ON THE PROBLEM OF MOLLUZZO FOR THE MODULUS 4

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Abstract

We solve the currently smallest open case in the 1976 problem of Molluzzo on $\mathbb{Z}/m\mathbb{Z}$, namely the case $m = 4$. This amounts to constructing, for all positive integers n congruent to 0 or 7 mod 8, a sequence of integers modulo 4 of length n generating, by Pascal's rule, a Steinhaus triangle containing 0,1,2,3 with equal multiplicities.

1. Introduction

The problem of Molluzzo in combinatorial number theory is about the existence of certain triangular arrays in $\mathbb{Z}/m\mathbb{Z}$. It was first formulated by Steinhaus in 1958 for $m = 2$ [9], and then generalized by Molluzzo in 1976 for $m \geq 3$ [8]. It is still widely open for most moduli m. The problem reads as follows. Given $m \geq 2$, for which $n \geq 1$ does there exist a triangle

$$\nabla = \begin{array}{ccccc} x_{1,1} & x_{1,2} & \cdots & x_{1,n-1} & x_{1,n} \\ & x_{2,1} & & x_{2,n-1} & \\ & \ddots & & ⋰ & \\ & & x_{n,1} & & \end{array}$$

with entries $x_{i,j}$ in $\mathbb{Z}/m\mathbb{Z}$, of side length n, satisfying the following two conditions:

(C1) Pascal's rule: every element of ∇ outside the first row is the sum of the two elements above it. That is, $x_{i,j} = x_{i-1,j} + x_{i-1,j-1}$ for all $2 \leq i \leq n$ and $1 \leq j \leq n - i$.

(C2) The elements of $\mathbb{Z}/m\mathbb{Z}$ all occur with the same multiplicity in ∇.

Definition 1. A triangle ∇ in $\mathbb{Z}/m\mathbb{Z}$ satisfying condition (C1) is called a Steinhaus triangle. It is said to be *balanced* if it satisfies condition (C2).

By (C1), a Steinhaus triangle is completely determined by its first row S, and may thus be denoted by ∇S. For instance, the sequence $S = 0100203$ in $\mathbb{Z}/4\mathbb{Z}$ generates the following Steinhaus triangle ∇S:

$$\begin{array}{ccccccccccccc}
0 & & 1 & & 0 & & 0 & & 2 & & 0 & & 3 \\
& 1 & & 1 & & 0 & & 2 & & 2 & & 3 & \\
& & 2 & & 1 & & 2 & & 0 & & 1 & & \\
& & & 3 & & 3 & & 2 & & 1 & & & \\
& & & & 2 & & 1 & & 3 & & & & \\
& & & & & 3 & & 0 & & & & & \\
& & & & & & 3 & & & & & &
\end{array}.$$

Observe that ∇S is balanced, since each element of $\mathbb{Z}/4\mathbb{Z}$ appears with the same multiplicity, namely 7.

An obvious necessary condition for the existence of a balanced Steinhaus triangle ∇ in $\mathbb{Z}/m\mathbb{Z}$ of side length n is given by

$$\binom{n+1}{2} \equiv 0 \bmod m. \tag{1}$$

Indeed, it follows from (C2) that m divides the multiset cardinality of ∇, which is $\binom{n+1}{2}$.

Is this necessary condition also sufficient? This is the more detailed content of Molluzzo's problem. While it seems reasonable to conjecture a positive answer in most cases, two counterexamples are known: in $\mathbb{Z}/15\mathbb{Z}$ and in $\mathbb{Z}/21\mathbb{Z}$, there is no balanced Steinhaus triangle of side length 5 and 6, respectively, even though the pairs $(m, n) = (15, 5)$ and $(21, 6)$ both satisfy condition (1). (See [1, p. 75] and [3, p. 293].)

Note that, given $m \geq 2$, the condition for $n \in \mathbb{N}$ to satisfy (1) only depends on the class of n mod m if m is odd, or mod $2m$ if m is even.

1.1. Known Results

Despite its apparent simplicity, the problem of Molluzzo is very challenging, as testified by the scarcity of available results. The only moduli m for which it has been completely solved so far are

- $m = 2$ in [7, 5, 6],
- $m = 3^i$ for all $i \geq 1$ in [1, 2],

- $m = 5, 7$ in [1, 4].

In each case, the necessary existence condition (1) turns out to be sufficient.

1.2. Contents

In this paper, we solve the currently smallest open case of the problem, namely the case $m = 4$. Our solution is presented in Section 2 and proved valid in Section 3. Here again, the necessary existence condition (1) is found to be sufficient. The construction method, which consists of attempting to lift to $\mathbb{Z}/4\mathbb{Z}$ specific known solutions in $\mathbb{Z}/2\mathbb{Z}$, is explained in Section 4. It will probably take some time before complete solutions emerge for more moduli. For this reason we propose, in a short concluding section, a hopefully more tractable version of the problem.

2. A Solution for $m = 4$

Here we solve Molluzzo's problem in the group $\mathbb{Z}/4\mathbb{Z}$. For $m = 4$, it is easy to see that the necessary condition (1) amounts to the following: for all $n \in \mathbb{N}$, we have

$$\binom{n+1}{2} \equiv 0 \bmod 4 \quad \Longleftrightarrow \quad n \equiv 0 \text{ or } 7 \bmod 8.$$

As in [5] for the case $m = 2$, our solution involves the concept of *strongly balanced* triangles. We first introduce a notation for initial segments of sequences.

Notation 2. Let $S = (x_i)_{i \geq 1}$ be a finite or infinite sequence, and let $l \geq 0$ be an integer not exceeding the length of S. We denote by $S[l] = (x_1, \ldots, x_l)$ the initial segment of length l of S.

Definition 3. Let S be a finite sequence of length $n \geq 0$ in $\mathbb{Z}/4\mathbb{Z}$. The Steinhaus triangle ∇S is said to be *strongly balanced* if, for every $0 \leq t \leq n/8$, the Steinhaus triangle $\nabla S[n - 8t]$ is balanced.

Here is our main result in this paper.

Theorem 4. *There exists a balanced Steinhaus triangle of length n in $\mathbb{Z}/4\mathbb{Z}$ if and only if $\binom{n+1}{2} \equiv 0 \bmod 4$. More precisely, consider the following infinite, eventually periodic sequences in $\mathbb{Z}/4\mathbb{Z}$:*

$$S_1 = 01220232(21211322003023231120023\,2)^{\infty},$$
$$S_2 = 21210130(20013202211200211022013\,0)^{\infty},$$

$$T_1 = 0120021(212202102023032200322021)^{\infty},$$
$$T_2 = 1000212(312223301210312003103232)^{\infty},$$
$$T_3 = 1200210(220101222032222103000210)^{\infty},$$
$$T_4 = 2102203(232002102021230022302203)^{\infty}.$$

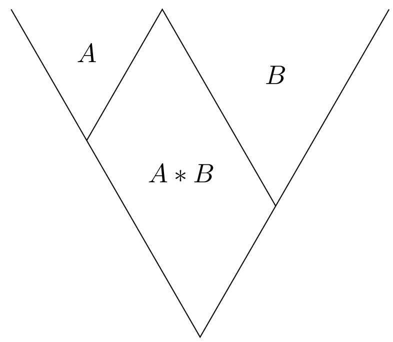

Figure 1: Defining $A * B$.

Then, for all integers i, j, k such that $1 \leq i \leq 2$, $1 \leq j \leq 4$ and $k \geq 0$, the Steinhaus triangles of the initial segments $S_i[8k]$ and $T_j[8k+7]$ are strongly balanced.

3. Proof of Theorem 4

We now prove the above theorem. Actually, only the statements concerning S_1 are treated in detail. The proof method for the other sequences S_2, T_1, T_2, T_3, T_4 is similar and only briefly commented.

Notation 5. Let A, B be two *blocks*, either triangles or lozenges. We denote by $A * B$ the unique parallelogram they would determine by Pascal's rule (C1) if they were adjacent in a large Steinhaus triangle (as may be freely assumed to be the case). See Figure 1.

Note that $A * B$ depends only on the right lower side of A and the left lower side of B, and is a lozenge if A, B have the same side length. For example:

- if $A = \begin{array}{ccc} 0 & & 1 \\ & 1 & \end{array}$ and $B = \begin{array}{ccc} 2 & & 3 \\ & 1 & \end{array}$, then $A * B = \begin{array}{ccc} & 3 & \\ 0 & & 0 \\ & 0 & \end{array}$ since $\begin{array}{ccccccc} 0 & & 1 & & 2 & & 3 \\ & 1 & & \mathbf{3} & & 1 & \\ & & \mathbf{0} & & \mathbf{0} & & \\ & & & \mathbf{0} & & & \end{array}$;
- if $A = \begin{array}{ccc} & 2 & \\ 2 & & 3 \\ & 1 & \end{array}$ and $B = \begin{array}{ccc} & 3 & \\ 0 & & 2 \\ & 2 & \end{array}$, then $A * B = \begin{array}{ccc} & 3 & \\ 0 & & 1 \\ & 1 & \end{array}$ since $\begin{array}{ccccccc} & 2 & & & & 3 & \\ 2 & & 3 & & 0 & & 2 \\ & 1 & & \mathbf{3} & & 2 & \\ & & \mathbf{0} & & \mathbf{1} & & \\ & & & \mathbf{1} & & & \end{array}$.

Consider now the four triangular blocks A_0, A_1, A_2, A_3 depicted in Figure 3. Taking the $*$ product of selected pairs, we define

$$B_i = A_i * A_{i+1} \text{ for } i = 0, 1, 2 \quad \text{and} \quad B_3 = A_3 * A_1. \tag{2}$$

Similarly, we set

$$C_i = B_i * B_{i+1} \text{ for } i = 0, 1, 2 \quad \text{and} \quad C_3 = B_3 * B_1. \tag{3}$$

Finally, we also need to set

$$D_0 = C_0 * C_1 \quad \text{and} \quad E_0 = D_0 * C_3. \tag{4}$$

The blocks A_i, B_i, C_i, D_0, E_0 $(i = 1, 2, 3)$ are displayed in Figure 3. In view of the following lemma, we shall refer to them as the *building blocks* of $\nabla S_1[8k]$.

Lemma 6. *For every integer $k \geq 0$, the Steinhaus triangle $\nabla S_1[8k]$ has the structure depicted in Figure 2, where A_i, B_i, C_i and D_0, E_0 are the blocks defined above.*

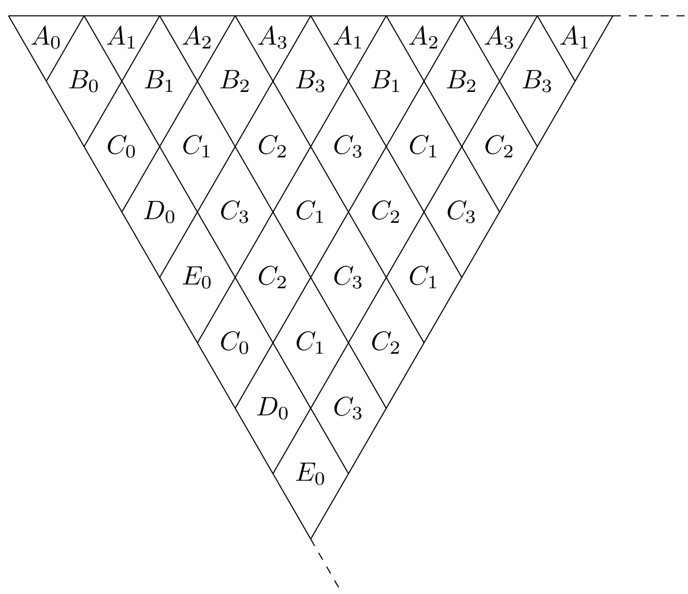

Figure 2: Structure of $\nabla S_1[8k]$

Proof. Recall that $S_1 = 01220232(212113220030232311200232)^\infty$. Thus, S_1 is made of an initial block $I = 01220232$ of length 8, and a period $P_1P_2P_3$ of length $24 = 3 \times 8$, where

$$P_1 = 21211322, \quad P_2 = 00302323, \quad P_3 = 11200232.$$

```
0 1 2 2 0 2 3 2
 1 3 0 2 2 1 1
  0 3 2 0 3 2
   3 1 2 3 1
    0 3 1 0
     3 0 1
      3 1
       0
```
A_0

```
2 1 2 1 1 3 2 2
 3 3 3 2 0 1 0
  2 2 1 2 1 1
   0 3 3 3 2
    3 2 2 1
     1 0 3
      1 3
       0
```
A_1

```
0 0 3 0 2 3 2 3
 0 3 3 2 1 1 1
  3 2 1 3 2 2
   1 3 0 1 0
    0 3 1 1
     3 0 2
      3 2
       1
```
A_2

```
1 1 2 0 0 2 3 2
 2 3 2 0 2 1 1
  1 1 2 2 3 2
   2 3 0 1 1
    1 3 1 2
     0 0 3
      0 3
       3
```
A_3

```
       0
      1 3
     3 0 1
    0 3 1 1
   0 3 0 2 0
  1 3 3 2 2 1
 2 0 2 1 0 3 2
2 2 2 3 1 3 1 2
 0 0 1 0 0 0 3
  0 1 1 0 0 3
   1 2 1 0 3
    3 3 1 3
     2 0 0
      2 0
       2
```
B_0

```
       2
      2 2
     3 0 1
    1 3 1 2
   2 0 0 3 2
  1 2 0 3 1 1
 0 3 2 3 0 2 0
0 3 1 1 3 2 2 1
 3 0 2 0 1 0 3
  3 2 2 1 1 3
   1 0 3 2 0
    1 3 1 2
     0 0 3
      0 3
       3
```
B_1

```
       0
      1 2
     3 3 3
    3 2 2 1
   0 1 0 3 2
  2 1 1 3 1 2
 0 3 2 0 0 3 2
1 3 1 2 0 3 1 1
 0 0 3 2 3 0 2
  0 3 1 1 3 2
   3 0 2 0 1
    3 2 2 1
     1 0 3
      1 3
       0
```
B_2

```
       0
      1 3
     3 0 1
    0 3 1 1
   2 3 0 2 0
  1 1 3 2 2 1
 0 2 0 1 0 3 2
3 2 2 1 1 3 1 2
 1 0 3 2 0 0 3
  1 3 1 2 0 3
   0 0 3 2 3
    0 3 1 1
     3 0 2
      3 2
       1
```
B_3

```
       2
      1 1
     0 2 0
    3 2 2 1
   2 1 0 3 2
  2 3 1 3 1 2
 2 1 0 0 0 3 2
0 3 1 0 0 3 1 1
 3 0 1 0 3 0 2
  3 1 1 3 3 2
   0 2 0 2 1
    2 2 2 3
     0 0 1
      0 1
       1
```
C_0

Figure 3: The building blocks of $\nabla S_1[8k]$

```
              2
            1   2
          0   3   2
        0   3   1   1
      2   3   0   2   0
    1   1   3   2   2   1
  0   2   0   1   0   3   2
3   2   2   1   1   3   1   2
  1   0   3   2   0   0   3
    1   3   1   2   0   3
      0   0   3   2   3
        0   3   1   1
          3   0   2
            3   2
              1
```

C_1

```
              0
            2   1
          0   3   2
        1   3   1   2
      2   0   0   3   2
    1   2   0   3   1   1
  0   3   2   3   0   2   0
0   3   1   1   3   2   2   1
  3   0   2   0   1   0   3
    3   2   2   1   1   3
      1   0   3   2   0
        1   3   1   2
          0   0   3
            0   3
              3
```

C_2

```
              2
            1   1
          0   2   0
        3   2   2   1
      0   1   0   3   2
    2   1   1   3   1   2
  0   3   2   0   0   3   2
1   3   1   2   0   3   1   1
  0   0   3   2   3   0   2
    0   3   1   1   3   2
      3   0   2   0   1
        3   2   2   1
          1   0   3
            1   3
              0
```

C_3

```
              0
            2   1
          0   3   2
        1   3   1   2
      0   0   0   3   2
    1   0   0   3   1   1
  2   1   0   3   0   2   0
3   3   1   3   3   2   2   1
  2   0   0   2   1   0   3
    2   0   2   3   1   3
      2   2   1   0   0
        0   3   1   0
          3   0   1
            3   1
              0
```

D_0

```
              2
            1   2
          0   3   2
        0   3   1   1
      0   3   0   2   0
    1   3   3   2   2   1
  2   0   2   1   0   3   2
2   2   2   3   1   3   1   2
  0   0   1   0   0   0   3
    0   1   1   0   0   3
      1   2   1   0   3
        3   3   1   3
          2   0   0
            2   0
              2
```

E_0

Figure 3: The building blocks of $\nabla S_1[8k]$ (continued)

Observe that $A_0 = \nabla I$ and $A_i = \nabla P_i$ for $i = 1, 2, 3$. This accounts for the top structure of $\nabla S_1[8k]$. Now, by definition we have

$$\begin{array}{llll} B_0 = A_0 * A_1, & B_1 = A_1 * A_2, & B_2 = A_2 * A_3, & B_3 = A_3 * A_1, \\ C_0 = B_0 * B_1, & C_1 = B_1 * B_2, & C_2 = B_2 * B_3, & C_3 = B_3 * B_1, \\ D_0 = C_0 * C_1, & E_0 = D_0 * C_3. & & \end{array}$$

It remains to show that $C_1 * C_2 = C_3$, $C_2 * C_3 = C_1$, $C_3 * C_1 = C_2$ and $E_0 * C_2 = C_0$. To do this, recall that the $*$ product of two blocks only depends on their lower sides, and observe on Figure 3 that

- the lower sides of C_1 coincide with those of B_3;
- the lower sides of C_2 coincide with those of B_1;
- the lower sides of C_3 coincide with those of B_2;
- the lower sides of E_0 coincide with those of B_0.

It follows that

$$\begin{array}{ll} C_1 * C_2 = B_3 * B_1 = C_3, & C_2 * C_3 = B_1 * B_2 = C_1, \\ C_3 * C_1 = B_2 * B_3 = C_2, & E_0 * C_2 = B_0 * B_1 = C_0, \end{array} \tag{5}$$

as desired. This completes the proof of the lemma. □

We are now in a position to prove our main result.

Proof of Theorem 4 for S_1. We shall prove, by induction on k, that the Steinhaus triangle $\nabla S_1[8k]$ is strongly balanced. This is true for $k = 0$. For $k \geq 1$, it suffices to show that $\nabla S_1[8k]$ is balanced. It will then automatically be strongly balanced since $\nabla S_1[8k - 8]$ is, by the induction hypothesis. Thus, we are assuming that $0, 1, 2, 3$ occur with the same multiplicity in $\nabla S_1[8k - 8]$, and we must show that this remains true in $\nabla S_1[8k]$.

Consider the multiset difference

$$T \;=\; \nabla S_1[8k] \setminus \nabla S_1[8k - 8],$$

a band of width 8 bordering the eastern side of $\nabla S_1[8k]$. To conclude the proof, we need only show that $0, 1, 2, 3$ occur with the same multiplicity in T. For any finite multiset X on $\mathbb{Z}/4\mathbb{Z}$, and for all $j \in \mathbb{Z}/4\mathbb{Z}$, let us denote by

$$\mathfrak{m}_X(j)$$

the occurrence multiplicity of j in X.

In order to determine the function $\mathfrak{m}_T$ on $\mathbb{Z}/4\mathbb{Z}$, we need to determine $\mathfrak{m}_X$ for the building blocks $X = A_i, B_i, C_i, D_0, E_0$. This is done in Table 1. Let now C

	A_0	A_1	A_2	A_3	B_0	B_1	B_2	B_3	C_0	C_1	C_2	C_3	D_0	E_0
0	9	5	8	7	20	16	15	16	17	15	17	16	20	19
1	9	10	9	10	15	15	16	17	17	16	15	17	15	14
2	9	11	8	11	14	16	15	14	17	17	15	16	14	17
3	9	10	11	8	15	17	18	17	13	16	17	15	15	14

Table 1: Multiplicities of $0, 1, 2, 3$ in each building block of $\nabla S_1[8k]$

denote the multiset union of C_1, C_2, C_3. That is, by definition we have

$$\mathfrak{m}_C(j) = \mathfrak{m}_{C_1}(j) + \mathfrak{m}_{C_2}(j) + \mathfrak{m}_{C_3}(j)$$

for all $j \in \mathbb{Z}/4\mathbb{Z}$. Looking at the columns below C_1, C_2, C_3 in Table 1, we see that

$$\mathfrak{m}_C(j) = 15 + 16 + 17 = 48 \tag{6}$$

for all $j \in \mathbb{Z}/4\mathbb{Z}$. We are now ready to show that $\mathfrak{m}_T$ is constant on $\mathbb{Z}/4\mathbb{Z}$. For this, we need to distinguish 3 cases, according to the class of k mod 3.

`Case 1:` $k = 3q$. Figure 2 shows that the building blocks making up T are A_2, B_1 and C_0 occurring once each, and C_1, C_2, C_3 occurring $q-1$ times each. Therefore, using Table 1 and (6), we get

$$\begin{aligned} \mathfrak{m}_T(j) &= \mathfrak{m}_{A_2}(j) + \mathfrak{m}_{B_1}(j) + \mathfrak{m}_{C_0}(j) + (q-1)\mathfrak{m}_C(j) \\ &= 41 + 48(q-1), \end{aligned}$$

for all $j \in \mathbb{Z}/4\mathbb{Z}$.

`Case 2:` $k = 3q + 1$. In this case, the building blocks making up T are A_3, B_2, C_1 and D_0 occurring once each, and C_1, C_2, C_3 occurring $q-1$ times each. Thus

$$\begin{aligned} \mathfrak{m}_T(j) &= \mathfrak{m}_{A_3}(j) + \mathfrak{m}_{B_2}(j) + \mathfrak{m}_{C_1}(j) + \mathfrak{m}_{D_0}(j) + (q-1)\mathfrak{m}_C(j) \\ &= 57 + 48(q-1), \end{aligned}$$

for all $j \in \mathbb{Z}/4\mathbb{Z}$.

`Case 3:` $k = 3q + 2$. Now, the building blocks making up T are A_1, B_3, C_2, C_3 and E_0 occurring once each, and C_1, C_2, C_3 occurring $q-1$ times each. Thus

$$\begin{aligned} \mathfrak{m}_T(j) &= \mathfrak{m}_{A_1}(j) + \mathfrak{m}_{B_3}(j) + \mathfrak{m}_{C_2}(j) + \mathfrak{m}_{C_3}(j) + \mathfrak{m}_{E_0}(j) + (q-1)\mathfrak{m}_C(j) \\ &= 73 + 48(q-1), \end{aligned}$$

for all $j \in \mathbb{Z}/4\mathbb{Z}$.

Hence $\mathfrak{m}_T$ is constant on $\mathbb{Z}/4\mathbb{Z}$ in each case, as desired. This completes the proof of Theorem 4 for the sequence S_1.

The proof for the sequences S_2, T_1, T_2, T_3, T_4 follows similar lines. To start with, the structure of each derived triangle is the same as in Figure 2. Indeed, let A_0, A_1, A_2, A_3 be the triangles constructed from the finite subsequences given in Table 2 for each sequence S_2, T_1, T_2, T_3, T_4. Let now $B_0, B_1, B_2, B_3, C_0, C_1, C_2, C_3, D_0$, E_0 be the blocks defined by the same formulae (2), (3) and (4) as in the proof for S_1. Then, as easily verified, the equalities (5) still hold. Finally, as for S_1, the conclusion of the proof follows from the knowledge of the multiplicities of $0, 1, 2, 3 \in \mathbb{Z}/4\mathbb{Z}$ for each block. For convenience, these multiplicities are made explicit in Table 3.

	A_0	A_1	A_2	A_3
S_2	$\nabla(21210130)$	$\nabla(20013202)$	$\nabla(21120021)$	$\nabla(10220130)$
T_1	$\nabla(0120021)$	$\nabla(21220210)$	$\nabla(20230322)$	$\nabla(00322021)$
T_2	$\nabla(1000212)$	$\nabla(31222330)$	$\nabla(12103120)$	$\nabla(03103232)$
T_3	$\nabla(1200210)$	$\nabla(22010122)$	$\nabla(20322221)$	$\nabla(03000210)$
T_4	$\nabla(2102203)$	$\nabla(23200210)$	$\nabla(20212300)$	$\nabla(22302203)$

Table 2: Definition of blocks A_0, A_1, A_2, A_3 for the sequences S_2, T_1, T_2, T_3, T_4.

4. The Construction Method

We now explain how our solution was constructed. Let $m_1, m_2 \geq 2$ be integers, with m_2 a multiple of m_1. Consider the canonical quotient map

$$\pi : \mathbb{Z}/m_2\mathbb{Z} \longrightarrow \mathbb{Z}/m_1\mathbb{Z}.$$

If ∇ is a Steinhaus triangle in $\mathbb{Z}/m_2\mathbb{Z}$, then $\pi(\nabla)$ is a Steinhaus triangle in $\mathbb{Z}/m_1\mathbb{Z}$. Moreover, if ∇ is balanced, then so is $\pi(\nabla)$, as all fibers of π have the same cardinality. Thus, an obvious strategy for constructing balanced Steinhaus triangles in $\mathbb{Z}/m_2\mathbb{Z}$ consists in trying to lift to $\mathbb{Z}/m_2\mathbb{Z}$ known balanced Steinhaus triangles in $\mathbb{Z}/m_1\mathbb{Z}$. This route is tricky, as illustrated by Theorems 10 and 11 below. It allowed us to solve the case $m = 4$ of Molluzzo's problem, but neither the case $m = 6$ nor the case $m = 8$ so far.

We shall restrict our attention to *strongly balanced* Steinhaus triangles. These were defined earlier in $\mathbb{Z}/4\mathbb{Z}$ only. We now generalize them to $\mathbb{Z}/m\mathbb{Z}$ for all even moduli.

Definition 7. Let $m \geq 2$ be an even modulus. Let S be a finite sequence of length $n \geq 0$ in $\mathbb{Z}/m\mathbb{Z}$. The Steinhaus triangle ∇S is said to be *strongly balanced* if, for every $0 \leq t \leq n/(2m)$, the Steinhaus triangle $\nabla S[n - 2mt]$ is balanced.

S_2 :

	A_0	A_1	A_2	A_3	B_0	B_1	B_2	B_3	C_0	C_1	C_2	C_3	D_0	E_0
0	9	8	6	10	17	19	14	15	16	13	20	15	20	15
1	9	9	10	11	16	11	19	20	20	18	10	20	9	14
2	9	12	12	8	13	13	18	17	16	19	12	17	12	15
3	9	7	8	7	18	21	13	12	12	14	22	12	23	20

T_1 :

	A_0	A_1	A_2	A_3	B_0	B_1	B_2	B_3	C_0	C_1	C_2	C_3	D_0	E_0
0	7	5	10	11	18	17	12	20	12	19	17	12	13	17
1	7	10	6	10	13	13	18	12	20	14	15	19	13	15
2	7	13	10	8	10	15	20	13	14	13	16	19	14	10
3	7	8	10	7	15	19	14	19	10	18	16	14	16	14

T_2 :

	A_0	A_1	A_2	A_3	B_0	B_1	B_2	B_3	C_0	C_1	C_2	C_3	D_0	E_0
0	7	10	9	10	13	18	16	15	12	15	18	15	14	13
1	7	10	8	10	13	18	16	16	13	15	17	16	14	12
2	7	7	8	8	16	15	16	16	16	16	15	17	15	16
3	7	9	11	8	14	13	16	17	15	18	14	16	13	15

T_3 :

	A_0	A_1	A_2	A_3	B_0	B_1	B_2	B_3	C_0	C_1	C_2	C_3	D_0	E_0
0	7	7	10	11	16	15	14	18	14	19	15	14	11	17
1	7	9	8	9	14	16	15	16	15	17	17	14	14	15
2	7	13	9	7	10	17	19	14	13	14	16	18	15	10
3	7	7	9	9	16	16	16	16	14	14	16	18	16	14

T_4 :

	A_0	A_1	A_2	A_3	B_0	B_1	B_2	B_3	C_0	C_1	C_2	C_3	D_0	E_0
0	7	8	12	7	15	13	19	18	14	16	15	17	13	13
1	7	9	7	8	14	17	19	13	15	15	14	19	13	16
2	7	10	8	12	13	19	13	15	12	16	18	14	14	14
3	7	9	9	9	14	15	13	18	15	17	17	14	16	13

Table 3: Multiplicities of $0, 1, 2, 3$ in each building block of $\nabla S_2[8k]$ and $\nabla T_i[8k+7]$ for $i \in \{1, 2, 3, 4\}$.

Note that this definition coincides with Definition 3 for $m = 4$. From now on, we assume that $m_1 = m$ is an even number, and that $m_2 = 2m_1$. The following notation helps to measure, roughly speaking, to what extent strong solutions in $\mathbb{Z}/m\mathbb{Z}$ can be lifted to strong solutions in $\mathbb{Z}/2m\mathbb{Z}$.

Notation 8. Let S be an infinite sequence in $\mathbb{Z}/m\mathbb{Z}$. For $n \geq 0$, let $a_n(S)$ denote the number of sequences T in $\mathbb{Z}/2m\mathbb{Z}$, of length n, such that

- ∇T is a strongly balanced Steinhaus triangle in $\mathbb{Z}/2m\mathbb{Z}$;
- $\pi(T) = S[n]$, the initial segment of length n in S.

We denote by $\mathcal{G}_S(t) = \sum_{n=0}^{\infty} a_n(S)t^n$ the generating function of the numbers $a_n(S)$.

We shall use this notation as a convenient device for exhibiting the value of the $a_n(S)$ for all n at once. For our present purposes, the favorable case occurs when $\mathcal{G}_S(t)$ is an infinite series, not just a polynomial. Indeed, $\mathcal{G}_S(t)$ is an infinite series if and only if there exists infinitely many strongly balanced Steinhaus triangles in $\mathbb{Z}/2m\mathbb{Z}$, which lift those in $\mathbb{Z}/m\mathbb{Z}$ generated by initial segments of S.

4.1. From $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}/4\mathbb{Z}$

Here we set $m = 2$. Several types of balanced Steinhaus triangles of length $4k$ or $4k + 3$ in $\mathbb{Z}/2\mathbb{Z}$ are known. See [7, 5, 6]. We focus here on the ones given in [5], which have the added property of being strongly balanced.

Theorem 9 ([5])**.** *Let $Q_1, \dots, Q_4$ and $R_1, \dots, R_{12}$ be the following eventually periodic sequences of $\mathbb{Z}/2\mathbb{Z}$:*

$Q_1 = 0100(001001011100)^\infty$,
$Q_2 = (010010000111)^\infty$,
$Q_3 = 0101(011000011000)^\infty$,
$Q_4 = 0101(101000101000)^\infty$,

$R_1 = 001(010000100001)^\infty$,
$R_2 = 0011110(001101010110)^\infty$,
$R_3 = 010(000101000010)^\infty$,
$R_4 = 0100001(010010111100001010111111)^\infty$,
$R_5 = 0100001(100100001001)^\infty$,
$R_6 = 0101011(010101100011)^\infty$,
$R_7 = 0101011(010111111010110100111101)^\infty$,
$R_8 = 010(101110110010)^\infty$,
$R_9 = 100(001000010100)^\infty$,
$R_{10} = 1000010(110001101010)^\infty$,
$R_{11} = 1111101(011000110101)^\infty$,
$R_{12} = 111(110110000111)^\infty$.

For all integers i, j, k such that $1 \leq i \leq 4$, $1 \leq j \leq 12$ and $k \geq 0$, the Steinhaus triangles $\nabla Q_i[4k]$ and $\nabla R_j[4k+3]$ are strongly balanced in $\mathbb{Z}/2\mathbb{Z}$.

Can we lift some initial segments of these sequences to sequences in $\mathbb{Z}/4\mathbb{Z}$ which generate strongly balanced Steinhaus triangles? To answer this question, we have determined by computer the numbers $a_n(S)$ for all 16 sequences S in Theorem 9 and all $n \geq 1$. In 11 out of the 16 cases, the numbers $a_n(S)$ turn out to vanish for all sufficiently large n, i.e., the series $\mathcal{G}_S(t)$ is just a polynomial. But remarkably, in the remaining 5 cases, the $a_n(S)$ turn out to be *ultimately periodic* and non-vanishing, so that the infinite series $\mathcal{G}_S(t)$ is actually a *rational function*. These 16 series are displayed below; the 5 infinite ones occur for the sequences $Q_1, Q_3, R_3, R_9, R_{10}$.

Theorem 10. *The generating functions $\mathcal{G}_S(t)$ of $Q_1, \ldots, Q_4$ and $R_1, \ldots, R_{12}$ are:*

$$\begin{aligned}
\mathcal{G}_{Q_1}(t) = {} & 1 + 8t^8 + 34t^{16} + 58t^{24} + 84t^{32} + 88t^{40} + 86t^{48} + 82t^{56} + 60t^{64} + 36t^{72} \\
& + + 34t^{80} + 28t^{88} + 16t^{96} + \frac{2t^{104}}{1-t^8},
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_{Q_2}(t) = {} & 1 + 4t^8 + 14t^{16} + 32t^{24} + 36t^{32} + 48t^{40} + 44t^{48} + 26t^{56} + 22t^{64} + 8t^{72} \\
& + 6t^{80} + 4t^{88} + 2t^{96},
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_{Q_3}(t) = {} & 1 + 8t^8 + 28t^{16} + 46t^{24} + 78t^{32} + 124t^{40} + 118t^{48} + 96t^{56} + 78t^{64} + 60t^{72} \\
& + 20t^{88} + 14t^{96} + 10t^{104} + 4t^{112} + 6t^{120} + 4t^{128} + 6t^{136} + 4t^{144} + 2t^{152} \\
& + 2t^{160} + 2t^{168} + 2t^{176} + 2t^{184} + 2t^{192} + 2t^{200} + 4t^{208} + \frac{2t^{216}}{1-t^8},
\end{aligned}$$

$$\mathcal{G}_{Q_4}(t) = 1 + 8t^8 + 26t^{16} + 42t^{24} + 66t^{32} + 62t^{40} + 52t^{48} + 36t^{56} + 26t^{64} + 12t^{72} + 6t^{80},$$

$$\mathcal{G}_{R_1}(t) = 0, \quad \mathcal{G}_{R_2}(t) = 0,$$

$$\begin{aligned}
\mathcal{G}_{R_3}(t) = {} & 10t^7 + 38t^{15} + 70t^{23} + 88t^{31} + 76t^{39} + 54t^{47} + 44t^{55} + 28t^{63} + 16t^{71} + 8t^{79} \\
& + 4t^{87} + 4t^{95} + 4t^{103} + 4t^{111} + 4t^{119} + 6t^{127} + 4t^{135} + 6t^{143} + \frac{4t^{151}}{1-t^8},
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_{R_4}(t) = {} & 10t^7 + 52t^{15} + 102t^{23} + 136t^{31} + 152t^{39} + 118t^{47} + 108t^{55} + 80t^{63} \\
& + 60t^{71} + 32t^{79} + 20t^{87} + 8t^{95} + 2t^{103},
\end{aligned}$$

$$\mathcal{G}_{R_5}(t) = 10t^7,$$

$$\begin{aligned}
\mathcal{G}_{R_6}(t) = {} & 10t^7 + 30t^{15} + 66t^{23} + 96t^{31} + 96t^{39} + 94t^{47} + 66t^{55} + 42t^{63} + 24t^{71} \\
& + 8t^{79} + 2t^{87} + 2t^{95},
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_{R_7}(t) = {} & 10t^7 + 60t^{15} + 138t^{23} + 204t^{31} + 304t^{39} + 266t^{47} + 246t^{55} + 148t^{63} + \\
& + 64t^{71} + 36t^{79} + 14t^{87} + 10t^{95} + 8t^{103},
\end{aligned}$$

$$\mathcal{G}_{R_8}(t) = 10t^7,$$

$$\begin{aligned}
\mathcal{G}_{R_9}(t) = {} & 10t^7 + 42t^{15} + 80t^{23} + 130t^{31} + 164t^{39} + 174t^{47} + 126t^{55} + 68t^{63} + 38t^{71} \\
& + 20t^{79} + 22t^{87} + 12t^{95} + 2t^{103} + 2t^{111} + 2t^{119} + 2t^{127} + 2t^{135} + 2t^{143} \\
& + 2t^{151} + 2t^{159} + 2t^{167} + 2t^{175} + 2t^{183} + 2t^{191} + 2t^{199} + 4t^{207} + \frac{2t^{215}}{1-t^8},
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_{R_{10}}(t) = {} & 10t^7 + 58t^{15} + 98t^{23} + 130t^{31} + 160t^{39} + 138t^{47} + 132t^{55} + 84t^{63} \\
& + 64t^{71} + 34t^{79} + 14t^{87} + 8t^{95} + 6t^{103} + 2t^{111} + 2t^{119} + 4t^{127} + \frac{2t^{135}}{1-t^8},
\end{aligned}$$

$$\mathcal{G}_{R_{11}}(t) = 4t^7 + 16t^{15} + 26t^{23} + 32t^{31} + 30t^{39} + 30t^{47} + 26t^{55} + 12t^{63} + 8t^{71} + 2t^{79},$$

$$\mathcal{G}_{R_{12}}(t) = 4t^7.$$

The origin of our sequences S_1, S_2, T_1, T_2, T_3, T_4, solving the problem of Molluzzo in $\mathbb{Z}/4\mathbb{Z}$, is now clear. Indeed, they are lifts to $\mathbb{Z}/4\mathbb{Z}$ of the 5 sequences $Q_1, Q_3, R_3, R_9, R_{10}$ in $\mathbb{Z}/2\mathbb{Z}$ with $\mathcal{G}_S(t)$ infinite. More precisely, we have

$$\pi(S_1) = Q_1, \;\; \pi(S_2) = Q_3, \;\; \pi(T_1) = \pi(T_4) = R_3, \;\; \pi(T_2) = R_{10}, \;\; \pi(T_3) = R_9,$$

as the reader may readily check.

4.2. From $\mathbb{Z}/4\mathbb{Z}$ to $\mathbb{Z}/8\mathbb{Z}$

Having solved the problem in $\mathbb{Z}/4\mathbb{Z}$ with Theorem 2, can we lift our solutions S_1, S_2, T_1, T_2, T_3, T_4 to sequences in $\mathbb{Z}/8\mathbb{Z}$ giving rise to infinitely many strongly balanced Steinhaus triangles? Unfortunately, the answer is no, as shown by the following computational result.

Theorem 11. *The generating functions $\mathcal{G}_S(t)$ of $S_1, S_2, T_1, T_2, T_3, T_4$ are polynomials only:*

$$\mathcal{G}_{S_1}(t) = 1 + 16t^{16} + 46t^{32} + 32t^{48} + 14t^{64},$$

$$\mathcal{G}_{S_2}(t) = 1 + 22t^{16} + 60t^{32} + 56t^{48} + 28t^{64} + 6t^{80},$$

$$\mathcal{G}_{T_1}(t) = 14t^{15} + 40t^{31} + 40t^{47} + 24t^{63} + 8t^{79} + 2t^{95} + 2t^{111},$$

$$\mathcal{G}_{T_2}(t) = 30t^{15} + 66t^{31} + 76t^{47} + 32t^{63} + 12t^{79},$$

$$\mathcal{G}_{T_3}(t) = 14t^{15} + 54t^{31} + 42t^{47} + 34t^{63} + 12t^{79} + 2t^{95},$$

$$\mathcal{G}_{T_4}(t) = 14t^{15} + 54t^{31} + 64t^{47} + 40t^{63} + 10t^{79} + 2t^{95}.$$

Summarizing, at this stage, it is not even known whether there exist infinitely many balanced Steinhaus triangles in $\mathbb{Z}/8\mathbb{Z}$.

5. A Weaker Version of the Problem

With our present solution of the case $m = 4$, the currently smallest open case of Molluzzo's problem becomes $m = 6$. The scarcity of solved instances ($m = 2, 3^k, 4, 5, 7$) motivates us to propose a weaker, more accessible version of the problem.

Problem. (The Weak Molluzzo Problem) Let $m \in \mathbb{N}$, $m \geq 2$. Are there infinitely many balanced Steinhaus triangles in $\mathbb{Z}/m\mathbb{Z}$?

The picture is brighter here. Indeed, the first author has shown in [1, 2, 3] that, *for each odd modulus m, there are infinitely many balanced Steinhaus triangles in $\mathbb{Z}/m\mathbb{Z}$.* Thus, the weak Molluzzo problem is affirmatively solved for all odd m, for

$m = 2$ and here for $m = 4$. On the other hand, it is widely open for all even moduli $m \geq 6$.

Somewhat similarly to the conjecture on the existence of Hadamard matrices of every order divisible by 4, the nature of the problem seems to lie less in the rarity of the solutions than in the difficulty of pinpointing easy-to-describe ones. For instance, in $\mathbb{Z}/6\mathbb{Z}$, there are exactly 94648 sequences of length 12 yielding a balanced Steinhaus triangle; up to automorphisms, they still total 23662 classes.

We note, finally, that all known solutions so far are by explicit constructions. However, the possibility of future nonconstructive existence results cannot be ruled out, for instance with the polynomial method of Alon-Tarsi.

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