EXTENDING NATHANSON HEIGHTS TO ARBITRARY FINITE FIELDS

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Abstract

In this paper, we extend the definition of the Nathanson height from points in projective spaces over $\mathbb{F}_p$ to points in projective spaces over arbitrary finite fields. If $[a_0 : \ldots : a_n] \in \mathbb{P}^d(\overline{\mathbb{F}_p})$, then the Nathanson height is

$$\mathrm{h}_p([a_0 : a_1 : \ldots : a_d]) = \min_{b \in \overline{\mathbb{F}_p}^*} \sum_{i=0}^{d} \mathrm{H}(ba_i)$$

where $H(a_i) = |\mathrm{N}(a_i)| + p(\deg(a_i) - 1)$ with N the field norm and $|\mathrm{N}(a_i)|$ the element of $\{0, 1, \ldots, p-1\}$ congruent to $\mathrm{N}(a_i)$ modulo p.

We investigate the basic properties of this extended height, provide some bounds, study its image on the projective line $\mathrm{h}_p(\mathbb{P}^1(\overline{\mathbb{F}_p}))$ and propose some questions for further research.

1. Introduction

The classical Nathanson height is a sort of measure of complexity of a point in a projective space over a finite field $\mathbb{F}_p$. For each prime p and dimension d we have a height function. These heights are functions defined given a finite field $\mathbb{F}_p$ and $d \in \mathbb{N}$

$$\mathrm{h}_p : \mathbb{P}^d(\mathbb{F}_p) \to \mathbb{Z}, \quad \mathrm{h}_p([a_0 : a_1 : \ldots : a_d]) = \min_{b \in \mathbb{F}_p^*} \sum_{i=0}^{d} |ba_i|$$

where $|a|$ is the element of $\{0, 1, \ldots, p-1\}$ such that $a \equiv |a| \pmod p$.

In [4], Nathanson and Sullivan defined these heights, and they studied the image of h_p. In [3], Nathanson continued studying these heights in the projective line, and O'Bryant gave an explicit formula in the projective line in [5]. Also, Batson extended the definition of Nathanson heights from points to linear subspaces of $\mathbb{P}^d(\mathbb{F}_p)$ in [1].

Nathanson and Sullivan proposed a number of problems in [4]. One of them was to find a reasonable definition of the height function for points in projective space over *arbitrary* finite fields. In this paper, we propose the following definition: let

$$\mathrm{H} : \overline{\mathbb{F}_p} \to \mathbb{R}, \qquad \mathrm{H}(a) = |\mathrm{N}(a)| + p(\deg(a) - 1),$$

where N is the field norm and $\deg(a)$ is $[\mathbb{F}_p(a) : \mathbb{F}_p]$. The Nathanson height is the function

$$\mathrm{h}_p : \mathbb{P}^d(\mathbb{F}_p) \to \mathbb{R}, \qquad \mathrm{h}_p([a_0 : a_1 : \ldots : a_d]) = \min_{b \in \overline{\mathbb{F}_p}^*} \sum_{i=0}^{d} \mathrm{H}(ba_i). \tag{1}$$

We describe how this article is organized. In the second section, we recall elementary facts about finite fields and prove some general properties of these extended Nathanson heights. In the third section, we study these heights in the projective line, and we conclude and propose directions for further research in the last section.

We are going to use standard conventions. In this paper, p is a prime. The field $\mathbb{F}_p$ is the finite field with p elements, and $\overline{\mathbb{F}_p}$ is its algebraic closure. If $\mathbb{F}$ is a finite field, then $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$.

2. Definition and Basic Properties

From now on, we call the Nathanson heights defined in (1) simply "heights." To prove some properties of the heights, we need to recall some facts of the finite fields (see [2]).

First, for all $n \in \mathbb{N}$ there is exactly one field of cardinality p^n, and we denote this field by $\mathbb{F}_{p^n}$. If we define the degree as above, then if $a, b \in \overline{\mathbb{F}_p}^*$ and $\deg(a) < \deg(b)$, then

$$\deg(a) < \deg(ab) = \deg(b). \tag{2}$$

Recall that if $a \in \overline{\mathbb{F}_p}$, then the norm $\mathrm{N}(a)$ is the product of all its conjugates, in particular $\mathrm{N}(a) = 0$ if and only if $a = 0$. For a finite extension of finite fields $\mathbb{F}/\mathbb{F}'$, if $a \in \mathbb{F}$, then we define

$$T_a : \mathbb{F} \to \mathbb{F} \ \text{ by } \ T_a(x) = ax \quad \text{and} \quad \mathrm{N}_{\mathbb{F}/\mathbb{F}'}(a) = \det(T_a).$$

The function $\mathrm{N}_{\mathbb{F}/\mathbb{F}'}$ is multiplicative, and if $\deg(a) = n$ and $\mathbb{F}/\mathbb{F}_{p^n}$ is a finite extension, then

$$\mathrm{N}_{\mathbb{F}/\mathbb{F}_p}(a) = \mathrm{N}(a)^{[\mathbb{F}:\mathbb{F}_{p^n}]}. \tag{3}$$

Recall that for all $n \in \mathbb{N}$

$$\mathrm{N}_{\mathbb{F}_{p^n}/\mathbb{F}_p}(\mathbb{F}_{p^n}^*) = \mathbb{F}_p{}^*.$$

Now, if $\mathbf{a} = [a_0 : a_1 : \ldots : a_d] \in \mathbb{P}^d(\overline{\mathbb{F}_p})$, then let n be the minimal natural number such that $\mathbf{a} \in \mathbb{P}^d(\mathbb{F}_{p^n})$.

Proposition 1. *Let $\mathbf{a} = [a_0 : \ldots : a_d] \in \mathbb{P}^d(\overline{\mathbb{F}_p})$ and let n be the minimal natural number such that $\mathbf{a}$ sits inside the finite projective space $\mathbb{P}^d(\mathbb{F}_{p^n})$. Then*

$$\mathrm{h}_p(\mathbf{a}) = \min_{\substack{b \in \overline{\mathbb{F}_p}^* \\ \deg(b) \leq n}} \sum_{i=0}^{d} \mathrm{H}(ba_i).$$

Proof. Assume that $a_0, \ldots, a_d \in \mathbb{F}_{p^n}$. It is easy to see that

$$\mathrm{h}_p(\mathbf{a}) \leq \min_{\substack{b \in \overline{\mathbb{F}_p}^* \\ \deg(b) \leq n}} \sum_{i=0}^{d} \mathrm{H}(ba_i)$$

since $\mathbb{F}_{p^n} \subseteq \overline{\mathbb{F}_p}$. For the other inequality, the norm of an element is at most $p-1$ so if $a, a' \in \overline{\mathbb{F}_p}$ satisfy $\deg(a) < \deg(a')$, then $\mathrm{H}(a) < \mathrm{H}(a')$. Thus, if $\deg(b) > n$, then (2) implies that for all a_i

$$\deg(a_i) \leq n < \deg(ba_i) \quad \text{and} \quad \mathrm{H}(a_i) < \mathrm{H}(ba_i)$$

so

$$\mathrm{h}_p([a_0 : a_1 : \ldots : a_d]) \leq \sum_{i=0}^{d} \mathrm{H}(a_i) < \sum_{i=0}^{d} \mathrm{H}(ba_i)$$

and the inequality follows. □

The proposition shows that our heights agree with the Nathanson heights in the case $\mathbb{F}_{p^n} = \mathbb{F}_p$.

Corollary 2. *If $\mathbf{a} = [a_0 : a_1 : \ldots : a_d] \in \mathbb{P}^d(\mathbb{F}_p)$, then*

$$\mathrm{h}_p(\mathbf{a}) = \min_{b \in \mathbb{F}_p{}^*} \sum_{i=0}^{d} |ba_i|$$

Proof. From Proposition 1

$$\mathrm{h}_p(\mathbf{a}) = \min_{\substack{b \in \overline{\mathbb{F}_p}^* \\ \deg(b) \leq 1}} \sum_{i=0}^{d} \mathrm{H}(ba_i) = \min_{b \in \mathbb{F}_p{}^*} \sum_{i=0}^{d} \mathrm{H}(ba_i)$$

but since $ba_i \in \mathbb{F}_p$, we have $\mathrm{H}(ba_i) = |ba_i|$. □

Remark. Recall that the norm and the degree are Galois invariant; Thus, the functions H and h_p are invariant under the action of $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$.

For $\mathbf{a} = [a_0 : a_1 : \ldots : a_d] \in \mathbb{P}^d(\overline{\mathbb{F}_p})$, let n be the minimal natural number such that $\mathbf{a} \in \mathbb{P}^d(\mathbb{F}_{p^n})$.

Proposition 3. *With the notation as above* $1 + (n-1)p \le \mathrm{h}_p(\mathbf{a}) \le 1 + (np-1)d$.

Proof. Assume $a_0, a_1, \dots a_d \in \mathbb{F}_{p^n}$. For the right-hand side inequality, for all a_i and $b \in \mathbb{F}_{p^n}^*$

$$|\mathrm{N}(ba_i)| \le p - 1 \quad \text{and} \quad \deg(ba_i) \le n$$

so if we assume without loss of generality $a_0 \neq 0$, then

$$\mathrm{h}_p(\mathbf{a}) = \mathrm{h}_p([1 : a_0^{-1}a_1 : \dots : a_0^{-1}a_d]) \le 1 + \sum_{i=1}^{d} \mathrm{H}(a_0^{-1}a_i) \le 1 + (np-1)d.$$

For the left-hand side inequality, we claim that for all $b \in \mathbb{F}_{p^n}^*$ there is an entry a_{i_b} such that $\deg(ba_{i_b}) = n$. In fact, the n is minimal so for all $m < n$ we have $[ba_0 : ba_1 : \dots : ba_d] \in \mathbb{P}^d(\mathbb{F}_{p^n}) \setminus \mathbb{P}^d(\mathbb{F}_{p^m})$, and thereby there is a $ba_i \in \mathbb{F}_{p^n}$ which is not contained in a smaller field.

By Proposition 1, there is a $b_0 \in \mathbb{F}_{p^n}$ such that $\mathrm{h}_p(\mathbf{a}) = \sum_{i=0}^{d} \mathrm{H}(b_0 a_i)$, and we have

$$\begin{aligned} \mathrm{h}_p(\mathbf{a}) &= \sum_{i=0}^{d} \mathrm{H}(b_0 a_i) \\ &= |\mathrm{N}(b_0 a_{i_{b_0}})| + p(n-1) + \sum_{i \neq i_{b_0}} \mathrm{H}(b_0 a_i) \qquad \text{by the claim above} \\ &\ge 1 + p(n-1). \end{aligned}$$

□

In the last proposition, if $n > 1$, then the right-hand side could be improved as follows:

$$\begin{aligned} \mathrm{h}_p(\mathbf{a}) = \sum_{i=0}^{d} \mathrm{H}(b_0 a_i) &= |\mathrm{N}(b_0 a_{i_{b_0}})| + p(n-1) + \sum_{i \neq i_{b_0}} \mathrm{H}(b_0 a_i) \\ &\ge 1 + p(n-1) + \sum_{i \neq i_{b_0}} \mathrm{H}(b_0 a_i) \\ &\ge 2 + p(n-1). \end{aligned} \tag{4}$$

The last inequality holds since $\mathrm{H}(b_0 a_i) = 0$ if and only if $b_0 a_i = 0$. This gives us that $\sum_{i \neq i_{b_0}} \mathrm{H}(b_0 a_i) = 0$ implies $\mathbf{a} \in \mathbb{P}^d(\mathbb{F}_p)$ and n would not be minimal. Furthermore, $\mathrm{N}_{\mathbb{F}_{p^n}/\mathbb{F}_p}(\mathbb{F}_{p^n}^*) = {\mathbb{F}_p}^*$, $\mathrm{N}_{\mathbb{F}_{p^n}/\mathbb{F}_p}$ is multiplicative, and $\frac{|\mathbb{F}_{p^n}^*|}{p-1} > |\bigcup_{m<n} \mathbb{F}_{p^m}^*|$ imply, by the pigeonhole principle, that there is $a \in \mathbb{F}_{p^n}^*$ such that $\deg(a) = n$ and $\mathrm{N}(a) = 1$. Thus, a satisfies

$$\mathrm{h}_p([1 : a : 0 : \dots : 0]) = \min_{\substack{b \in \overline{\mathbb{F}_p}^* \\ \deg(b) \le n}} (\mathrm{H}(b) + \mathrm{H}(ba)) = 2 + p(n-1)$$

and the lower bound (4) is the best possible.

3. Projective Line

Some research has been done to find an explicit formula for the heights (see [3], [4] and [5]). In this section, we study the heights in the projective line, and we find some formulas.

In this section, $\mathbf{a} = [a_0 : a_1] \in \mathbb{P}^1(\overline{\mathbb{F}_p})$ and n is the minimal natural number such that $\mathbf{a} \in \mathbb{P}^1(\mathbb{F}_{p^n})$. If $a_0 = 0$ or $a_1 = 0$, then $\mathrm{h}_p(\mathbf{a}) = 1$, so we assume $a_0 \neq 0$ and $a_1 \neq 0$ hereafter. Define $a_* = a_0^{-1} a_1$,

$$\mathcal{A}_0 = \{|b| + |b^n \mathrm{N}(a_*)| : b \in {\mathbb{F}_p}^*\}$$

and

$$\mathcal{A}_1 = \{|b| + |b^n \mathrm{N}(a_*^{-1})| : b \in {\mathbb{F}_p}^*\}.$$

Proposition 4. *With the notation as above,*

$$\mathrm{h}_p(\mathbf{a}) = p(n-1) + \min(\mathcal{A}_0 \cup \mathcal{A}_1).$$

Moreover, if $(p-1)|(n-1)$*, then*

$$\mathrm{h}_p(\mathbf{a}) = p(n-1) + \min\{\mathrm{h}_p([1 : \mathrm{N}(a_*)]), \mathrm{h}_p([1 : \mathrm{N}(a_*^{-1})])\},$$

and if $(p-1)|n$*, then*

$$\mathrm{h}_p(\mathbf{a}) = p(n-1) + \min\{1 + |\mathrm{N}(a_*)|, 1 + |\mathrm{N}(a_*^{-1})|\}.$$

Proof. Recall, as in Proposition 2, that $\mathbf{a} \in \mathbb{P}^1(\mathbb{F}_{p^n})$ implies that for all $b \in \mathbb{F}_{p^n}^*$ we have $\deg(ba_0) = n$ or $\deg(ba_1) = n$, and since $|\mathrm{N}(ba_0)|$ and $|\mathrm{N}(ba_1)|$ are less or equal to $p-1$, we have that

$$\mathrm{h}_p(\mathbf{a}) \leq \mathrm{H}(1) + \mathrm{H}(a_*) = p(n-1) + 1 + |\mathrm{N}(a_*)| \leq pn$$

and thereby the minimum of $\mathrm{H}(ba_0) + \mathrm{H}(ba_1)$ with $\deg(b) \leq n$ is achieved when $\deg(ba_0) + \deg(ba_1) \leq n+1$, i.e., when $\deg(ba_0) + \deg(ba_1) = n+1$. Since $\deg(ba_0) + \deg(ba_1) = n+1$ occurs, in this case, if and only if $ba_0 \in \mathbb{F}_p$ or $ba_1 \in \mathbb{F}_p$, we conclude that

$$\begin{aligned}
\mathrm{h}_p(\mathbf{a}) &= \min\{\mathrm{H}(b) + \mathrm{H}(ba_*), \mathrm{H}(b) + \mathrm{H}(ba_*^{-1}) : \ b \in \mathbb{F}_p\} \\
&= p(n-1) + \min\{|\mathrm{N}(b)| + |\mathrm{N}(ba_*)|, |\mathrm{N}(b)| + |\mathrm{N}(ba_*^{-1})| : \ b \in \mathbb{F}_p\} \\
&= p(n-1) + \min(\mathcal{A}_0 \cup \mathcal{A}_1),
\end{aligned}$$

with the last equality following from properties of the norm recalled in Section 2.

For the second claim, if $(p-1)|(n-1)$, then

$$|b^n \mathrm{N}(a_*)| = |b\mathrm{N}(a_*)| \quad \text{and} \quad |b^n \mathrm{N}(a_*^{-1})| = |b\mathrm{N}(a_*^{-1})|$$

so $\min \mathcal{A}_0 = \mathrm{h}_p([1 : \mathrm{N}(a_*)])$ and $\min \mathcal{A}_1 = \mathrm{h}_p([1 : \mathrm{N}(a_*^{-1})])$, and the claim follows.

Finally, if $(p-1) \mid n$, then

$$|b^n \mathrm{N}(a_*)| = |\mathrm{N}(a_*)| \text{ and } |b^n \mathrm{N}(a_*^{-1})| = |\mathrm{N}(a_*^{-1})|,$$

so $\min \mathcal{A}_0 = 1 + |\mathrm{N}(a_*)|$ and $\min \mathcal{A}_1 = 1 + |\mathrm{N}(a_*^{-1})|$, and the claim follows. □

4. Conclusion and Further Research

The heights defined above generalize the heights defined by Nathanson and Sullivan [4]. Roughly, this generalization measures the complexity of $\mathbf{a} \in \mathbb{P}^d(\overline{\mathbb{F}_p})$; in other words, if $\mathbf{a}$ has a lot of entries which are not zeroes and the minimum $n \in \mathbb{N}$ such that $ap \in \mathbb{P}^d(\mathbb{F}_{p^n})$ is big, then $\mathrm{h}_p(\mathbf{a})$ is big. In the case of a projective line, Proposition 2 implies that $\mathrm{h}_p(\mathbf{a}) \in [(n-1)p+1, np]$ if and only if n is the minimum natural such that $\mathbf{a} \in \mathbb{P}^1(\mathbb{F}_{p^n})$.

Some problems are the following:

1. Does there exist an easier formula for h_p, even in the case of the projective line?

2. O'Bryant proved in [5] that asymptotically, as $p \to \infty$, the image of $\mathrm{h}_p(\mathbb{P}^1(\mathbb{F}))$ is roughly equal to $\{0\} \cup \{\frac{p}{n} : n \in \mathbb{N}\}$. How about the set $\mathrm{h}_p(\mathbb{P}^1(\overline{\mathbb{F}}))$? Can the set $\mathrm{h}_p(\mathbb{P}^1(\overline{\mathbb{F}}))$ be asymptotically roughly equal to $\{0\} \cup \{pk + \frac{p}{n} : n \in \mathbb{N}, k \in \mathbb{N} \cup \{0\}\}$?

3. Study the counting functions $\mathrm{H}_p^d : \mathbb{N} \to \mathbb{N}$, $\mathrm{H}_p^d(m) = \{\mathbf{a} \in \mathbb{P}^d(\overline{\mathbb{F}_p}) : \mathrm{h}_p(\mathbf{a}) \leq m\}$.

4. Generalize the heights h_p in the direction of the p-adic fields. Can we generalize the heights proposed by Nathanson and Sullivan in [4] in such a way that they are related to the classic global fields heights?

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References

[1] J. Batson, *Nathanson heights in finite vector spaces*, J. Number Theory **128** (2008), no. 9, 2616-2632.

[2] P. Morandi, *Field and Galois Theory*, Springer-Verlag, Graduate Texts in Mathematics **167** (2008), New York 1996.

[3] M. B. Nathanson, *Height on the finite projective line*, Int. J. Number Theory **5** (2009), no. 1, 55-65.

[4] M. B. Nathanson and B. D. Sullivan, *Heights in the finite projective space, and a problem on directed graphs*, preprint available at arXiv:math.NT/0703418, 2007.

[5] K. O'Bryant, *Gaps in the Spectrum of Nathanson Heights of the Projective Points*, Integers **7** (2007) A38.