move, we left the game position in a PPD state. We will follow the following strategy during stage 3:

- Place the largest minority pile onto the preferred pile.

- If this results in a single minority pile, by remarks below, our position is SMP — move to stage 4.

- Otherwise, we remain in stage 3.

Note that we start with all minority piles having height 1. If our opponent ever creates a minority pile of height 2, on our move, we add it to the preferred pile. We subsequently note that throughout stage 3, after each of our moves, the color we have called minority is still the minority — that is, of the piles besides the preferred pile, the minority color occurs less frequently. For if at any point, our opponent tries to switch minority and majority by placing a minority token on a majority token, we then add that pile to the preferred pile.

Finally, we note that during stage 3, no majority pile is ever as tall as the preferred pile. Each turn, the preferred pile grows in height by at least 1. Assume it takes k moves on our part to create a single minority pile. This preferred pile is now at least $k + 4$ tokens high. However, starting with piles of height 1, it takes at least j moves to create a pile of height $j + 1$. After our last move, our opponent will have had k moves (starting with their first move after entering stage 3). Thus, the highest possible majority pile is $k + 1$. Clearly, no majority pile has size equal to the preferred pile, and thus the position is SMP.

Stage 4: Having reached an SMP position, after our opponent's move we play as follows: make any move that leaves the game in an SMP position.

First note that on our last move, we placed the last minority pile on the preferred pile. At this point, there was at least one majority pile (since, as noted above, the minority color is preserved during stage 3). Morever, since after our move, there are an even number of piles, either we reduced to two piles total (in which case we won) or there were at least three majority piles.

Thus, on our first move in stage 4, there are at least two majority piles. Note that starting from an SMP position, since no piles have height equal to the preferred pile, our opponent can create at most one pile of height equal to the preferred pile on their turn.

We claim that from now on in stage 4, we either win the game or can make a move that leaves the game in an SMP position. We break down the possibilities in cases:

- If there are exactly two majority piles, we place one on the other and win. (Both resulting piles — the minority and the majority — cannot have the same height, since we have an odd number of tokens.)

- Otherwise, there are at least four majority piles (since the number of majority piles is even). If one pile has height equal to the preferred pile, we place this pile on another majority pile. By our note above, this was the only majority pile equal in height to the preferred pile.

- Otherwise, we have at least four majority piles with none equal in height to the preferred pile. If among the majority piles, there are two different heights,

then there exists some move (placing a majority pile on a majority pile) that does not create a pile equal in height to the minority pile. If all the majority piles have the same height, there is still such a move unless this common height is half the height of the preferred pile. So assume all the majority piles have height p and the preferred pile has height $2p$. But then, since we have an even number of majority piles, we have a even total number of tokens, contradicting our assumption.   □

6. Even Number of Tokens, More than Two Red Tokens

Conjecture 4 $(2, b + r, \{b, r\})$ *is a win for the second player if $r > 2$ and $b + r$ is even.*

As seen in Table 1, the second player wins all games examined to date where the number of tokens is even and the number of red tokens is greater than two. As of this paper, the authors have not found a proof for this conjecture. However, there are some clear losing game states which the second player wishes to avoid. The first (a disjoint partition state) can occur midgame; the second (a subsets of cases where only one pile exists of one color) occurs in the endgame.

6.1. Disjoint Partitions

Assume we start Babylon with an even number $2m$ of tokens. We describe a game state as being a *disjoint partition state* after a first player's move if it has the form $s_1, ..., s_i : t_1, ..., t_j$ where

- $i + j$ is odd (showing the first player has just played)

- $\sum_{k=1}^{i} s_k = \sum_{k=1}^{j} t_k = m$

- for all k, l (where $1 \le k \le i$ and $1 \le l \le j$) $s_k \ne t_l$

For example, the game position $2, 3 : 5$ would be a disjoint partition state. Note that the second player (on the next move) can only combine the piles of size 2 and 3, leaving two piles of size 5 for the first player to combine and win the game.

Lemma 5 *If the first player leaves a disjoint partition state, then regardless of the second player's move, the first player can either win the game or return the game to a disjoint partition state.*

Proof. We note that the second player cannot combine two piles of different colors, thus, the second player, on their move, can only combine two piles of the same color. If, after this move, no pile of one color has a height equal to a pile of the other color,

find the unique pile of greatest height. If this height is m, then the first player will win (since the last move of the game will be combining the two piles of different colors). If the height of the largest pile is not m, then there exists some other pile of that color. Add that pile to the largest pile — there is still no duplication in pile height, and the game is in a disjoint partition state.

If, after the second player's move, a pile of one color has height equal to a pile of another color, find the piles of greatest height. If there is only one, add the appropriate duplicate pile to the pile of greatest height — the game is in a disjoint partition state. If there are two piles of greatest height, they must be the duplicate piles just created. If they both have height m, the first player wins on the next move. If both piles have a common height less than m, the first player adds one pile to a pile of greatest height, and the game is in a disjoint partition state.   □

Thus, during the game, a second player needs to avoid any move that will allow the first player to create a disjoint partition state.

6.2. Endgame Concerns

Again, we assume we start the game with $2m$ tokens as the second player. We are not assuming we have an equal number of each color token. We therefore win if the ending position is two different piles with different colors on top and different heights.

We assume we have reached the point where, for one of the colors, there is only one pile (which we will call the *preferred pile*). We assume we have just made a move (so there is an even number of piles). The game position satisfies one of the following configurations:

1. There exists a pile of the other color equal in height to the preferred pile. On our opponent's move, they will move that pile onto the preferred pile. This creates a situation where there are an odd number of piles of only one color. Since the game will now end in an even number of moves, and we are next to move, this is a losing position.

2. The preferred pile has height equal to m (that is, half the number of tokens). Then, the only moves remaining are the piling up of the piles of the other color and the last move of placing one pile of height m on the other. Thus, this is a losing position.

3. Any pile (including the preferred pile) has height greater than m and no pile of the other color is equal in height to the preferred pile. Note that there must be at least one other pile of the other color (since we have an even number of piles). If this largest pile is the preferred pile, this is an obvious winning position. If the largest pile is of the other color, we can always on our turn place a pile on top of this largest pile. (In fact, if our opponent creates a

pile equal in height to the preferred pile, we must place this new pile on the largest pile.) Since the preferred pile will never be covered, this is a winning position.

4. The preferred pile has height less than m, no other pile has height equal to the preferred pile, but some pile has height greater than the preferred pile. Regardless of our opponent's move, we can always place a pile on top of this larger pile. (In fact, if our opponent creates a pile equal in height to the preferred pile, we must place this new pile on the larger pile.) Since the preferred pile will never be covered, this is a winning position.

5. The preferred pile has height less than m and all other piles have height less than the preferred pile.

We will now explore whether the last configuration is a winning or losing position (and therefore assume we are in the last configuration).

Recall that we lose the game if the ending position is one pile. This only occurs if the preferred pile is placed under another pile of the same height. Therefore, we will lose the game if, after our move, there is a pile of the same height as the preferred pile.

Assume on our move that there are only two piles of the other color (besides the preferred pile). The only legal move is to combine these two piles (since neither are equal to the height of the preferred pile). But the resulting pile cannot have height equal to the preferred pile, since the preferred pile has height less than m and there are $2m$ tokens total. So our move does not leave a pile of the same height as the preferred pile.

Assuming there are more than two piles of the other color, note that if we have at least three piles of the other color with at least two distinct heights, on our move, we can always combine two of these piles and not leave a pile equal in height to the preferred pile. Even if all of the piles of the other color have the same height, we can combine two of these piles and not leave a pile equal in height to the preferred pile unless the common height of the piles of the other color is half the height of the preferred pile.

To explore this case, we introduce the notation $i : j_1, j_2, j_3, \ldots$ to indicate that we have a preferred pile of height i and piles of the other color of heights j_1, j_2, ... Using this notation, we do not on our move want to start with the position $2k : k, k, \ldots, k$. Thus, we do not want to leave the position $2k : k, k, \ldots, k, k - j, j$ for some j after our move.

We therefore make the following definition:

Definition Assume that we started with an even number of tokens $2m$, we are the second player, and that there is now only one pile of one color (the *preferred pile*). Assume in addition that the height of the preferred pile is less than m and all other

piles have height less than the preferred pile. We will call our position *safe* if either:

- the preferred pile has an odd height, or

- the preferred pile has height equal to $2k$ for some k and the game position is not $2k : k, k, \ldots, k, k - j, j$ for some $0 < j < k$.

Lemma 6 *If we leave a safe position after our move, on our next move, we will leave either a safe position or a configuration of type 3 or 4 (which are winning positions).*

Proof. Note that we will have a next move since our opponent has no piles of height equal to the preferred pile. Also note that before our next move, there will be an odd number of piles and thus an even number of piles of the other color. In particular, there will be at least two piles of the other color.

If the height of the preferred pile is odd and our opponent creates a pile of height equal to the preferred pile, on our move we add another pile of that color to the new pile. We are in configuration 3 or 4. If the opponent makes any other move, note that after that move, there are at least two piles of the other color. If there are exactly two piles, we combine them and win the game. If there are more than two piles, then a move exists that will create a pile not equal in size to the preferred pile. Thus, our position is safe, or we are in configuration 3 or 4.

If the preferred pile has height equal to $2k$ for some k and the game position is not $2k : k, k, \ldots, k, k - j, j$ for some $0 < j < k$, then we consider our opponent's possible moves:

- If our opponent creates a pile of height $2k$, then we place another pile of the same color on top. We are in configuration 3 or 4.

- If our opponent creates a pile of height greater than $2k$, then we can make any move and we will be in configuration 3 or 4.

- If our opponent creates a pile of height less than $2k$ but greater than k, then we can make any move and we will be safe or in configurations 3 or 4.

- Suppose our opponent creates a pile of height k or less. On our move, we have the following possibilities:

    - If we can create a pile of height greater than k but not equal to $2k$, we do so. We have left a safe position.

    - If we can only create piles of height $2k$, the game position must be $2k : k, k, \ldots, k$. But then we must have left the position $2k : k, k, \ldots, k, k-j, j$ for some j, contradicting our assumption that we left a safe position.

		Blue									
		1	2	3	4	5	6	7	8	9	10
	1	N	N	N	N	N	N	N	N	N	N
	2		N	P	N	N	N	P	N	P	N
Red	3			N	N	N	P	N	P	N	P
	4				N	P	N	P	N	P	N
	5					N	P	N	P	N	N
	6						N	P	N	N	N
	7							N	N	N	P
	8								N	P	N
	9									N	P
	10										N

Table 2: N and P classifications for Babylon games $(3, b + r + 1, \{b, r, 1\})$

- If we can only create piles of height k, then the game position must be $2k : \frac{k}{2}, \frac{k}{2}, ..., \frac{k}{2}$. Any move we make will result in a safe position unless we started with the position $2k : \frac{k}{2}, \frac{k}{2}, \frac{k}{2}, \frac{k}{2}$. If we start with this position, we must produce $2k : k, \frac{k}{2}, \frac{k}{2}$. But then the preferred pile equals one-half of the total number of tokens, which contradicts our assumption.
- If we can only create piles of height less than k, then there must be no piles of height k and any move we make results in a safe position.

Therefore, if we leave a safe position, on our next move we can either move to a winning position or leave a safe position.   □

The above discussion summarizes the 'endgame' for this version of Babylon and outlines a possible 'midgame' strategy: we should strive to reach configurations 3 or 4 or, if we reach configuration 5, we must do so in a safe position.

7. Hypotheses and Conclusions

At this point, we have only preliminary computations for extensions to the general cases with three or more colors. Table 2 shows the complexity introduced by adding one token of color yellow.

This appears similar to the results in Table 1, except for games where T is a multiple of 8, which disrupts the checkerboard pattern observed in the two-color games. Based on our earlier results, we can prove the following two corollaries.

Corollary 7 $(3, b + 2, \{b, 1, 1\})$ *is a win for the first player if $b + 2$ is even.*

Proof. If $b = 2$, the first player should place one blue token on the other. The only move left for the second player is to combine the red and yellow tokens, leaving the

first player to win the game. If $b > 2$, we first note that once a color is no longer on the top of any pile, there is no way to uncover that color through game play. The first player should then cover either the yellow or red token with a blue token. This game is now indistinguishable from $(2, b + 2, \{b, 2\})$, and from Theorem 4.2, we know that this game is a win for the first player.   □

Corollary 8 $(3, b + r + 1, \{b, r, 1\})$ *is a win for the first player if* $b + r + 1$ *is odd.*

Proof. The first player should place a blue token on a yellow token. This game is now indistinguishable from $(2, b + r + 1, \{b, r + 1\})$ where our strategy was to place a blue token on a red token in the first move. Thus from Theorem 5.1, we know that this game is a win for the first player.   □

In conclusion, we have a full characterization and strategies for the cases with one or two red tokens and all cases with all odd number of tokens. Based on experimental evidence, we believe that the game with an even number of tokens (with more than two red tokens) is always a second player win, but we have no proof or strategy. We have shown two partial results that may inform play in the two color game for an even number of tokens, and two corollaries for the three color game. We continue to explore these smaller games to build an eventual strategy for the original formulation of $(4, 12, \{3, 3, 3, 3\})$.

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