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THE NUMBER OF PERMUTATIONS WITH PRESCRIBED UP-DOWN STRUCTURE AS A FUNCTION OF TWO VARIABLES

Vladimir Shevelev
Dept. of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva, Israel
shevelev@bgu.ac.il

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Abstract

We consider the number of permutations with prescribed up-down structure as a function of two arguments: the number n of elements and the introduced up-down index k of a permutation. We consider sets of permutations for which k is a fixed number and when k is a function of n. In the first case the number of permutations is a polynomial in n, the degree of which is defined by k.

1. Introduction

D. Andre [3] first considered (1881) the problem of enumerating the alternating permutations $\pi = (\pi_1, \dots \pi_n)$ of the numbers $1, 2, \dots, n$ for which ups and downs are alternating:

$$\pi_1 < \pi_2 > \pi_3 < \dots$$

This problem has a highly aesthetic solution: the exponential generating function of such permutations is the sum of tangent and secant. But only after almost a century (1968) I. Niven [14] considered the general problem of the enumerating the permutations with given up-down structure.

Definition 1. For a permutation $\pi = (\pi_1, \dots, \pi_n)$, the sequence $(q_1, q_2, \dots, q_{n-1})$, where

$$q_j = sign(\pi_{j+1} - \pi_j) = \begin{cases} 1, & if \ \ \pi_{j+1} > \pi_j \\ -1, & if \ \ \pi_{j+1} < \pi_j \end{cases}, \tag{1}$$

is called Niven's signature.

For example, $a = (2, 1, 5, 4, 3)$ has the signature $(-1, 1, -1, -1)$. Denote by $[q_1, q_2, \dots, q_{n-1}]$ the number of permutations having Niven signature $(q_1, q_2, \dots, q_{n-1})$. In view of symmetry we have

$$[q_1, q_2, \dots, q_{n-1}] = [-q_{n-1}, -q_{n-2}, \dots, -q_1]. \tag{2}$$

Niven obtained the following basic result.

Theorem 2. *([14]). In the signature $(q_1, q_2, \ldots, q_{n-1})$, let the indices of the positive q_i be $s_1 < s_2 < \ldots < s_m$ (if such q_i do not exist then assume $m = 0$). In addition, set $s_0 = 0$, $s_{m+1} = n$. Then*

$$[q_1, q_2, \ldots, q_{n-1}] = detN, \tag{3}$$

where $N = \{n_{ij}\}$ is the square matrix of order $m + 1$ in which

$$n_{ij} = \binom{s_i}{s_{j-1}}, \quad i, j = 1, 2, \ldots, m + 1. \tag{4}$$

Since Niven's celebrated result, there have been many articles on this subject. We mention only eleven papers in chronological order: N. G. Bruijn, 1970 [6], H. O. Foulkes, 1976 [10], L. Carlitz, 1978 [7], G. Viennot, 1979 [23], C. L. Mallows and L .A. Shepp, 1985 [12], V. Arnold, 1991 [4], V. S. Shevelev, 1996 [18], G. Szpiro, 2001 [21], B. Shapiro, M. Shapiro and A. Vainshtein, 2005 [17], F. C. S. Brown, T. M. A. Fink and K. Willbrand, 2007 [5], R. Stanley, 2007 [21].

Let us introduce an *index* of the Niven signature by the following way.

Definition 3. The integer $k = k_n$ is called the index of the signature $(q_1, q_2, \ldots, q_{n-1})$ if the $(n-1)$-digit binary representation of k is

$$k = \sum_{i=1}^{n-1} q_i^{!} 2^{n-i-1}, \tag{5}$$

where

$$q_i^{!} = \begin{cases} 1, & if \;\; q_i = 1, \\ 0, & otherwise \end{cases} \quad . \tag{6}$$

Denote by $S_n^{(k)}$ the set of permutations of elements $1, 2, \ldots, n$ having the index k, and put

$$\left\{ {n \atop k} \right\} = \left| S_n^{(k)} \right|. \tag{7}$$

It is clear that in (7) we have $n \geq 1$ and $0 \leq k \leq 2^{n-1} - 1$.

Note that, in view of the fact that $[q_1, q_2, \ldots, q_{n-1}] = [-q_1, -q_2, \ldots, -q_{n-1}]$,

$$\left\{ {n \atop k} \right\} = \left\{ {n \atop 2^{n-1} - 1 - k} \right\}. \tag{8}$$

We say that indices k and $\overline{k} = 2^{n-1} - 1 - k$ are *conjugate indices.*

In case of a fixed k, we show that $\left\{ n \atop k \right\}$ is a polynomial in n of degree $\lfloor \log_2(2k) \rfloor$. For different values of k, we call these polynomials *up-down polynomials.* In general, when k depends on n, we call $\left\{ n \atop k \right\}$ *an up-down coefficient.*

Recall that the number of permutations $1, \dots, n$ in which exactly t elements are greater than the previous element (i.e., permutations with t ascents) is given by the Eulerian number (see, e.g., [9], p.243; [11], p.267-272)

$$A(n,t) = \sum_{i=0}^{t} (-1)^i \binom{n+1}{i} (t+1-i)^n, \ \ 0 \le t \le n-1. \tag{9}$$

Let s(k) denote the number of $1's$ in the binary expansion of k. Then, by the definition, we have an expansion of $A(n,t)$ over polynomials in n :

$$A(n,t) = \sum_{j: \ s(j)=t} \left\{ n \atop j \right\}. \tag{10}$$

Let us draw an analogy between binomial coefficients and up-down coefficients;

1a. $\binom{n}{k}$ is the number of subsets of the cardinality k of a set of n elements.

1b. $\left\{ n \atop k \right\}$ is the number of permutations of n elements having the index k.

2a. Each subset of a set of n elements is contained in the number of $\binom{n}{k}$ subsets for some value of k.

2b. Each permutation of n elements is contained in the number of $\left\{ n \atop k \right\}$ permutations for some value of the index k.

3a. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

3b. $\sum_{k=0}^{2^{n-1}-1} \left\{ n \atop k \right\} = n!$

4a. $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

4b. $\sum_{k=0}^{2^n-1} (-1)^k \left\{ n \atop k \right\} = 0$.

5a. $\sum_{0 \le k \le n, \ \ k \ \ even} \binom{n}{k} = \sum_{0 \le k \le n, \ \ k \ \ odd} \binom{n}{k} = 2^{n-1}$.

5b. $\sum_{0 \le k \le 2^{n-1}-1, \ \ k \ \ even} \left\{ n \atop k \right\} = \sum_{0 \le k \le 2^{n-1}-1, \ \ k \ \ odd} \left\{ n \atop k \right\} = \frac{n!}{2}, \ \ n \ge 2$.

6a. $\binom{n}{n-k} = \binom{n}{k}$.

6b. $\left\{ {n \atop k} \right\} = \left\{ {n \atop 2^{n-1} - 1 - k} \right\}$.

7a. $\binom{n}{0} = \binom{n}{n} = 1$.

7b. $\left\{ {n \atop 0} \right\} = \left\{ {n \atop 2^{n-1} - 1} \right\} = 1$.

The latter equality corresponds to the identity permutation $(1, \ldots, n)$ and its reverse permutation $(n, \ldots, 1)$.

8. The central binomial coefficients and the "central" numbers $\left\{ {n \atop k} \right\}$ are equal to one another:

$$\left\{ {2n \atop 2^n - 1} \right\} = \binom{2n-1}{n-1} = \binom{2n-1}{n}; \quad \left\{ {2n+1 \atop 2^n - 1} \right\} = \binom{2n}{n}.$$

We prove these and other properties of the up-down coefficients in different sections of this paper and in [19].

9. Both $\left\{ {n \atop k} \right\}$ and $\binom{n}{k}$ are polynomials in n, if k does not depend on n, and, generally speaking, are not polynomials, otherwise.

Example 4. In case of the alternating permutations $\pi_1 < \pi_2 > \pi_3 < \ldots$, we have the sequence of indices $k = \kappa_{n-1}, \ n = 1, 2, 3, \ldots$ such that

$$\kappa_0 = 0, \ \ \kappa_1 = 1, \ \ \kappa_2 = 2, \ \ \kappa_3 = 5, \ \ \kappa_4 = 10, \ \ \kappa_5 = 21, \ldots .$$

Here $\kappa_n - \kappa_{n-2} = 2^{n-1}, \ \ n \geq 3$, whence

$$\kappa_{n-1} = \frac{2^{n+1} - 3 + (-1)^n}{6}, \ \ n = 1, 2, \ldots \tag{11}$$

and, according to (11), we put also $\kappa_{-1} = 0$. Thus, from the classical Andre's result we have

$$\sum_{n=0}^{\infty} \left\{ {n \atop \kappa_{n-1}} \right\} \frac{x^n}{n!} = \tan x + \sec x, \tag{12}$$

where we put $\left\{ {0 \atop 0} \right\} = 1$.

From the latter formula we find some values of $\left\{ n \atop m \right\}$:

$$\left\{ 1 \atop 0 \right\} = 1,\ \left\{ 2 \atop 1 \right\} = 1,\ \left\{ 3 \atop 2 \right\} = 2,\ \left\{ 4 \atop 5 \right\} = 5,\ \left\{ 5 \atop 10 \right\} = 16,\ \left\{ 6 \atop 21 \right\} = 61,\ \ldots \qquad (13)$$

It is well-known that these values are explicitly expressed by absolute values of the Bernoulli and Euler numbers [4], [8]. More exactly, for $n = 2m$ and $n = 2m - 1$ we have correspondingly

$$\left\{ 2m \atop \frac{2^{2m}-1}{3} \right\} = |E_{2m}|,\ \left\{ 2m-1 \atop \frac{2^{2m-1}-2}{3} \right\} = \frac{|B_{2m}|}{2m}(2^{2m}-1)2^{2m}. \qquad (14)$$

In this paper we develop a theory of up-down coefficients. In particular, we obtain generator functions, explicit formulas and numerous recursions for them, and study some interesting arithmetic properties. We also study zeros of up-down polynomials and give characteristic conditions when a given polynomial in n is up-down. Finally, we pose some open problems.

2. An Explicit Formula for Up-Down Coefficients

Let $k \in [2^{t-1}, 2^t)$ and the $(n-1)$-digit binary expansion of k have a form:

$$k = \underbrace{0\ldots0}_{n-t-1}\ 1\ \underbrace{0\ldots0}_{s_2-s_1-1}\ 1\ \underbrace{0\ldots0}_{s_3-s_2-1}\ 1\ldots1\ \underbrace{0\ldots0}_{s_m-s_{m-1}-1}\ 1\ \underbrace{0\ldots0}_{t-s_m}, \qquad (15)$$

where $1 = s_1 < s_2 < \ldots < s_m$ are places of 1's *after* $n-t-1$ 0's before the first 1.

Put

$$k^* = 2^{n-1} - k - 1 = \sum_{i=0}^{n-2} q_i 2^i. \qquad (16)$$

By Equation 7 and Theorem 2, we have

$$\left\{ n \atop k \right\} = \left\{ n \atop k^* \right\} = \begin{vmatrix} \binom{s_1}{0} & \binom{s_1}{s_1} & \binom{s_1}{s_2} & \cdots & \binom{s_1}{s_{m-2}} & \binom{s_1}{s_{m-1}} & \binom{s_1}{s_m} \\ \binom{s_2}{0} & \binom{s_2}{s_1} & \binom{s_2}{s_2} & \cdots & \binom{s_2}{s_{m-2}} & \binom{s_2}{s_{m-1}} & \binom{s_2}{s_m} \\ \binom{s_3}{0} & \binom{s_3}{s_1} & \binom{s_3}{s_2} & \cdots & \binom{s_3}{s_{m-2}} & \binom{s_3}{s_{m-1}} & \binom{s_3}{s_m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \binom{s_m}{0} & \binom{s_m}{s_1} & \binom{s_m}{s_2} & \cdots & \binom{s_m}{s_{m-2}} & \binom{s_m}{s_{m-1}} & \binom{s_m}{s_m} \\ \binom{n}{0} & \binom{n}{s_1} & \binom{n}{s_2} & \cdots & \binom{n}{s_{m-2}} & \binom{n}{s_{m-1}} & \binom{n}{s_m} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & \binom{s_2}{s_1} & 1 & \dots & 0 & 0 & 0 \\ 1 & \binom{s_3}{s_1} & \binom{s_3}{s_2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & \binom{s_m}{s_1} & \binom{s_m}{s_2} & \dots & \binom{s_m}{s_{m-2}} & \binom{s_m}{s_{m-1}} & 1 \\ 1 & \binom{n}{s_1} & \binom{n}{s_2} & \dots & \binom{n}{s_{m-2}} & \binom{n}{s_{m-1}} & \binom{n}{s_m} \end{vmatrix}. \tag{17}$$

Later on, we write the index k of the permutation signature in a more natural form; consider permutations from the reverse side, then the points of descents $t_1 > t_2 > \ldots > t_m \geq 1$ turn into points of ascents. Let $\overline{k}$ be the conjugate index with 1's in its binary expansion on the places t_i, $i = 1, \ldots, m$, and 0's on the other places. Then

$$\overline{k} = 2^{t_1 - 1} + 2^{t_2 - 1} + \cdots + 2^{t_m - 1}, \;\; t_1 > t_2 > \ldots > t_m \geq 1. \tag{18}$$

Note that, in view of (8), if

$$k := \overline{k}, \tag{19}$$

then the numbers of the corresponding permutations are equal to one another. Therefore, using (17)-(19) and replacing s_i by t_{m-i+1}, we obtain the following result.

Theorem 5. *If*

$$k = 2^{t_1 - 1} + 2^{t_2 - 1} + \ldots + 2^{t_m - 1}, \;\; t_1 > t_2 > \ldots > t_m \geq 1, \tag{20}$$

then

$$\left\{ n \atop k \right\} = \begin{vmatrix} 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & \binom{t_{m-1}}{t_m} & 1 & \dots & 0 & 0 & 0 \\ 1 & \binom{t_{m-2}}{t_m} & \binom{t_{m-2}}{t_{m-1}} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & \binom{t_1}{t_m} & \binom{t_1}{t_{m-1}} & \dots & \binom{t_1}{t_3} & \binom{t_1}{t_2} & 1 \\ 1 & \binom{n}{t_m} & \binom{n}{t_{m-1}} & \dots & \binom{n}{t_3} & \binom{n}{t_2} & \binom{n}{t_1} \end{vmatrix}. \tag{21}$$

Corollary 6. *If k does not depend on n, then $\left\{ n \atop k \right\}$ is a polynomial in n of degree $t_1 = \lfloor \log_2(2k) \rfloor$.*

In the last determinant let us replace the orders of elements in rows and columns by opposite ones. Then we obtain (21) in the form

$$\left\{ n \atop k \right\} = \begin{vmatrix} \binom{n}{t_1} & \binom{n}{t_2} & \binom{n}{t_3} & \dots & \binom{n}{t_{m-1}} & \binom{n}{t_m} & 1 \\ 1 & \binom{t_1}{t_2} & \binom{t_1}{t_3} & \dots & \binom{t_1}{t_{m-1}} & \binom{t_1}{t_m} & 1 \\ 0 & 1 & \binom{t_2}{t_3} & \dots & \binom{t_2}{t_{m-1}} & \binom{t_2}{t_m} & 1 \\ 0 & 0 & 1 & \dots & \binom{t_3}{t_{m-1}} & \binom{t_3}{t_m} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & 1 & \binom{t_{m-1}}{t_m} & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{vmatrix}. \tag{22}$$

Corollary 7. *We have*

$$\left\{ 0 \atop k \right\} = (-1)^m = \tau_k, \tag{23}$$

where $\{\tau_j\}_{j\geq 0} = \{1, -1, -1, 1, -1, 1, 1, -1, \dots\}$ *is the Thue-Morse sequence [13, 2].*

From (22) the following arithmetic property of numbers $\left\{ n \atop k \right\}$ follows.

Theorem 8. *If* n *is prime, then*

$$\left\{ n \atop k \right\} \equiv \tau_k \pmod{n}. \tag{24}$$

Moreover, if all divisors of n*, other than 1, are larger than* $1 + \lfloor \log_2 k \rfloor$*, then* (24) *holds as well.*

Corollary 9. *If* n *is prime, then*

$$A(n,t) \equiv 1 \pmod{n}. \tag{25}$$

Proof. There are $\binom{n-1}{t}$ indices containing exactly t 1's. Thus, by (10) and (24), we have

$$A(n,t) \equiv (-1)^t \binom{n-1}{t} \pmod{n}.$$

The latter is an integer of the form $(-1)^t(\frac{n_1 n}{t!} + (-1)^t) \equiv 1 \pmod{n}$. □

Remark 10. It is evident that the validity of (24) does not depend on whether k is a constant or a function of n.

Remark 11. Theorem 3, for the first time, was proved by the author in [18] in a similar way.

Remark 12. Recently, in the case of prime n, a weak form of Theorem 8 was given in [5] by replacing τ_k by ± 1.

In view of a rather simple structure of determinant (22), we are able to calculate it using permanents.

Theorem 13. *For k (as defined in (20)) we have*

$$\left\{ {n \atop k} \right\} = \tau_k \left(1 + \sum_{p=1}^{m} (-1)^p \sum_{1 \le i_1 < i_2 < \ldots < i_p \le m} \binom{n}{t_{i_1}} \prod_{r=2}^{p} \binom{t_{i_{r-1}}}{t_{i_r}} \right). \tag{26}$$

Proof. The number of diagonals of matrix (22) having no 0's is equal to the permanent of the following $(m+1) \times (m+1)$ matrix:

$$C_{m+1} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ 1 & 1 & 1 & \ldots & 1 & 1 & 1 \\ 0 & 1 & 1 & \ldots & 1 & 1 & 1 \\ 0 & 0 & 1 & \ldots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 \end{pmatrix}.$$

Decomposing $perC_{m+1}$ by the last row we find

$$perC_{m+1} = 2perC_m = 2^2 perC_{m-1} = \ldots = 2^m.$$

Denote by A the $(m+1) \times (m+1)$ matrix (22) and consider the $(m \times m)$ upper-triangle submatrix T with the main diagonal composed of 1's. Let us choose p 1's of the main diagonal of T in its rows $(1 \le)\ i_1 < i_2 < \ldots < i_p \le m$. To this choice corresponds a diagonal of A composed of the other $(m-p)$ 1's of the main diagonal of T and the unit in the last column of A which is the continuation of the i_p-th row of T and elements $\binom{n}{t_{i_1}}, \binom{t_{i_1}}{t_{i_2}}, \binom{t_{i_2}}{t_{i_3}}, \ldots, \binom{t_{i_{p-1}}}{t_{i_p}}$, such that finally we have $m-p+1+p = m+1$ elements of A which are in different rows and columns. As a result, we obtain $\sum_{p=0}^{m} \binom{m}{p} = 2^m$, i.e., *all* diagonals of A having no 0's (note that, $p=0$ corresponds to the choice of the empty subset of 1's of the main diagonal of T, i.e., in this case, all these 1's and the unit in the first row of A form the only diagonal of 1's). Therefore, we have

$$perA = 1 + \sum_{p=1}^{m} \sum_{1 \le i_1 < i_2 < \ldots < i_p \le m} \binom{n}{t_{i_1}} \prod_{r=2}^{p} \binom{t_{i_{t-1}}}{t_{i_r}}.$$

It is left to find the number of cycles of the permutation corresponding to diagonal of the elements

$$\binom{n}{t_{i_1}}, \binom{t_{i_1}}{t_{i_2}}, \binom{t_{i_2}}{t_{i_3}}, \ldots, \binom{t_{i_{p-1}}}{t_{i_p}}$$

and 1 in position $(i_p+1, \ m+1)$. Put, for symmetry, $n = t_0$. Then the matrix (22) becomes "quasi-Toeplitz" in the sense that, for a fixed $j \geq 0$, the diagonal $(a_{i, \ i+j})_{i\geq 1}$ contains only non-zero elements of the form:

$$\binom{t_0}{t_{i_{j+1}}}, \ \binom{t_1}{t_{i_{j+2}}}, \ldots$$

with the equal differences $(j+1)-0 = (j+2)-1 = \cdots = j+1$. Note that every non-zero element of the form $(a_{i, \ i+j})$ belongs to a cycle of the form $[(a_{i, \ i+j}), 1, \ldots, 1]$ with j 1's in the positions $(2,1), \ (3,2), \ldots, (j+1, j)$, i.e., it belongs to a cycle of length $j+1$. Thus the considered elements are in cycles of lengths $i_1, \ i_2-i_1, \ldots, i_p-i_{p-1}, \ m+1-i_p$ (the last length corresponds to 1 in the position $(i_p+1, \ m+1)$) so we have $p+1$ distinct cycles. Indeed, two cycles either coincide or are disjoint. The total length of all cycles is $i_1+(i_2-i_1)+\cdots+(i_p-i_{p-1})+m+1-i_p = m+1$. If some two cycles coincide, then these two cycles are considered as one and, in this case, the total sum will be less than $m+1$, which is impossible for a diagonal. Thus we have exactly $p+1$ cycles of the considered diagonal. The latter means that the parity of the corresponding permutation is $(-1)^{(m+1)-(p+1)} = (-1)^{m-p} = \tau_k(-1)^p$. This completes the proof of formula (26). □

Corollary 14. *We have the following representation of up-down coefficient as* $\tau_k = (-1)^m$ *plus a linear combination of binomial coefficients:*

$$\left\{ {n \atop k} \right\} = (-1)^m + \sum_{p=1}^{m} c_p \binom{n}{t_p}, \tag{27}$$

where

$$c_p = (-1)^m \left(-1 + \sum_{j=p+1}^{m} \binom{t_p}{t_j} - \sum_{p+1\leq j<l\leq m} \binom{t_p}{t_j}\binom{t_j}{t_l} + \cdots + (-1)^{m-p-1} \prod_{j=p+1}^{m} \binom{t_{j-1}}{t_j} \right). \tag{28}$$

In particular,

$$\begin{aligned} c_m &= (-1)^{m+1} \\ c_{m-1} &= (-1)^m \left(-1 + \binom{t_{m-1}}{t_m} \right) \\ c_{m-2} &= (-1)^m \left(-1 + \binom{t_{m-2}}{t_{m-1}} + \binom{t_{m-2}}{t_m} - \binom{t_{m-2}}{t_{m-1}}\binom{t_{m-1}}{t_m} \right) \end{aligned} \tag{29}$$

etc. These formulas play an important role in constructions of the next section.

Example 15. Let $k = 2^m - 1$. Then, according to (20) − (21), we have $t_m = 1, \quad t_{m-1} = 2, \quad \ldots, \quad t_1 = m$ and

$$c_m = (-1)^{m+1}$$

$$c_{m-1} = (-1)^m(-1+2) = (-1)^m$$

$$c_{m-2} = (-1)^m(-1+3+3-3\cdot 2) = (-1)^{m+1}$$

and, by induction, $c_m = -c_{m-1} = c_{m-2} = \ldots = (-1)^{m-1}c_1 = (-1)^{m-1}$, i.e., $c_p = (-1)^{p-1}$.

Thus, we have

$$\left\{ n \atop 2^m - 1 \right\} = (-1)^m + \sum_{p=1}^{m} (-1)^{p-1} \binom{n}{m-p+1} = \sum_{j=0}^{m} (-1)^{m-j} \binom{n}{j} = \binom{n-1}{m}. \tag{30}$$

(the latter identity is proved easily by induction over m).

In particular, putting $n = 2m$ and $n = 2m+1$, we have

$$\left\{ 2m \atop 2^m - 1 \right\} = \binom{2m-1}{m}, \quad \left\{ 2m+1 \atop 2^m - 1 \right\} = \binom{2m}{m}. \tag{31}$$

This proves the analogy for the central up-down coefficients and the central binomial coefficients.

Remark 16. On the other hand, applying Theorem 2 directly to the index with binary expansion (15) (without transformation (16)), we find that

$$\left\{ n \atop k \right\} = \begin{vmatrix} 1 & 1 & 0 & \ldots & 0 \\ 1 & \binom{n-t+s_2-1}{s_2-1} & 1 & \ldots & 0 \\ 1 & \binom{n-t+s_3-1}{s_3-1} & \binom{n-t+s_3-1}{s_3-s_2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \binom{n-t+s_m-1}{s_m-1} & \binom{n-t+s_m-1}{s_m-s_2} & \ldots & 1 \\ 1 & \binom{n}{t} & \binom{n}{t+1-s_2} & \ldots & \binom{n}{t+1-s_m} \end{vmatrix} \tag{32}$$

and, in the same way, we get

$$\left\{ n \atop k \right\} = (-1)^m + \sum_{p=1}^{m} (-1)^{m-p} \sum_{1 \le i_1 < i_2 < \ldots < i_p \le m} \binom{n}{t+1-s_{i_p}} \prod_{r=2}^{p} \binom{n-t+s_{i_r}-1}{s_{i_r}-s_{i_{r-1}}}. \tag{33}$$

Comparing (20) and (15) with $t = t_1$, we have

$$s_m = t_1 - t_m + 1, s_{m-1} = t_1 - t_{m-1} + 1, \ldots, s_2 = t_1 - t_2 + 1, s_1 = t_1 - t_1 + 1 = 1. \quad (34)$$

It is easy to check directly the following identity

$$\binom{n}{t_{i_p}} \prod_{r=2}^{p} \binom{n - t_{i_r}}{t_{i_{r-1}} - t_{i_r}} = \binom{n}{t_{i_1}} \binom{t_{i_1}}{t_{i_2}} \binom{t_{i_2}}{t_{i_3}} \cdots \binom{t_{i_{p-1}}}{t_{i_p}}. \quad (35)$$

Now from (33) − (35), (26) follows. Note that (33) was essentially discovered first in our paper [18].

Remark 17. Note that, the comparison of (21) and (32) with (34) gives an interesting identity for determinants:

$$\begin{vmatrix} 1 & 1 & 0 & \ldots & 0 \\ 1 & \binom{n-t_2}{n-t_1} & 1 & \ldots & 0 \\ 1 & \binom{n-t_3}{n-t_1} & \binom{n-t_3}{n-t_2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \binom{n-t_m}{n-t_1} & \binom{n-t_m}{n-t_2} & \ldots & 1 \\ 1 & \binom{n}{n-t_1} & \binom{n}{n-t_2} & \ldots & \binom{n}{n-t_m} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 1 & \binom{t_{m-1}}{t_m} & 1 & \ldots & 0 & 0 & 0 \\ 1 & \binom{t_{m-2}}{t_m} & \binom{t_{m-2}}{t_{m-1}} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & \binom{t_1}{t_m} & \binom{t_1}{t_{m-1}} & \ldots & \binom{t_1}{t_3} & \binom{t_1}{t_2} & 1 \\ 1 & \binom{n}{t_m} & \binom{n}{t_{m-1}} & \ldots & \binom{n}{t_3} & \binom{n}{t_2} & \binom{n}{t_1} \end{vmatrix}. \quad (36)$$

when the transformation of all elements different from $0, 1$ by the rule $n - t_i \mapsto t_{m-i+1}$, $i = 1, \ldots, m, m+1$, with the conventions $t_{m+1} = 0$, $t_0 = n$, leads to "sweeping" of the dependance on n in the bordered triangular submatrix.

E.g., for $m = 2$, (36) reduces to an easily verifiable identity

$$\binom{n - t_2}{t_1 - t_2} \binom{n}{t_2} = \binom{t_1}{t_2} \binom{n}{t_1}.$$

Consider the representation of $\left\{ n \atop k \right\}$ in form (22) in case of the alternating permutations. In this case $k = \kappa_{n-1}$ (11). Using (14) for $n = 2m$, we get an explicit formula for the Euler numbers $E_{2m},\ m \geq 1$ (cf.[1], Table 23.2, [20], sequence A000364). Taking into account the sign $(-1)^m$ of E_{2m}, we transpose the column of 1's in (22) to the first column.

Corollary 18. *We have*

$$E_{2m} = \begin{vmatrix} 1 & \binom{2m}{2m-1} & \binom{2m}{2m-3} & \binom{2m}{2m-5} & \cdots & \binom{2m}{1} \\ 1 & 1 & \binom{2m-1}{2m-3} & \binom{2m-1}{2m-5} & \cdots & \binom{2m-1}{1} \\ 1 & 0 & 1 & \binom{2m-3}{2m-5} & \cdots & \binom{2m-3}{1} \\ 1 & 0 & 0 & 1 & \cdots & \binom{2m-5}{1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}.$$

Analogously, using (14) for $n = 2m-1$, we have for the numbers

$$b_{2m} = \frac{B_{2m}}{2m}(2^{2m}-1)2^{2m}, \quad m \geq 2, \tag{37}$$

(cf.[1], Table 23.2) the following representation.

Corollary 19. *We have*

$$b_{2m} = \begin{vmatrix} 1 & \binom{2m-1}{2m-2} & \binom{2m-1}{2m-4} & \binom{2m-1}{2m-6} & \cdots & \binom{2m-1}{2} \\ 1 & 1 & \binom{2m-2}{2m-4} & \binom{2m-2}{2m-6} & \cdots & \binom{2m-2}{2} \\ 1 & 0 & 1 & \binom{2m-4}{2m-6} & \cdots & \binom{2m-4}{2} \\ 1 & 0 & 0 & 1 & \cdots & \binom{2m-6}{2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}.$$

The latter numbers are called the "tangent numbers" ([20], sequence A000182).

Example 20. For $m = 1$, we have

$$E_2 = \begin{vmatrix} 1 & \binom{2}{1} \\ 1 & 1 \end{vmatrix} = -1,$$

for $m = 2$, we have

$$\frac{B_4}{4} \cdot 16 \cdot 15 = \begin{vmatrix} 1 & \binom{3}{2} \\ 1 & 1 \end{vmatrix} = -2,$$

which corresponds to $B_4 = -\frac{1}{30}$.

For $m = 2$, we also have

$$E_4 = \begin{vmatrix} 1 & \binom{4}{3} & \binom{4}{1} \\ 1 & 1 & \binom{3}{1} \\ 1 & 0 & 1 \end{vmatrix} = 5,$$

for $m = 3$, we find

$$\frac{B_6}{6} \cdot 64 \cdot 63 = \begin{vmatrix} 1 & \binom{5}{4} & \binom{5}{2} \\ 1 & 1 & \binom{4}{2} \\ 1 & 0 & 1 \end{vmatrix} = 16,$$

which corresponds to $B_6 = \frac{1}{42}$.

3. Positive Integer Zeros of the Up-Down Polynomials: Characteristic Conditions for an Up-Down Polynomial

We consider $\left\{ n \atop k \right\}$ from a more formal point of view as a polynomial in n the values of which do not always have a combinatorial sense. E.g., this occurs when $n-1$ has smaller digits than k. In this case $\left\{ n \atop k \right\}$ could take zero and negative values.

We start with the following simple result.

Theorem 21. *If $k = 2^{t_1-1} + 2^{t_2-1} + \ldots + 2^{t_m-1},\ t_1 > t_2 > \ldots > t_m \geq 1$ then the integers $t_1, t_2, \ldots, t_m$ are roots of the up-down polynomial $\left\{ n \atop k \right\}$.*

Proof. Substituting in (22) $n = t_j$, we see that the first row coincides with the $(j+1)th$ row, $j = 1, 2, \ldots, m$. □

So, the first polynomials of the form

$$\left\{ n \atop k \right\}^* = \frac{\left\{ n \atop k \right\}}{(n-t_1)(n-t_2)\cdots(n-t_m)}$$

are (cf. Appendix):

$$1,\ 1,\ \frac{n+1}{2},\ \frac{1}{2},\ n^2+2,\ \frac{n+1}{3},\ \frac{2n+1}{6},\ \frac{1}{6},$$
$$\frac{n^3-2n^2+3n+6}{24},\ \frac{n^2-n+2}{8},\ \frac{5n^2+3}{24},\ etc.$$

It is easy to see that the sequence of degrees of these polynomials, for $k \geq 1$, is sequence of numbers of 0's in binary expansion of k (cf. sequences A080791, A023416 in [20]).

Later we prove that $n = t_j,\ \ j = 1, \ldots, m$, are the *only* positive integer zeros of $\left\{ n \atop k \right\}$. Now we obtain an additional result.

Theorem 22. *Let $l \in \mathbb{N}$. If*

$$k \equiv 2^{l-1} + j \mod 2^l, \tag{38}$$

where $j \in [0, 2^{l-1})$, then $\left\{ l \atop k \right\} = 0$.

Proof. We have $k = v2^l + 2^{l-1} + j,\quad v \geq 0,\quad 0 \leq j \leq 2^{l-1} - 1$. Therefore, if $k = 2^{t_1-1} + 2^{t_2-1} + \ldots + 2^{t_m-1}$, then there exists $h \in [1, m]$ such that $l = t_h$. Thus, by Theorem 5, we have $\left\{ l \atop k \right\} = \left\{ t_h \atop k \right\} = 0$. □

Remark 23. Note that the converse of Theorem 22 implies the converse of Theorem 21.

As a corollary from Theorem 6, we get also the following statement.

Theorem 24. *Let $k = 2^{t_1-1} + 2^{t_2-1} + \ldots + 2^{t_m-1}$ and $1 \leq i \leq \log_2(2k)$. If $i \neq t_p,\ \ p = 1, 2, \ldots, m$, then $\left\{ i \atop k - 2^{i-1} \right\} = 0$.*

Proof. Let $t_l < i < t_{l-1}$. Then

$$k - 2^{i-1} = 2^{t_1-1} + 2^{t_2-1} + \cdots + \left(2^{t_{l-1}-1} - 2^{i-1}\right) + 2^{t_l-1} + \ldots + 2^{t_m-1}$$

$$= 2^{t_1-1} + 2^{t_2-1} + \ldots + \left(2^{t_{l-1}-2} + 2^{t_{l-1}-3} + \ldots + 2^i + 2^{i-1}\right) + 2^{t_l-1} + \ldots + 2^{t_m-1}$$

$$\equiv 2^{i-1} + 2^{t_l-1} + \ldots + 2^{t_m-1} \pmod{2^i}$$

and the theorem directly follows from Theorem 6. □

The following theorem gives another algorithm for the evaluation of up-down polynomials.

Theorem 25. *Representation* (27)

$$\left\{ n \atop k \right\} = c_1 \binom{n}{t_1} + c_2 \binom{n}{t_2} + \ldots + c_m \binom{n}{t_m} + (-1)^m, \tag{39}$$

is unique with coefficients defined by the following system of the linear equations:

$$\begin{cases} c_m + (-1)^m = 0 \\ c_{m-1} + \binom{t_{m-1}}{t_m} c_m + (-1)^m = 0 \\ c_{m-2} + \binom{t_{m-2}}{t_{m-1}} c_{m-1} + \binom{t_{m-2}}{t_m} c_m + (-1)^m = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ c_1 + \binom{t_1}{t_2} c_2 + \binom{t_1}{t_3} c_3 + \ldots + \binom{t_1}{t_m} c_m + (-1)^m = 0. \end{cases} \tag{40}$$

Proof. Substituting $n = t_m, t_{m-1}, \ldots, t_1$ in (39) and using Theorem 5, we obtain the system (40). □

Example 26. Let us find $\left\{ n \atop 26 \right\}$. We have $26 = 2^{5-1} + 2^{4-1} + 2^{2-1}$.

Thus, $t_1 = 5,\ \ t_2 = 4,\ \ t_3 = 2,\ \ m = 3$. By (40),

$$\begin{cases} c_3 - 1 = 0 \\ c_2 + 6c_3 - 1 = 0 \\ c_1 + 5c_2 + 10c_3 - 1 = 0 \end{cases},$$

whence $c_1 = 16,\ \ c_2 = -5,\ \ c_3 = 1$. Consequently, by (39)

$$\left\{ n \atop 26 \right\} = 16 \binom{n}{5} - 5 \binom{n}{4} + \binom{n}{2} - 1.$$

Corollary 27. *Conversely, if we have a polynomial*

$$P(n) = a_1 \binom{n}{t_1} + a_2 \binom{n}{t_2} + \ldots + a_m \binom{n}{t_m} + (-1)^m,$$

such that $P(t_i) = 0,\ \ i = 1, \ldots, m$, *then* $P(n) = \left\{ n \atop k \right\}$, *where* k *is* $2^{t_1 - 1} + \ldots + 2^{t_m - 1}$.

Proof. The system $P(t_i) = 0$ with respect to coefficients $a_i,\ \ i = 1, \ldots, m$, coincides with (40), evidently having a unique solution. Thus $a_i = c_i,\ \ i = 1, \ldots, m$. □

Let $P(n)$ be a polynomial. It is evident that the condition

$$P(n) = C \binom{n}{k}$$

with a constant C is satisfied if and only if $P(r) = 0, \quad r = 0, 1, \ldots, k-1$ where $k = degP(n)$. Concerning $\left\{ n \atop k \right\}$ we have the following result. Put $\Delta P(n) = P(n) - P(n-1)$ and let $\Delta^r P(n)$ be the $r-th$ difference of $P(n)$.

Theorem 28. *For a polynomial $P(n)$ there exists a nonnegative integer k and a constant $C \neq 0$ such that*

$$P(n) = C \left\{ n \atop k \right\} \tag{41}$$

if and only if the following conditions satisfied:

$$P(0) \neq 0, \quad (\Delta^r P(r))P(r) = 0, \quad r = 1, 2, \ldots, l, \tag{42}$$

where $l = degP(n)$.

Proof. Let (42) hold. Note that we have

$$P(n) = P(0) + \sum_{r=1}^{l} \Delta^r P(r) \binom{n}{r}. \tag{43}$$

Indeed, put $P(n) = a_0 + \sum_{r=1}^{l} a_r \binom{n}{r}$. Then we consecutively find $a_0 = P(0)$,

$$\Delta P(n) = \sum_{r=1}^{l} a_r \binom{n-1}{r-1}, \quad a_1 = \Delta P(1), \ldots,$$

i.e.,

$$\Delta^t P(n) = \sum_{r=1}^{l} a_r \binom{n-t}{r-t}, \quad a_t = \Delta^t P(t), \quad t = 0, \ldots, l.$$

If all $\Delta^r P(r) = 0, \quad r = 1, \ldots, l$, then we put $k = 0, \quad C = P(0)$. If $\Delta^r P(r) \neq 0$, for $r = t_1 > t_2 > \ldots > t_m \geq 1$, then by (43)

$$P(n) = P(0) + \sum_{i=1}^{m} b_{m+1-i} \binom{n}{t_i},$$

where $b_i = \Delta^{t_i} P(t_i), \quad i = 1, 2, \ldots, m$. Putting $a_i = (-1)^m \frac{b_i}{P(0)}$ for $i = 1, 2, \ldots, m$, we have

$$\frac{(-1)^m}{P(0)} P(n) = (-1)^m + \sum_{i=1}^{m} a_{m+1-i} \binom{n}{t_i} \tag{44}$$

and, according to (42), $P(t_i) = 0, \quad i = 1, 2, \ldots, m$. Thus, by Corollary 27 for $a_i := a_{m+1-i}, \quad i = 1, \ldots, m$, the polynomial (44) is $\left\{ n \atop k \right\}$ with

$$k = 2^{t_1 - 1} + 2^{t_2 - 1} + \ldots + 2^{t_m - 1}.$$

The converse statement, evidently, follows from Theorem 25. □

Example 29. Consider $P(n) = n^3 - 3n^2 + 2n - 6$. We have

$$\Delta P(n) = 3n^2 - 9n + 6, \ \ \Delta^2 P(n) = 6n - 12, \ \ \Delta^3 P(n) = 6,$$

such that

$$(\Delta P(1))P(1) = (\Delta^2 P(2))P(2) = (\Delta^3 P(3))P(3) = 0.$$

Therefore, we conclude that $P(n) = C\begin{Bmatrix} n \\ k \end{Bmatrix}$. Now we easily find C. Since $r = 3$ is the smallest value of r for which $\Delta^r P(r) \neq 0$, then $t_m = 3$ with $m = 1$. Therefore, $k = 4$ and $C = -P(0) = 6$. Thus $P(n) = 6\begin{Bmatrix} n \\ 4 \end{Bmatrix}$.

4. Asymptotic Formula for Up-Down Polynomials

An asymptotic formula for up-down polynomials is based on the following lemma which gives a representation of coefficients c_p in (39) by other up-down polynomials.

Lemma 30. *If $k = 2^{t_1-1} + 2^{t_2-1} + \cdots + 2^{t_m-1}$, $t_1 > t_2 > \cdots > t_m$, then for coefficients c_p in* (39) *we have*

$$c_p = \begin{Bmatrix} t_p \\ k - 2^{t_p-1} \end{Bmatrix}.$$

Proof. Putting $k_1 = k - 2^{t_p-1}$, we have

$$k_1 = 2^{t_1-1} + 2^{t_2-1} + \cdots + 2^{t_{p-1}-1} + 2^{t_{p+1}-1} + \cdots + 2^{t_m-1}.$$

Using, for k_1 (26), and substituting $n = t_p$, we obtain

$$\begin{Bmatrix} t_p \\ k - 2^{t_p-1} \end{Bmatrix} = (-1)^{m-1}\left(1 - \sum_{i=p+1}^{m}\binom{t_p}{t_i} + \sum_{p+1 \le i < j \le m}\binom{t_p}{t_i}\binom{t_i}{t_j} - \ldots\right).$$

Comparison of this with (28) gives the lemma. □

Now we are able to give an asymptotic formula for up-down polynomials.

Theorem 31. *For $n \to \infty$, we have:*

1) *if $t_2 < t_1 - 1$, then*

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \frac{n^{t_1}}{t_1!}\begin{Bmatrix} t_1 \\ k - 2^{t_1-1} \end{Bmatrix}(1 - \binom{t_1}{2}\frac{1}{n} + O(\frac{1}{n^2})); \tag{45}$$

2) *if $t_2 = t_1 - 1$, then*

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \frac{n^{t_1}}{t_1!}(\begin{Bmatrix} t_1 \\ k - 2^{t_1-1} \end{Bmatrix} + \frac{1}{n}(t_1\begin{Bmatrix} t_1 - 1 \\ k - 2^{t_1-2} \end{Bmatrix} - \binom{t_1}{2}\begin{Bmatrix} t_1 \\ k - 2^{t_1-1} \end{Bmatrix}) + O(\frac{1}{n^2})). \tag{46}$$

Proof. By Lemma 30, we have

$$\left\{ {n \atop k} \right\} = c_1 \binom{n}{t_1} + c_2 \binom{n}{t_2} + o(n^{t_2}) = \left\{ {t_1 \atop k - 2^{t_1 - 1}} \right\} \binom{n}{t_1} + \left\{ {t_2 \atop k - 2^{t_2 - 1}} \right\} \binom{n}{t_2} + o(n^{t_2}).$$

Now the theorem easily follows from the asymptotic formula for the binomial coefficients $\binom{n}{t}$, $t \geq 2$:

$$\binom{n}{t} = \frac{n^t}{t!} - \frac{n^{t-1}}{2(t-2)!} + O(n^{t-2}).$$

□

5. Two Recursions for Up-Down Coefficients

The first recursion for up-down coefficients we find directly from (39) and Lemma 30.

Theorem 32. *(The first recursion) If* $k = 2^{t_1 - 1} + 2^{t_2 - 1} + \ldots + 2^{t_m - 1}$, *then*

$$\left\{ {n \atop k} \right\} = (-1)^m + \sum_{p=1}^{m} \left\{ {t_p \atop k - 2^{t_p - 1}} \right\} \binom{n}{t_p}. \tag{47}$$

Example 33. Knowing $\left\{ {n \atop j} \right\}$, $j \leq 20$, to find $\left\{ {n \atop 21} \right\}$.

We have

$$21 = 2^{5-1} + 2^{3-1} + 2^{1-1}, \quad t_1 = 5, \quad t_2 = 3, \quad t_3 = 1.$$

By (47), we obtain

$$\left\{ {n \atop 21} \right\} = -1 + \left\{ {5 \atop 5} \right\} \binom{n}{5} + \left\{ {3 \atop 17} \right\} \binom{n}{3} + \left\{ {1 \atop 20} \right\} \binom{n}{1}. \tag{48}$$

Knowing the formulas (cf., Appendix)

$$\left\{ {n \atop 5} \right\} = 2\binom{n}{3} - \binom{n}{1} + 1,$$

$$\left\{ {n \atop 17} \right\} = 4\binom{n}{5} - \binom{n}{1} + 1,$$

$$\left\{ {n \atop 20} \right\} = 9\binom{n}{5} - \binom{n}{3} + 1,$$

we get

$$\left\{ {5 \atop 5} \right\} = 2 \cdot 10 - 5 + 1 = 16,$$

$$\left\{ {3 \atop 17} \right\} = -2,$$

$$\left\{ {1 \atop 20} \right\} = 1$$

and, by (48), we find

$$\left\{ {n \atop 21} \right\} = 16 \binom{n}{5} - 2\binom{n}{3} + \binom{n}{1} - 1.$$

The second, even more simple, recursion is based on Theorems 24 and 25.

Theorem 34. *(The second recursion) Let* $k = 2^{t_1 - 1} + 2^{t_2 - 1} + \ldots + 2^{t_m - 1},\ t_1 > t_2 > \ldots > t_m \geq 1$. *Then, for* $l > t_1$, *we have*

$$\left\{ {n \atop k + 2^{l-1}} \right\} = \left\{ {l \atop k} \right\} \binom{n}{l} - \left\{ {n \atop k} \right\}. \tag{49}$$

Proof. By (47), we have

$$\left\{ {n \atop k + 2^{l-1}} \right\} = (-1)^{m+1} + b_1 \binom{n}{t_m} + b_2 \binom{n}{t_{m-1}} + \ldots + b_m \binom{n}{t_1} + b_{m+1} \binom{n}{l}, \tag{50}$$

where $b_i,\ \ i = 1, 2, \ldots, m+1$, are defined by the following system:

$$\begin{cases} b_1 + (-1)^{m+1} = 0 \\ \binom{t_{m-1}}{t_m} b_1 + b_2 + (-1)^{m+1} = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \binom{t_1}{t_m} b_1 + \binom{t_1}{t_{m-1}} b_2 + \ldots + \binom{t_1}{t_2} b_{m-1} + b_m + (-1)^{m+1} = 0 \\ \binom{l}{t_m} b_1 + \binom{l}{t_{m-1}} b_2 + \ldots + \binom{l}{t_1} b_m + b_{m+1} + (-1)^{m+1} = 0 \end{cases} . \tag{51}$$

By comparing the first m equations of (51) with (48), we conclude that

$$b_i = -c_i,\ \ i = 1, 2, \ldots, m \tag{52}$$

and, by the $(m+1)-$th equation of (51), taking into account (47), we find that

$$b_{m+1} = -\left\{ {l \atop k} \right\}. \tag{53}$$

Now from (47) , (50), (52) and (53) we obtain

$$\left\{ {n \atop k} \right\} + \left\{ {n \atop k + 2^{l-1}} \right\} = \left\{ {l \atop k} \right\} \binom{n}{l}$$

and (49) follows. □

Example 35. Starting with $\left\{ n \atop 0 \right\} = 1$ and putting $k = 0,\ \ l = 1$, we obtain

$$\left\{ n \atop 1 \right\} = \left\{ 1 \atop 0 \right\} \binom{n}{1} - \left\{ n \atop 0 \right\} = \binom{n}{1} - 1.$$

Furthermore, we consecutively find

$$\textit{putting}\ \ k = 0,\ \ l = 2,\ \left\{ n \atop 2 \right\} = \left\{ 2 \atop 0 \right\} \binom{n}{2} - \left\{ n \atop 0 \right\} = \binom{n}{2} - 1,$$

$$\textit{putting}\ \ k = 1,\ \ l = 2,\ \left\{ n \atop 3 \right\} = \left\{ 2 \atop 1 \right\} \binom{n}{2} - \left\{ n \atop 1 \right\} = \binom{n}{2} - \binom{n}{1} + 1,$$

$$\textit{putting}\ \ k = 0,\ \ l = 3,\ \left\{ n \atop 4 \right\} = \left\{ 3 \atop 0 \right\} \binom{n}{3} - \left\{ n \atop 0 \right\} = \binom{n}{3} - 1,$$

$$\textit{putting}\ \ k = 1,\ \ l = 3,\ \left\{ n \atop 5 \right\} = \left\{ 3 \atop 1 \right\} \binom{n}{3} - \left\{ n \atop 1 \right\} = 2\binom{n}{3} - \binom{n}{1} + 1,\ \ \textit{etc.}$$

6. Arithmetical Properties of the Numbers of Alternating Permutations

Denote by $a(n)$ the number of alternating (Andre's) permutations of numbers $1, \ldots, n$. We saw (Example 4) that

$$a(n) = \left\{ n \atop \frac{2^{n+1} - 3 + (-1)^n}{6} \right\}. \tag{54}$$

Now we prove the following results.

Theorem 36. 1) *If p is odd prime, then* $a(p) \equiv (-1)^{(p-1)/2} \pmod{p}$;

2) *For every nonnegative n,* $a(2^n) \equiv 1 \pmod{2^n}$;

3) *If p is odd prime, then* $a(2p) \equiv 1 \pmod{2p}$.

Proof of 1). In view of (54), we should prove that

$$\left\{ p \atop \frac{2^p - 2}{3} \right\} \equiv (-1)^{(p-1)/2} \pmod{p}. \tag{55}$$

Since $k = \kappa_{p-1} = \frac{2^p-2}{3}$ is the index of alternative permutations of p elements, then its binary expansion is $\kappa_{p-1} = 1010\ldots10$ with $\frac{(p-1)}{2}$ 1's and $\frac{(p-1)}{2}$ 0's. Therefore,

$$\tau_{\kappa_{p-1}} = (-1)^{(p-1)/2}. \tag{56}$$

Now 1) follows from Theorem 8. □

Corollary 37. *Let p be an odd prime and let* $|B_{p+1}| = \alpha_{p+1}/\beta_{p+1}$ (with $(\alpha_{p+1}, \beta_{p+1}) = 1$). *Then*

$$12\alpha_{p+1} + (-1)^{(p+1)/2}\beta_{p+1} \equiv 0 \pmod{p}. \tag{57}$$

Proof. Denote $\frac{B_{2n}}{2n}(2^{2n}-1)2^{2n}$ by b_{2n}. Using (14), for $m = \frac{p+1}{2}$, and (55), we have

$$|b_{p+1}| = \frac{|B_{p+1}|}{p+1}(2^{p+1}-1)2^{p+1} \equiv (-1)^{(p-1)/2} \pmod{p}.$$

Furthermore, since

$$(p+1)\beta_{p+1}|b_{p+1}| = \alpha_{p+1}(4\cdot 2^{p-1}-1)4\cdot 2^{p-1} \equiv 12\alpha_{p+1} \pmod{p},$$

then $(-1)^{(p-1)/2}\beta_{p+1} \equiv 12\alpha_{p+1} \pmod{p}$, and the corollary follows. □

We will need several lemmas.

Lemma 38. *If* $1 \le 2i-1 \le 2^m$, *then*

$$\binom{2^m}{2i-1} \equiv 0 \pmod{2^m}.$$

Proof. Let $\alpha(m)$ be exponent such that

$$2^{\alpha(m)} || \ m. \tag{58}$$

Denote by $s(m)$ the number of 1's in the binary expansion of m. It is known (cf. [15]) that,

$$\alpha(m!) = (m - s(m)). \tag{59}$$

From (59) we obtain:

$$\alpha(\binom{m}{x}) = \alpha(m!) - \alpha(x!) - \alpha((m-x)!) =$$

$$((m - s(m)) - (x - s(x)) - (m - x - s(m-x)) = (s(x) + s(m-x) - s(m)). \tag{60}$$

Using (60), we find

$$\alpha(\binom{2^n}{2i-1}) = s(2i-1) + s(2^n - (2i-1)) - s(2^n).$$

Note that, evidently we have $s(2^n - (2i-1)) = n + 1 - s(2i-1)$. Therefore,

$$\alpha(\binom{2^n}{2i-1}) = n + 1 - 1 = n$$

and, according to (58), the lemma follows. □

Lemma 39. *For $n \geq 1$ and $0 \leq j \leq 2^{n-1} - 2$,*

$$\left\{ \begin{matrix} 2^n \\ \frac{2^{2^n-2j}-1}{3} \end{matrix} \right\} \equiv - \left\{ \begin{matrix} 2^n \\ \frac{2^{2^n-2j-2}-1}{3} \end{matrix} \right\} \pmod{2^n}.$$

Proof. Note that,

$$\frac{2^{2^n-2j}-1}{3} = \frac{2^{2^n-2j-2}-1}{3} + 2^{2^n-2j-2}$$

and

$$2^n - 2j - 1 > 2^n - 2j - \log_2 3 > \log_2 (2\frac{2^{2^n-2j-2}-1}{3}).$$

Therefore, we can apply Theorem 34 with $l = 2^n - 2i - 1$. Thus

$$\left\{ \begin{matrix} 2^n \\ \frac{2^{2^n-2j}-1}{3} \end{matrix} \right\} = \left\{ \begin{matrix} 2^n \\ \frac{2^{2^n-2j-2}-1}{3} + 2^{2^n-2j-2} \end{matrix} \right\} =$$

$$\left\{ \begin{matrix} 2^n - 2j - 1 \\ \frac{2^{2^n-2j-2}-1}{3} \end{matrix} \right\} \binom{2^n}{2^n - 2j - 1} - \left\{ \begin{matrix} 2^n \\ \frac{2^{2^n-2j-2}-1}{3} \end{matrix} \right\},$$

and the lemma follows from Lemma 38 (for $i = j + 1$). □

Lemma 40. *If p is an odd prime, then for $2i - 1 \leq 2p$,*

$$\binom{2p}{2i-1} \equiv \begin{cases} 0 & \text{if } i \neq \frac{p+1}{2} \\ 2 & \text{if } i = \frac{p+1}{2} \end{cases} \pmod{2p}.$$

Proof. We have

$$\sum_{1 \leq i \leq p} \binom{2p}{2i-1} = 2^{2p-1} \equiv 2 \pmod{2p}$$

by Fermat's theorem. Furthermore, using the identity

$$(2p - 2i + 1) \binom{2p}{2i-1} = 2p \binom{2p-1}{2i-1},$$

we see that, if $i < \frac{p+1}{2}$, then $\binom{2p}{2i-1} \equiv 0 \pmod{2p}$. Thus, for $i = \frac{p+1}{2}$, we have $\binom{2p}{p} \equiv 2 \pmod{2p}$. The case of $i \geq \frac{p+1}{2}$ is symmetric. □

Lemma 41. *If p is an odd prime, then for $0 \leq j \leq (p-1)/2$,*

$$\left\{ \begin{matrix} 2p \\ \frac{2^{2p-2j}-1}{3} \end{matrix} \right\} \equiv - \left\{ \begin{matrix} 2p \\ \frac{2^{2p-2j-2}-1}{3} \end{matrix} \right\} \pmod{2p},$$

$$\left\{ \begin{matrix} 2p \\ \frac{2^{p-1-2j}-1}{3} \end{matrix} \right\} \equiv - \left\{ \begin{matrix} 2p \\ \frac{2^{p-3-2j}-1}{3} \end{matrix} \right\} \pmod{2p}.$$

Proof. This is the same as the proof of Lemma 39, in view of Lemma 40 and the equalities

$$\frac{2^{2p-2j}-1}{3} = \frac{2^{2p-2j-2}-1}{3} + 2^{2p-2j-2},$$
$$\frac{2^{p-1-2j}-1}{3} = \frac{2^{p-3-2j}-1}{3} + 2^{p-3-2j}.$$

□

We now complete the proof of Theorem 36.

Proof of Theorem 36, Parts 2 and 3. For (2), firstly, note that, in case $n = 1$, $a(2) = \left\{ {2 \atop 1} \right\} = 1 \equiv 1 \pmod 2$. Furthermore, using Lemma 39, we have modulo 2^n, $n \geq 2$,

$$a(2^n) = \left\{ {2^n \atop \frac{2^{2^n}-1}{3}} \right\} \equiv -\left\{ {2^n \atop \frac{2^{2^n-2}-1}{3}} \right\} \equiv \left\{ {2^n \atop \frac{2^{2^n-4}-1}{3}} \right\} \equiv \cdots \equiv -\left\{ {2^n \atop 1} \right\}.$$

Now the statement follows in view of

$$\left\{ {m \atop 1} \right\} = \binom{m}{1} - 1.$$

For (3), using Lemma 41, we have modulo $2p$, $p \geq 3$,

$$a(2p) = \left\{ {2p \atop \frac{2^{2p}-1}{3}} \right\} \equiv -\left\{ {2p \atop \frac{2^{2p-2}-1}{3}} \right\} \equiv \cdots \equiv (-1)^{\frac{p-1}{2}} \left\{ {2p \atop \frac{2^{p+1}-1}{3}} \right\}. \tag{61}$$

Now, using Theorem 34 and Lemma 40, we have

$$\left\{ {2p \atop \frac{2^{p+1}-1}{3}} \right\} = \left\{ {2p \atop \frac{2^{p-1}-1}{3} + 2^{p-1}} \right\} \binom{2p}{p} - \left\{ {2p \atop \frac{2^{p-1}-1}{3}} \right\}$$
$$\equiv 2 \left\{ {p \atop \frac{2^{p-1}-1}{3}} \right\} - \left\{ {2p \atop \frac{2^{p-1}-1}{3}} \right\}. \tag{62}$$

But, using the proof of (1), we have

$$\frac{p-1}{2} = s\left(\frac{2^p-2}{3}\right) = s\left(\frac{2^{p-1}-1}{3}\right).$$

Therefore, by Theorem 8, we have

$$\left\{ {p \atop \frac{2^{p-1}-1}{3}} \right\} \equiv (-1)^{(p-1)/2} \pmod p. \tag{63}$$

Again, using Lemma 41, by (61)-(63), we have

$$a(2p) \equiv 2 - (-1)^{(p-1)/2} \left\{ {2p \atop \frac{2^{p-1}-1}{3}} \right\} \equiv$$

$$2+(-1)^{(p-1)/2}\left\{\begin{matrix}2p\\ \frac{2^{p-3}-1}{3}\end{matrix}\right\} \equiv \cdots \equiv 2+\left\{\begin{matrix}2p\\ 1\end{matrix}\right\} = 2+\binom{2p}{1}-1 \equiv 1 \pmod{2p}.$$

□

Corollary 42. *Let $n = 2p+1$, where p is odd primes. Then*

$$|E_{n-1}| \equiv 1 \pmod{n-1}. \qquad (64)$$

Proof. Use (14) with $m = \frac{n-1}{2}$ and part (3) of Theorem 36. □

7. Generating Function for the Up-Down Coefficients

Note that, formula (49) gives the possibility to add to k any powers of 2 more than 2^{t_1}. Therefore, using some iterations of (49), one can formally get any $k_1 > k$. Thus we have a natural way to define $\left\{\begin{matrix}n\\ k\end{matrix}\right\}$ for $k > 2^{n-1}-1$. Moreover, the following lemma shows that the series $\sum_{k=0}^{\infty}\left\{\begin{matrix}n\\ k\end{matrix}\right\}x^k$ converges for $|x| < 1$, for every n.

Lemma 43. *For a fixed n, the sequence $\{\left\{\begin{matrix}n\\ k\end{matrix}\right\}\}_{k\geq 0}$ is bounded.*

Proof. Using (26), we have

$$\left\{\begin{matrix}1\\ k\end{matrix}\right\} = (-1)^m\left(1-\binom{1}{t_m}\right) = \begin{cases}(-1)^m, & if \;\; t_m \geq 2,\\ 0, & if \;\; t_m = 1\end{cases} \quad ;$$

$$\left\{\begin{matrix}2\\ k\end{matrix}\right\} = (-1)^m\left(1-\binom{2}{t_{m-1}}-\binom{2}{t_m}+\binom{2}{t_{m-1}}\binom{2}{t_m}\right) =$$

$$= \begin{cases}(-1)^m, & if \;\; t_m \geq 3\\ 0, & if \;\; t_m = 2\\ 0, & if \;\; t_m = 1, \;\; t_{m-1} = 2,\\ (-1)^{m-1}, & if \;\; t_m = 1, \;\; t_{m-1} \geq 3\end{cases} \quad ;$$

$$\left\{\begin{matrix}3\\ k\end{matrix}\right\} = (-1)^m\left(1-\binom{3}{t_{m-2}}-\binom{3}{t_{m-1}}-\binom{3}{t_m}+\binom{3}{t_{m-2}}\binom{3}{t_{m-1}}+\right.$$

$$\left.+\binom{3}{t_{m-2}}\binom{3}{t_m}+\binom{3}{t_{m-1}}\binom{3}{t_m}-\binom{3}{t_{m-2}}\binom{3}{t_{m-1}}\binom{3}{t_m}\right) =$$

$$= \begin{cases} (-1)^m, \ \ if \ \ t_m \geq 4 \\ 0, \ \ if \ \ t_m = 3 \\ 0, \ \ if \ \ t_m = 2, \ \ t_{m-1} = 3 \\ 2(-1)^{m-1}, \ \ if \ \ t_m = 2, \ \ t_{m-1} > 3 \\ 0, \ \ if \ \ t_m = 1, \ \ t_{m-1} = 2, \ \ t_{m-2} = 3 \\ 4(-1)^m, \ \ if \ \ t_m = 1, \ \ t_{m-1} = 2, \ \ t_{m-2} > 3 \\ 0, \ \ if \ \ t_m = 1, \ \ t_{m-1} = 3, \ \ t_{m-2} > 3 \\ 2(-1)^{m-1}, \ \ if \ \ t_m = 1, \ \ t_{m-1} > 3 \end{cases} ;$$

etc.

We see that, for every fixed n, we have a finite number of distinct values of the sequence $\left\{ \left\{ {n \atop k} \right\} \right\}_{k \geq 0}$. Therefore it is bounded by a constant $C(n)$. □

Denote, for any $n \in \mathbb{N}$,

$$F(n,x) = \sum_{k=0}^{\infty} \left\{ {n \atop k} \right\} x^k, \ \ |x| < 1. \tag{65}$$

Put

$$\tau(x) = \sum_{k=0}^{\infty} \tau_k x^k, \ \ |x| < 1, \tag{66}$$

where τ is the Thue-Morse sequence (23).

Theorem 44. *For every $n \in \mathbb{N}$, the quotient $\frac{F(n,x)}{\tau(x)}$ is a rational function.*

Proof. When p runs through $1, \ldots, m$, then, in view of (20), we have $1 \leq t_m \leq t_p \leq t_1 = \lfloor \log_2(2k) \rfloor$. Therefore, one can write (47) in the form

$$\left\{ {n \atop k} \right\} = \tau_k + \sum_{1 \leq i \leq \log_2(2k)} \binom{n}{i} \left\{ {i \atop k - 2^{i-1}} \right\}, \ \ k \geq 1. \tag{67}$$

Note that, for $k = 0$, the sum is 0, and we have $\left\{ {n \atop 0} \right\} = 1 = \tau_0$. Therefore, by (65) and (67), we find

$$F(n,x) = \sum_{k=0}^{\infty} \left\{ {n \atop k} \right\} x^k = \tau(x) + \sum_{k \geq 1} \sum_{1 \leq i \leq \log_2(2k)} \binom{n}{i} \left\{ {i \atop k - 2^{i-1}} \right\} x^k$$

$$= \tau(x) + \sum_{i=1}^{n} \binom{n}{i} \sum_{k=2^{i-1}}^{\infty} x^k \left\{ {i \atop k - 2^{i-1}} \right\} = \tau(x) + \sum_{i=1}^{n} \binom{n}{i} \sum_{r=0}^{\infty} \left\{ {i \atop r} \right\} x^{r+2^{i-1}}$$

$$= \tau(x) + \sum_{i=1}^{n} \binom{n}{i} x^{2^{i-1}} F(i,x), \ \ \ |x| < 1. \tag{68}$$

(68) gives us a recursion formula for $F(n,x)$:

$$\left(1-x^{2^{n-1}}\right)F(n,x)=\tau(x)+\sum_{i=1}^{n-1}\binom{n}{i}x^{2^{i-1}}F(i,x),\quad |x|<1. \tag{69}$$

Put

$$F(n,x)=\tau(x)\frac{P_n(x)}{(1-x)(1-x^2)\dots(1-x^{2^{n-2}})}\qquad (|x|<1). \tag{70}$$

Then we obtain a recursion formula for $P_n(x)$:

$$P_n(x)=\frac{1}{1-x^{2^{n-1}}}\Bigg((1-x)(1-x^2)\dots(1-x^{2^{n-1}}) + \sum_{i=1}^{n-1}\binom{n}{i}(1-x^{2^i})(1-x^{2^{i+1}})\dots(1-x^{2^{n-1}})\,x^{2^{i-1}}P_i(x)\Bigg). \tag{71}$$

Here it is not expedient to cancel $1-x^{2^{n-1}}$ without additional conventions. In particular, (71) yields

$$P_1(x)=\frac{1}{1-x}(1-x)=1$$

$$P_2(x)=\frac{1}{1-x^2}((1-x)(1-x^2)+2(1-x^2)x)=1+x$$

$$P_3(x)=1+2x+2x^2+x^3,$$

$$P_4(x)=1+3x+5x^2+3x^3+3x^4+5x^5+3x^6+x^7,$$

$$P_5(x)=1+4x+9x^2+6x^3+9x^4+16x^5+11x^6+4x^7+$$
$$4x^8+11x^9+16x^{10}+9x^{11}+6x^{12}+9x^{13}+4x^{14}+x^{15},$$

$$P_6(x)=1+5x+14x^2+10x^3+19x^4+35x^5+26x^6+10x^7+14x^8+40x^9+$$
$$+61x^{10}+35x^{11}+26x^{12}+40x^{13}+19x^{14}+5x^{15}+5x^{16}+19x^{17}+40x^{18}+$$
$$+26x^{19}+35x^{20}+61x^{21}+40x^{22}+14x^{23}+10x^{24}+26x^{25}+35x^{26}+19x^{27}+$$
$$+10x^{28}+14x^{29}+5x^{30}+x^{31},\ \ etc.$$

By a simple induction, we see that $P_n(x)$ is a polynomial in x of degree $2^{n-1}-1$. Thus, the theorem follows from (70). □

But (70) gives us more. Since

$$(1-x)(1-x^2)\dots(1-x^{2^{n-2}})=\sum_{k=0}^{2^{n-1}-1}\tau_k x^k=\tau(x)+o(x^{2^{n-1}-1}),$$

then, from (70) it follows that $P_n(x)=F(n,x)(1+o(x^{2^{n-1}-1}))$ and, since $P_n(x)$ is a polynomial of degree $2^{n-1}-1$, then, by (65), we conclude that the following statement is true.

Theorem 45. *Polynomial $P_n(x)$ which is defined recursively by (71) is equal to*

$$P_n(x) = \sum_{k=0}^{2^{n-1}-1} \left\{ {n \atop k} \right\} x^k, \tag{72}$$

and, for every $n \in \mathbb{N}$, we have an identity

$$\frac{\sum\limits_{k=0}^{2^{n-1}-1} \left\{ {n \atop k} \right\} x^k}{\sum\limits_{k=0}^{\infty} \left\{ {n \atop k} \right\} x^k} = \frac{1}{\prod\limits_{i=n-1}^{\infty} (1 - x^{2^i})}. \tag{73}$$

8. Another Type of Recursion for Up-Down Coefficients

For any $k \in \mathbb{N}$, let us consider the set A_k of those positive integers $i \le \log_2(2k)$ for which $\left\lfloor \frac{k}{2^i} - \frac{1}{2} \right\rfloor = \left\lceil \frac{k+1}{2^i} - 1 \right\rceil$. The common values of these expressions denote by $\lambda(k;i)$:

$$\lambda(k;i) = \left\lfloor \frac{k}{2^i} - \frac{1}{2} \right\rfloor = \left\lceil \frac{k+1}{2^i} - 1 \right\rceil. \tag{74}$$

Theorem 46. *We have*

$$\left\{ {n \atop k} \right\} = \tau_k + \sum_{i \in A_k} \binom{n}{i} \left\{ {i \atop k - 2^{i-1} - \lambda(k;i)2^i} \right\} \tau_{\lambda(k;,i)}. \tag{75}$$

Proof. Taking into account (72) and comparing $Coef_{x^k},\ k \le 2^{n-1} - 1$, in both sides of (71), we find

$$\left\{ {n \atop k} \right\} = \tau_k + \sum_{i=1}^{n-1} \binom{n}{i} \sum_{l} \tau_l \left\{ {i \atop k - 2^{i-1} - 2^i l} \right\},$$

where the summing is over those values of $l \ge 0$ for which

$$2^{i-1} + 2^i l \le k, \ \ k - 2^{i-1} - 2^i l \le degP_i(x) = 2^{i-1} - 1.$$

Consequently,

$$l \in \left[\frac{k+1}{2^i} - 1, \ \frac{k}{2^i} - \frac{1}{2} \right].$$

Nevertheless, the length of this segment equals $\frac{1}{2} - \frac{1}{2^i}$. This means that it could contain not more than one integer value of $l \ge 0$. It does contain it only in the case $\left\lfloor \frac{k}{2^i} - \frac{1}{2} \right\rfloor = \left\lceil \frac{k+1}{2^i} - 1 \right\rceil \ge 0$. □

Example 47. Let $k = 2^m$. Then $i \le \log_2(2k) = m+1$. If $i \le m$, then $\left\lfloor \frac{k}{2^i} - \frac{1}{2} \right\rfloor = 2^{m-i} - 1$, while $\left\lceil \frac{k+1}{2^i} - 1 \right\rceil = 2^{m-i}$. It is left to consider the case $i = m+1$ for which

$$\left\lfloor \frac{k}{2^i} - \frac{1}{2} \right\rfloor = \left\lceil \frac{k+1}{2^i} - 1 \right\rceil = 0.$$

Thus, by (75), we have

$$\begin{Bmatrix} n \\ 2^m \end{Bmatrix} = -1 + \binom{n}{m+1} \begin{Bmatrix} m+1 \\ 0 \end{Bmatrix} = \binom{n}{m+1} - 1. \tag{76}$$

9. Converse of Theorem 22

We have seen (Theorem 22) that

$$\begin{Bmatrix} a \\ b \end{Bmatrix} = 0, \quad if \;\; b \equiv 2^{a-1} + c \pmod{2^a}, \;\; c \in [0, 2^{a-1}). \tag{77}$$

The converse of Theorem 22 is based on the following result.

Theorem 48. *Let $a \in \mathbb{N}$, $\; r \ge a$, $\; c \in [0, 2^{a-1})$. Then, for $b = 2^r l + c$, where l is odd, we have*

$$\begin{Bmatrix} a \\ b \end{Bmatrix} = \tau_l \begin{Bmatrix} a \\ c \end{Bmatrix}. \tag{78}$$

Proof. Let $k = 2^{t_1 - 1} + 2^{t_2 - 1} + \ldots + 2^{t_m - 1}$, $\; t_1 > t_2 > \ldots > t_m \ge 1$. Comparing (47) and (75), we conclude that

$$A_k = \{t_1, t_2, \ldots, t_m\}, \tag{79}$$

and, for $i \in A_k$,

$$\begin{Bmatrix} i \\ k - 2^{i-1} - \lambda(k;i)2^i \end{Bmatrix} \tau_{\lambda(k;i)} = \begin{Bmatrix} i \\ k - 2^{i-1} \end{Bmatrix}. \tag{80}$$

In particular, for $i = t_j$, we have

$$\lambda(k; t_j) = \left\lfloor \frac{k}{2^{t_j}} - \frac{1}{2} \right\rfloor = \left\lceil \frac{k+1}{2^{t_j}} - 1 \right\rceil.$$

Thus

$$\begin{aligned} \lambda(k; t_j) &= \left\lfloor \frac{2^{t_1-1} + 2^{t_2-1} + \cdots + 2^{t_j-1} + \cdots + 2^{t_m-1}}{2^{t_j}} - \frac{1}{2} \right\rfloor \\ &= 2^{t_1 - t_j - 1} + 2^{t_2 - t_j - 1} + \cdots + 2^{t_{j-1} - t_j - 1} \end{aligned} \tag{81}$$

and, consequently,

$$\tau_{\lambda(k;\; t_j)} = (-1)^{j-1}. \tag{82}$$

Therefore, by (82) (for $i = t_j$) and (81), we find

$$(-1)^{j-1}\left\{\begin{matrix} t_j \\ k - 2^{t_1-1} - 2^{t_2-1} - \cdots - 2^{t_{j-1}-1} - 2^{t_l-1} \end{matrix}\right\} = \left\{\begin{matrix} t_j \\ k - 2^{t_j-1} \end{matrix}\right\}, \tag{83}$$

or, taking into account that $k = 2^{t_1-1} + \cdots + 2^{t_m-1}$,

$$(-1)^{j-1}\left\{\begin{matrix} t_j \\ 2^{t_{j+1}-1} + \cdots + 2^{t_m-1} \end{matrix}\right\} =$$

$$\left\{\begin{matrix} t_j \\ 2^{t_1-1} + \ldots + 2^{t_{j-1}-1} + 2^{t_{j+1}-1} + \ldots + 2^{t_m-1} \end{matrix}\right\}. \tag{84}$$

Putting here

$$t_j = a, \;\; 2^{t_{j+1}-1} + \cdots + 2^{t_m-1} = c \in [0, 2^{a-1}),$$
$$t_{j-1} - 1 = r \geq a, \;\; 2^{t_1-1} + \ldots + 2^{t_{j-1}-1} = 2^r l,$$

where

$$l = 2^{t_1-1-r} + 2^{t_2-1-r} + \cdots + 2^{t_{j-1}-1-r} = 2^{t_1-1-r} + 2^{t_2-1-r} + \cdots + 1,$$

such that $\tau_l = (-1)^{j-1}$, and $b = 2^r l + c$, we write (84) in the form of (78). □

Since in Theorem 48, $b - c = 2^r l$, then $\tau_l = \tau_{b-c}$. Therefore, Theorem 48, for $b = k$, one can write in the following form.

Corollary 49. *We have*

$$\left\{\begin{matrix} a \\ k \end{matrix}\right\} = \tau_{k-i}\left\{\begin{matrix} a \\ i \end{matrix}\right\}, \;\; k \equiv i \pmod{2^a}, \;\; i = 0, 1, 2, \ldots, 2^{a-1} - 1.$$

It is worth adding that, by (23),

$$\left\{\begin{matrix} 0 \\ k \end{matrix}\right\} = \tau_k. \tag{85}$$

In particular, taking into account (77) and (85), we obtain the following sequences:

$$\left\{\begin{matrix} 0 \\ k \end{matrix}\right\} : 1,\ -1,\ -1,\ 1,\ -1,\ 1,\ 1,\ -1,\ \ldots;$$

$$\left\{\begin{matrix} 1 \\ k \end{matrix}\right\} : \mathbf{1}, 0, -1, 0, -1, 0, 1, 0, -1, 0, 1, 0, 1, 0, -1, \ldots;$$

$$\left\{\begin{matrix} 2 \\ k \end{matrix}\right\} : \mathbf{1}, \mathbf{1}, 0, 0, -1, -1, 0, 0, -1, -1, 0, 0, 1, 1, \ldots;$$

$$\left\{\begin{matrix} 3 \\ k \end{matrix}\right\} : \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, 0, 0, 0, 0, -1, -2, -2, -1, 0, 0, 0, 0, \ldots;$$

$$\left\{{4 \atop k}\right\} : \mathbf{1}, \mathbf{3}, \mathbf{5}, \mathbf{3}, \mathbf{3}, \mathbf{5}, \mathbf{3}, \mathbf{1}, \underbrace{0, \ldots, 0}_{8}, -1, -3, -5, -3, -3, \ldots; \tag{86}$$

$$\left\{{5 \atop k}\right\} \mathbf{1}, \mathbf{4}, \mathbf{9}, \mathbf{6}, \mathbf{9}, \mathbf{16}, \mathbf{11}, \mathbf{4}, \mathbf{4}, \mathbf{11}, \mathbf{16}, \mathbf{9}, \mathbf{6}, \mathbf{9}, \mathbf{4}, \mathbf{1}, \underbrace{0, \ldots, 0}_{16}, -1, \ldots;$$

$$\left\{{6 \atop k}\right\} \mathbf{1}, \mathbf{5}, \mathbf{14}, \mathbf{10}, \mathbf{19}, \mathbf{35}, \mathbf{26}, \mathbf{10}, \mathbf{14}, \mathbf{40}, \mathbf{61}, \mathbf{35}, \mathbf{26}, \mathbf{40}, \mathbf{19}, \mathbf{5}, \mathbf{5}, \mathbf{19}, \mathbf{40}, \mathbf{26}, \mathbf{35}, \mathbf{61},$$
$$\mathbf{40}, \mathbf{14}, \mathbf{10}, \mathbf{26}, \mathbf{35}, \mathbf{19}, \mathbf{10}, \mathbf{14}, \mathbf{5}, \mathbf{1}, \underbrace{0, \ldots, 0}_{32}, -1, -5, -14, -10, -19, -35, \ldots,$$

etc.

We see that, if the signs in the sequence $(\left\{{a \atop k}\right\})$ are ignored, then it becomes to periodic sequence with period 2^a, such that, by (77), the second part of the period containing terms of the form $2^{a-1} + c,\ \ c = 1, \ldots, 2^{a-1}$, consists of 0's. It is left to show that all terms $\left\{{a \atop k}\right\},\ \ k = 0, 1, \ldots, 2^{a-1} - 1$, of the first part of the period are positive. For this, it is sufficient to note that there exists at least one permutation of elements $1, \ldots, a$ with a given index $k \in [0, 2^{a-1} - 1]$ ([14], Lemma 2). This gives the converse of Theorem 6 and, consequently, of Theorem 5. Thus we have the following statement.

Theorem 50. *If $k = 2^{t_1 - 1} + 2^{t_2 - 1} + \ldots + 2^{t_m - 1},\ t_1 > t_2 > \ldots > t_m \geq 1$, then $t_i,\ i = 1, 2, \ldots, m$, are the only positive integer roots of the polynomials $\left\{{n \atop k}\right\}$.*

Note that, the bold-faced numbers are coefficients of the corresponding polynomials $P_i(x),\ \ i = 1, 2, \ldots$, which are defined by the recursion (71). Consider the concatenation sequence of all of the bold-faced numbers (cf. [20], sequence A060351)

$$\left\{{1 \atop 0}\right\}; \left\{{2 \atop 0}\right\}, \left\{{2 \atop 1}\right\}; \left\{{3 \atop 0}\right\}, \left\{{3 \atop 1}\right\}, \left\{{3 \atop 2}\right\}, \left\{{3 \atop 3}\right\}; \left\{{4 \atop 0}\right\}, \left\{{4 \atop 1}\right\}, \left\{{4 \atop 2}\right\},$$
$$\left\{{4 \atop 3}\right\}, \left\{{4 \atop 4}\right\}, \left\{{4 \atop 5}\right\}, \left\{{4 \atop 6}\right\}, \left\{{4 \atop 7}\right\}; \ldots. \tag{87}$$

It is easy to see that this sequence can be written in the explicit form:

$$\{\left\{{\lfloor \log_2 k \rfloor + 1 \atop k - 2^{\lfloor \log_2 k \rfloor}}\right\}\}_{k=1}^{\infty}. \tag{88}$$

This sequence is closely connected with asymptotics of $\left\{{n \atop k}\right\}$ (cf. (45)-(46)). Thus we conclude that the first coefficients of $\left\{{n \atop k}\right\},\ \ k \geq 1$, as linear combinations of binomial coefficients, form sequence (88) (cf., Appendix).

10. On Full Cycles of $1, \ldots, n$ with a Given Index

The following theorem is very well-known.

Theorem 51. (cf, e.g.,[16]). *The number of full cycles of n elements $1, 2, \ldots, n$ equals $(n-1)!$.*

Denote by $\begin{bmatrix} n \\ k \end{bmatrix}$ the number of full cycles of n elements of index k. We call it the *up-down coefficient for cycles.*

Numerous experiments show that, for $n \geq 3$, very similar expressions hold for numbers $\begin{bmatrix} n \\ k \end{bmatrix}$ and $\begin{Bmatrix} n \\ k \end{Bmatrix}$. Our conjecture is the following.

Conjecture 52. Let $n \geq 3$ and $t_1 = t_1(k)$ be defined by (20). If all divisors of n, other than 1, are larger than t_1, then

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{1}{n}\left(\begin{Bmatrix} n \\ k \end{Bmatrix} - \begin{Bmatrix} 0 \\ k \end{Bmatrix}\right), \tag{89}$$

where, according to (85), $\begin{Bmatrix} 0 \\ k \end{Bmatrix} = \tau_k$; otherwise, (89) is a good approximation of $\begin{bmatrix} n \\ k \end{bmatrix}$.

Note that, in the conditions of Conjecture 52, in view of (24), the fraction in (89) is an integer. Moreover, from (22) we find that

$$\begin{Bmatrix} n \\ k \end{Bmatrix} - \begin{Bmatrix} 0 \\ k \end{Bmatrix} = \begin{vmatrix} \binom{n}{t_1} & \binom{n}{t_2} & \binom{n}{t_3} & \ldots & \binom{n}{t_{m-1}} & \binom{n}{t_m} & 0 \\ 1 & \binom{t_1}{t_2} & \binom{t_1}{t_3} & \ldots & \binom{t_1}{t_{m-1}} & \binom{t_1}{t_m} & 1 \\ 0 & 1 & \binom{t_2}{t_3} & \ldots & \binom{t_2}{t_{m-1}} & \binom{t_2}{t_m} & 1 \\ 0 & 0 & 1 & \ldots & \binom{t_3}{t_{m-1}} & \binom{t_3}{t_m} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & \binom{t_{m-1}}{t_m} & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 \end{vmatrix}. \tag{90}$$

Matrix (90) differs from matrix (22) only in the last element of the first row. This element corresponds to the only diagonal in matrix (22) composed of 1's. The corresponding term in the determinant (90) is $(-1)^m = \tau_k = \begin{Bmatrix} 0 \\ k \end{Bmatrix}$. However, now we prove the following statement about the essentially more complicated structure of numbers $\{\begin{bmatrix} n \\ k \end{bmatrix}\}$.

Theorem 53. *If Conjecture 52 is true, then, in contrast to sequence* $(\left\{ n \atop k \right\})_{n\geq 0}$, *at least for some values of* k, *the sequence* $\{ \left[n \atop k \right] \}_{n\geq 0}$ *is not polynomial.*

Proof. Indeed, let sequence $\{ \left[n \atop k \right] \}$ be polynomial for every k: $\left[n \atop k \right] = P^{(k)}(n)$. Denoting polynomial (90) by $Q^{(k)}(n)$ and considering all prime values of n, we conclude that $P^{(k)}(n) \equiv Q^{(k)}(n)/n$. Since $\left[n \atop k \right]$ is an integer for every n, we have $Q^{(k)}(n)/n$ is an integer-valued polynomial in n for every k. Let k be of the form $k = 2^{p-1}$, where p is prime. Then $t_1 = p$, and in (22) $m = 1$, i.e., we have a 2×2 matrix:

$$\left\{ n \atop 2^{p-1} \right\} = \begin{vmatrix} \binom{n}{p} & 1 \\ 1 & 1 \end{vmatrix} = \binom{n}{p} - 1.$$

Hence, by (90),

$$Q(n)/n = \frac{1}{n} \begin{vmatrix} \binom{n}{p} & 0 \\ 1 & 1 \end{vmatrix} = \frac{(n-1)(n-2)\dots(n-p+1)}{p!} \quad .$$

Let n be multiple of p. We see that, for such n, $Q(n)/n$ is not an integer. We have a contradiction which completes our proof. □

Let us consider an analog of the Eulerian number $A^*(n, t)$ enumerating full cycles of elements $1, \dots, n$, having exactly t ascents.

Theorem 54. *If Conjecture 52 is true, then, for prime* $n \geq 3$, *we have*

$$A^*(n,t) = \frac{1}{n}\left(A(n,t) + (-1)^{t-1}\binom{n-1}{t} \right). \tag{91}$$

Proof. According to (89) and (10),

$$A^*(n,t) = \sum_{0\leq j\leq 2^{n-1}-1:\ s(j)=t} \frac{\left\{ n \atop j \right\} - \tau_j}{n} =$$

$$\frac{1}{n}(\sum_{0\leq j\leq 2^{n-1}-1:\ s(j)=t} (\left\{ n \atop j \right\} - \sum_{0\leq j\leq 2^{n-1}-1:\ s(j)=t} (-1)^t))$$

and (91) follows. □

11. Some Open Problems

1. We conjecture that all real roots of the up-down polynomials are rational.

2. We conjecture that a polynomial $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$, $k \geq 1$, has only real roots if and only if the number of 0's in the binary expansion of k less that 2.

In view of Theorem 21, this condition is sufficient (since there is no place for two conjugate complex roots). Therefore, it is left to prove its necessity.
We have polynomials $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ with only real roots for

$$k = 1, 2, 3, 5, 6, 7, 11, 13, 14, 15, 23, 27, 29, 30, 31, \ldots$$

(cf. sequence A089633 [20]).

3. Investigate sequence $\{k_m\}$ for which the polynomials $\left\{ \begin{matrix} n \\ k_m \end{matrix} \right\}$ have a root $n = -1$. The first values of k_m are: $2, 5, 8, 11, 23, \ldots$

The following 3 conjectures are connected with sequence (13), which we shall denote by $\{a_n\}$.

4. We conjecture that $\max_k \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = a_n$.

5. Let $D_n^{(a)}$ be the number of alternating permutations without fixed points (i.e., $\pi(i) \neq i, \;\; i = 1, 2, \ldots, n$). We conjecture that

$$\lim_{n\to\infty} \frac{D_n^{(a)}}{a_n} = e^{-1}.$$

6. Let $S^{(a)}(n, l)$ be the number of alternating permutations having l cycles (the absolute value of the "alternating" Stirling numbers of the first kind). We conjecture that, for a fixed l,

$$\lim_{n\to\infty} \frac{nS^{(a)}(n, l)}{a_n (\ln n)^{l-1}} = \frac{1}{(l-1)!}.$$

The latter means that, for each l, the events "a permutation is alternative" and "a permutation has l cycles" are asymptotically independent.

7. In connection with Theorem 36, we call an odd composite number m a *zig-zag pseudoprime,* if $a(m) \equiv (-1)^{\frac{m-1}{2}} \pmod{m}$ ([20], sequence A180942). Is it true that every Carmichael number ([20], sequence A002997) is also zig-zag pseudoprime?

8. In connection with Corollary 37, we call an odd composite number m a *B-pseudoprime,* if $12\alpha_{m+1} + (-1)^{(m+1)/2}\beta_{m+1} \equiv 0 \pmod{m}$ ([20], sequence A180943). Explain the following phenomenon: among the first fifty B-pseudoprimes there are only two of the form $4l - 1$.

9. Find a proof of Conjecture 52 for cycles (Section 10).

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Appendix

Below is a list of the first thirty-two up-down polynomials.

$$\left\{\begin{matrix} n \\ 0 \end{matrix}\right\} = 1 \qquad \left\{\begin{matrix} n \\ 1 \end{matrix}\right\} = \binom{n}{1} - 1 \qquad \left\{\begin{matrix} n \\ 2 \end{matrix}\right\} = \binom{n}{2} - 1$$

$$\left\{\begin{matrix} n \\ 3 \end{matrix}\right\} = \binom{n}{2} - \binom{n}{1} + 1 \qquad \left\{\begin{matrix} n \\ 4 \end{matrix}\right\} = \binom{n}{3} - 1$$

$$\left\{\begin{matrix} n \\ 5 \end{matrix}\right\} = 2\binom{n}{3} - \binom{n}{1} + 1 \qquad \left\{\begin{matrix} n \\ 6 \end{matrix}\right\} = 2\binom{n}{3} - \binom{n}{2} + 1$$

$$\left\{\begin{matrix} n \\ 7 \end{matrix}\right\} = \binom{n}{3} - \binom{n}{2} + \binom{n}{1} - 1 \qquad \left\{\begin{matrix} n \\ 8 \end{matrix}\right\} = \binom{n}{4} - 1$$

$$\left\{\begin{matrix} n \\ 9 \end{matrix}\right\} = 3\binom{n}{4} - \binom{n}{1} + 1 \qquad \left\{\begin{matrix} n \\ 10 \end{matrix}\right\} = 5\binom{n}{4} - \binom{n}{2} + 1$$

$$\left\{\begin{matrix} n \\ 11 \end{matrix}\right\} = 3\binom{n}{4} - \binom{n}{2} + \binom{n}{1} - 1 \qquad \left\{\begin{matrix} n \\ 12 \end{matrix}\right\} = 3\binom{n}{4} - \binom{n}{3} + 1$$

$$\left\{\begin{matrix} n \\ 13 \end{matrix}\right\} = 5\binom{n}{4} - 2\binom{n}{3} + \binom{n}{1} - 1 \qquad \left\{\begin{matrix} n \\ 14 \end{matrix}\right\} = 3\binom{n}{4} - 2\binom{n}{3} + \binom{n}{2} - 1$$

$$\left\{\begin{matrix} n \\ 15 \end{matrix}\right\} = \binom{n}{4} - \binom{n}{3} + \binom{n}{2} - \binom{n}{1} + 1 \qquad \left\{\begin{matrix} n \\ 16 \end{matrix}\right\} = \binom{n}{5} - 1$$

$$\left\{\begin{matrix} n \\ 17 \end{matrix}\right\} = 4\binom{n}{5} - \binom{n}{1} + 1 \qquad \left\{\begin{matrix} n \\ 18 \end{matrix}\right\} = 9\binom{n}{5} - \binom{n}{2} + 1$$

$$\left\{\begin{matrix} n \\ 19 \end{matrix}\right\} = 6\binom{n}{5} - \binom{n}{2} + \binom{n}{1} - 1 \qquad \left\{\begin{matrix} n \\ 20 \end{matrix}\right\} = 9\binom{n}{5} - \binom{n}{3} + 1$$

$$\left\{\begin{matrix} n \\ 21 \end{matrix}\right\} = 16\binom{n}{5} - 2\binom{n}{3} + \binom{n}{1} - 1$$

$$\left\{\begin{matrix} n \\ 22 \end{matrix}\right\} = 11\binom{n}{5} - 2\binom{n}{3} + \binom{n}{2} - 1$$

$$\left\{\begin{matrix} n \\ 23 \end{matrix}\right\} = 4\binom{n}{5} - \binom{n}{3} + \binom{n}{2} - \binom{n}{1} + 1 \qquad \left\{\begin{matrix} n \\ 24 \end{matrix}\right\} = 4\binom{n}{5} - \binom{n}{4} + 1$$

$$\left\{\begin{matrix} n \\ 25 \end{matrix}\right\} = 11\binom{n}{5} - 3\binom{n}{4} + \binom{n}{1} - 1$$

$$\left\{\begin{matrix} n \\ 26 \end{matrix}\right\} = 16\binom{n}{5} - 5\binom{n}{4} + \binom{n}{2} - 1$$

$$\left\{ \begin{matrix} n \\ 27 \end{matrix} \right\} = 9\binom{n}{5} - 3\binom{n}{4} + \binom{n}{2} - \binom{n}{1} + 1$$

$$\left\{ \begin{matrix} n \\ 28 \end{matrix} \right\} = 6\binom{n}{5} - 3\binom{n}{4} + \binom{n}{3} - 1$$

$$\left\{ \begin{matrix} n \\ 29 \end{matrix} \right\} = 9\binom{n}{5} - 5\binom{n}{4} + 2\binom{n}{3} - \binom{n}{1} + 1$$

$$\left\{ \begin{matrix} n \\ 30 \end{matrix} \right\} = 4\binom{n}{5} - 3\binom{n}{4} + 2\binom{n}{3} - \binom{n}{2} + 1$$

$$\left\{ \begin{matrix} n \\ 31 \end{matrix} \right\} = \binom{n}{5} - \binom{n}{4} + \binom{n}{3} - \binom{n}{2} + \binom{n}{1} - 1$$

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