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THE $(r_1, \ldots, r_p)$-STIRLING NUMBERS OF THE SECOND KIND

Miloud Mihoubi[1]
Faculty of Mathematics, University of Science and Technology Houari Boumediene, Algiers, Algeria.
mmihoubi@usthb.dz or miloudmihoubi@gmail.com

Mohammed Said Maamra[1]
Faculty of Mathematics, University of Science and Technology Houari Boumediene, Algiers, Algeria.
mmaamra@usthb.dz or mmaamra@yahoo.fr

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Abstract

Let $R_1, \ldots, R_p$ be subsets of the set $[n] = \{1, \ldots, n\}$ with $|R_i| = r_i$ and $R_i \cap R_j = \varnothing$ for all $i, j = 1, \ldots, p$, $i \neq j$. The $(r_1, \ldots, r_p)$-Stirling number of the second kind, $p \geq 1$, introduced in this paper and denoted by $\left\{ {n \atop k} \right\}_{r_1,\ldots,r_p}$, counts the number of partitions of the set $[n]$ into k classes (or blocks) such that the elements in each R_i, $i = 1, \ldots, p$, are in different classes (or blocks). Combinatorial and algebraic properties of these numbers are explored.

1. Introduction

The $(r_1, \ldots, r_p)$-Stirling numbers of the second kind represent a certain generalization of the Stirling and r-Stirling numbers of the second kind. The Stirling number of the second kind, denoted by $\left\{ {n \atop k} \right\}$, counts the number of partitions of the set $[n]$ into k non-empty disjoint subsets. An excellent introduction to these numbers can be found in [8]. The r-Stirling number of the second kind, denoted by $\left\{ {n \atop k} \right\}_r$, counts the number of partitions of the set $[n]$ into k non-empty disjoint subsets such that the numbers $1, 2, \ldots, r$ are in different subsets. Combinatorial interpretations and algebraic properties of these numbers can be found in [3]. Several authors studied these numbers and their role in probability, approximations, congruences and other frameworks. For example, Chrysaphinou [5] studied Touchard polynomials and their connections with the r-Stirling numbers and other numbers, Hsu et al. [9] studied the properties and approximations for a family of Stirling numbers, Mező

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[11, 10] studied the r-Bell numbers and the maximum of the r-Stirling numbers and Mihoubi et al. [12] gave some properties with respect to the r-Stirling numbers. One can see the references [4, 6, 13] for more applications and results on these numbers. Our generalization leads us to study an extension of the r-Stirling numbers, in which we may establish

- a generalization of the Dobiński and Stirling formulas,
- a combinatorial interpretation of the coefficient of $z^{\underline{k}},\ k = 0, 1, 2, \ldots,$ in the polynomial $(z + r_p)^n\,(z + r_p)^{\underline{r_1}} \cdots (z + r_p)^{\underline{r_{p-1}}}$, where $z^{\underline{n}} = z\,(z-1)\cdots(z-n+1)\,,\ n \geq 1$, and $z^{\underline{0}} = 1$,
- inequalities generalize those given by Bouroubi [2] on the single variable Bell polynomials,
- some properties for the $(r_1, \ldots, r_p)$-Stirling numbers of the second kind.

The $(r_1, \ldots, r_p)$-Stirling numbers of the second kind are defined as follows:

Definition 1. Let $R_1, \ldots, R_p$ be subsets of the set $[n]$ with $|R_i| = r_i$ and $R_i \cap R_j = \varnothing$ for all $i, j = 1, \ldots, p$, $i \neq j$. The $(r_1, \ldots, r_p)$-Stirling number of the second kind, $p \geq 1$, denoted by $\left\{ {n \atop k} \right\}_{r_1,\ldots,r_p}$, counts the number of partitions of the set $[n]$ into k non-empty subsets such that the elements of each of the p sets

$$R_1 := [r_1]\,,\ R_2 := [r_1 + r_2] \setminus [r_1]\,, \ldots,\ R_p := [r_1 + \cdots + r_p] \setminus [r_1 + \cdots + r_{p-1}]$$

are in distinct subsets.

From this definition, one can verify easily that the $(r_1, \ldots, r_p)$-Stirling numbers of the second kind satisfy

$$\begin{aligned}
&\left\{ {n \atop k} \right\}_{r_1,\ldots,r_p} = 0, \qquad n < r_1 + \cdots + r_p \text{ or } k < \max(r_1, \ldots, r_p)\,,\\
&\left\{ {n \atop k} \right\}_{r_1,\ldots,r_p} = \left\{ {n \atop k} \right\} \quad \text{if } r_1, \ldots, r_p \in \{0, 1\}\,,\\
&\left\{ {n \atop k} \right\}_{r_1,\ldots,r_p} = \left\{ {n \atop k} \right\}_{r_p} \quad \text{if } r_1, \ldots, r_{p-1} \in \{0, 1\}\,,\\
&\left\{ {n \atop k} \right\}_{r_1,\ldots,r_p} = \left\{ {n \atop k} \right\}_{r_1,\ldots,r_p,0} = \left\{ {n \atop k} \right\}_{r_1,\ldots,r_p,1},\\
&\left\{ {n \atop k} \right\}_{r_1,\ldots,r_p} = \left\{ {n \atop k} \right\}_{r_{\sigma(1)},\ldots,r_{\sigma(p)}} \quad \text{for all permutations } \sigma \text{ on the set } \{1, \ldots, p\}\,.
\end{aligned}$$

Therefore, by the symmetry of the $(r_1, \ldots, r_p)$-Stirling numbers to respect to $r_1, \ldots,$ r_p, we can suppose $r_1 \leq r_2 \leq \cdots \leq r_p$ and throughout this paper we use the notations

$$D^n_{z=z_0} := \frac{d^n}{dz^n}|_{z=z_0}, \quad \mathbf{r}_p := (r_1, \dots, r_p), \quad |\mathbf{r}_p| := r_1 + \cdots + r_p, \text{ and}$$
$$P_t(z; \mathbf{r}_p) := (z + r_p)^t (z + r_p)^{\underline{r_1}} \cdots (z + r_p)^{\underline{r_{p-1}}}, \quad t \in \mathbb{R}.$$

2. The $\mathbf{r}_p$-Stirling Numbers of the Second Kind

2.1. Combinatorial Recurrence Relations

Broder [3] introduced the r-Stirling numbers of the second kind and showed that these numbers satisfy

$$\begin{aligned}
&\left\{ n \atop k \right\}_r = 0, \quad n < r, \\
&\left\{ r \atop k \right\}_r = \delta_{r,k}, \\
&\left\{ n \atop k \right\}_r = \left\{ n \atop k \right\}_{r-1} - (r-1) \left\{ n-1 \atop k \right\}_{r-1}, \quad n \geq r \geq 1, \\
&\left\{ n \atop k \right\}_r = k \left\{ n-1 \atop k \right\}_r + \left\{ n-1 \atop k-1 \right\}_r, \quad n > r.
\end{aligned}$$

The $\mathbf{r}_p$-Stirling numbers of the second kind satisfy recurrence relations similar to those of the r-Stirling and the regular Stirling numbers of the second kind with modified initial conditions; see Theorems 4 and 5 given below. To start, we give a theorem in which we express the r-Stirling numbers in terms of the Stirling numbers.

Theorem 2. *We have*

$$\left\{ n \atop k \right\}_r = \frac{1}{(k-r)!} \sum_{i=0}^{r} \binom{r}{i} \left\{ n-r \atop k-i \right\} (k-i)!.$$

Proof. For $i = 0, \dots, r$, there are $\binom{r}{i}$ ways to form i singletons using the elements in $\{1, \dots, r\}$ and $\left\{ n-r \atop k-i \right\}$ ways to partition the set $\{r+1, \dots, n\}$ into $k-i$ subsets. The $r-i$ elements of the set $\{1, \dots, r\}$ not already used can be inserted in the $k-i$ subsets in $(k-i) \cdots ((k-i) - (r-i) + 1) = \frac{(k-i)!}{(k-r)!}$ ways. Then, the number of partitions of the set $\{1, \dots, n\}$ into k subsets such that the elements of the set $\{1, \dots, r\}$ are in different subsets is $\left\{ n \atop k \right\}_r = \sum_{i=0}^{r} \binom{r}{i} \left\{ n-r \atop k-i \right\} \frac{(k-i)!}{(k-r)!}, \quad n \geq r.$ ☐

Theorem 2 can be translated to the $\mathbf{r}_p$-case as follows:

Theorem 3. *Let $\mathbf{r}_{p,\alpha} = (r_1, \ldots, r_{\alpha-1}, r_{\alpha+1}, \ldots, r_p)$, $1 \le \alpha \le p$. The $\mathbf{r}_p$-Stirling numbers of the second kind satisfy*

$$\left\{ {n \atop k} \right\}_{\mathbf{r}_p} = \frac{1}{(k - r_\alpha)!} \sum_{j=0}^{r_\alpha} \binom{r_\alpha}{j} \left\{ {n - r_\alpha \atop k - j} \right\}_{\mathbf{r}_{p,\alpha}} (k-j)!.$$

Proof. By the symmetry of $r_1, \ldots, r_p$, we consider only the case $\alpha = p$. For $i = 0, \ldots, r_p$, there are $\binom{r_p}{i}$ ways to form i singletons using the elements in R_p and $\left\{ {n \atop k} \right\}_{\mathbf{r}_{p-1}}$ ways to partition the set $[n] \setminus R_p$ into $k-i$ subsets such that the elements of each R_i, $i = 1, \ldots, p-1$, are in different subsets. The $r_p - i$ elements of the set R_p not already used can be inserted in the $k - i$ subsets in $\frac{(k-i)!}{(k-r_p)!}$ ways. Then, the number of partitions of the set $[n]$ into k subsets such that the elements in each R_i, $i = 1, \ldots, p$, are in different subsets is $\left\{ {n \atop k} \right\}_{\mathbf{r}_p} = \sum_{j=0}^{r_\alpha} \binom{r_\alpha}{j} \left\{ {n-r_\alpha \atop k-j} \right\}_{\mathbf{r}_{p,\alpha}} \frac{(k-j)!}{(k-r_\alpha)!}$. □

Theorem 4. *The $\mathbf{r}_p$-Stirling numbers of the second kind satisfy*

$$\left\{ {n \atop k} \right\}_{\mathbf{r}_p} = k \left\{ {n-1 \atop k} \right\}_{\mathbf{r}_p} + \left\{ {n-1 \atop k-1} \right\}_{\mathbf{r}_p}, \quad n > |\mathbf{r}_p|.$$

Proof. To form a partition of the set $[n]$ into k non-empty subsets, we can form a partition of the set $[n-1]$ into k non-empty subsets by adding the element n to any of the k subsets, or we form a partition of the set $[n-1]$ into $k-1$ non-empty subsets by adding the subset $\{n\}$. Obviously, for $n > |\mathbf{r}_p|$, the distribution of the elements of the sets R_i, $i = 1, \ldots, p$, into different subsets is not influenced by this process. □

Theorem 5. *Let $\mathbf{e}_i$ be the i^{th} vector of the canonical basis of $\mathbb{R}^p$. Then, for all $i = 1, \ldots, p$, the $\mathbf{r}_p$-Stirling numbers of the second kind satisfy*

$$\left\{ {n \atop k} \right\}_{\mathbf{r}_p} = \left\{ {n \atop k} \right\}_{\mathbf{r}_p - \mathbf{e}_i} - (r_i - 1) \left\{ {n-1 \atop k} \right\}_{\mathbf{r}_p - \mathbf{e}_i}, \quad n \ge |\mathbf{r}_p|, \ r_1 \cdots r_p \ge 1.$$

Proof. For all $i = 1, \ldots, p$, the identity of the theorem can be written as

$$(r_i - 1) \left\{ {n-1 \atop k} \right\}_{\mathbf{r}_p - \mathbf{e}_i} = \left\{ {n \atop k} \right\}_{\mathbf{r}_p - \mathbf{e}_i} - \left\{ {n \atop k} \right\}_{\mathbf{r}_p}, \quad n \ge |\mathbf{r}_p|, \ r_1 \cdots r_p \ge 1.$$

The number $\left\{ {n \atop k} \right\}_{\mathbf{r}_p - \mathbf{e}_i} - \left\{ {n \atop k} \right\}_{\mathbf{r}_p}$ counts the number of partitions of $[n]$ into k non-empty subsets such that the sets $R_1, \cdots, R_{i-1}, R_i \setminus \{|\mathbf{r}_i|\}, R_{i+1}, \ldots, R_p$ are in different subsets but $|\mathbf{r}_i|$ is not. But this number is equal to $(r_i - 1) \left\{ {n-1 \atop k} \right\}_{\mathbf{r}_p - \mathbf{e}_i}$ because such partitions can be obtained in $r_i - 1$ ways from partitions of $[n] \setminus \{|\mathbf{r}_i|\}$ into k non-empty subsets such that the above sets are in different subsets by including $|\mathbf{r}_i|$ in any of the $|\mathbf{r}_i| - 1$ subsets of the subsets containing the elements of the set $R_i \setminus \{|\mathbf{r}_i|\}$. □

2.2. Generating Functions for the $\mathbf{r}_p$-Stirling Numbers

Broder [3] showed that the exponential generating function of the r-Stirling numbers is given by

$$\sum_{n\geq 0}\left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r \frac{t^n}{n!} = \frac{1}{k!}\left(e^t-1\right)^k \exp\left(rt\right)$$

and these numbers satisfy

$$\left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}\left(-1\right)^{k-j}\left(j+r\right)^n,$$

After that, Mező [10, 11] defined the r-Bell polynomial by

$$B_n\left(z;r\right) = \sum_{k=0}^{n}\left\{ \begin{matrix} n+r \\ k+r \end{matrix} \right\}_r z^k$$

and showed that it can be written as

$$B_n\left(z;r\right) = \exp\left(-z\right)\sum_{i=0}^{\infty}\frac{\left(i+r\right)^n}{i!}z^i.$$

These two last identities represent, respectively, extensions of the known Stirling and Dobiński formulas and can be written in the $\mathbf{r}_p$-case as it is showed in Theorem 8 given below. Now, to give the exponential generating function of the $\mathbf{r}_p$-Stirling numbers, let us give it for $p=2$.

Theorem 6. *For $r \leq s$ we have*

$$\sum_{n\geq 0}\left\{ \begin{matrix} n+r+s \\ k+s \end{matrix} \right\}_{r,s} \frac{t^n}{n!} = \frac{1}{k!}\exp\left(rt\right) D^r_{x=\exp(t)-1}\left(x^k\left(x+1\right)^s\right).$$

Proof. For $k \leq r$ we have

$$\begin{aligned}
\sum_{n\geq 0}\left\{ \begin{matrix} n+r+s \\ k+s \end{matrix} \right\}_{r,s} \frac{t^n}{n!} &= \sum_{n\geq 0}\frac{1}{k!}\sum_{j=0}^{s}\binom{s}{j}\left(k+s-j\right)!\left\{ \begin{matrix} n+r \\ k+s-j \end{matrix} \right\}_r \frac{t^n}{n!} \\
&= \sum_{n\geq 0}\frac{1}{k!}\sum_{j=r-k}^{s}\binom{s}{j}\left(k+j\right)!\left\{ \begin{matrix} n+r \\ k+j \end{matrix} \right\}_r \frac{t^n}{n!} \\
&= \frac{1}{k!}\sum_{j=r-k}^{s}\binom{s}{j}\left(k+j\right)!\sum_{n\geq k+j-r}\left\{ \begin{matrix} n+r \\ k+j-r+r \end{matrix} \right\}_r \frac{t^n}{n!} \\
&= \frac{1}{k!}\exp\left(rt\right)\sum_{j=r-k}^{s}\binom{s}{j}\frac{\left(k+j\right)!}{\left(k+j-r\right)!}\left(\exp\left(t\right)-1\right)^{k+j-r} \\
&= \frac{1}{k!}\exp\left(rt\right) D^r_{x=\exp(t)-1}\left(x^k\left(x+1\right)^s\right),
\end{aligned}$$

and for $k > r$ we have

$$\sum_{n\geq 0}\left\{ \begin{matrix} n+r+s \\ k+s \end{matrix} \right\}_{r,s}\frac{t^n}{n!} = \sum_{n\geq k-r}\left\{ \begin{matrix} n+r+s \\ k+s \end{matrix} \right\}_{r,s}\frac{t^n}{n!}$$

$$= \sum_{n\geq k-r}\frac{1}{k!}\sum_{j=0}^{s}\binom{s}{j}(k+s-j)!\left\{ \begin{matrix} n+r \\ k+s-j \end{matrix} \right\}_{r}\frac{t^n}{n!}$$

$$= \sum_{n\geq k-r}\frac{1}{k!}\sum_{j=0}^{s}\binom{s}{j}(k+j)!\left\{ \begin{matrix} n+r \\ k+j \end{matrix} \right\}_{r}\frac{t^n}{n!}$$

$$= \frac{1}{k!}\sum_{j=0}^{s}\binom{s}{j}(k+j)!\sum_{n\geq k-r+j}\left\{ \begin{matrix} n+r \\ k+j \end{matrix} \right\}_{r}\frac{t^n}{n!}$$

$$= \frac{1}{k!}\exp(rt)\sum_{j=0}^{s}\binom{s}{j}\frac{(k+j)!}{(k+j-r)!}(\exp(t)-1)^{k+j-r}$$

$$= \frac{1}{k!}\exp(rt)\,D^{r}_{x=\exp(t)-1}\left(x^k(x+1)^s\right).$$

□

Theorem 6 can be written to the $\mathbf{r}_p$-case as follows:

Theorem 7. *For $r_1 \leq \cdots \leq r_p$ we have*

$$\sum_{n\geq 0}\left\{ \begin{matrix} n+|\mathbf{r}_p| \\ k+r_p \end{matrix} \right\}_{\mathbf{r}_p}\frac{t^n}{n!}$$

$$= \frac{1}{k!}\exp(r_1 t)\,D^{r_1}_{x_1=\exp(t)-1}D^{r_2}_{x_2=x_1}\cdots D^{r_{p-1}}_{x_{p-1}=x_{p-2}}\left(x^k_{p-1}\prod_{i=1}^{p-1}(x_i+1)^{r_{i+1}}\right).$$

Proof. By induction on p. By using Theorem 6, the theorem is true for $p = 2$. Assuming that the assertion is true for $p \geq 2$ and let

$$A = \frac{1}{k!}\exp(r_1 t)\,D^{r_1}_{x_1=\exp(t)-1}D^{r_2}_{x_2=x_1}\cdots D^{r_p}_{x_p=x_{p-1}}\left(x^k_{p}\prod_{i=1}^{p}(x_i+1)^{r_{i+1}}\right).$$

For $p+1$ we have

$$A = \frac{1}{k!}\sum_{j=0}^{r_{p+1}}\binom{r_{p+1}}{j}\exp\left(r_1 t\right)\times$$
$$D^{r_1}_{x_1=\exp(t)-1}D^{r_2}_{x_2=x_1}\cdots D^{r_p}_{x_p=x_{p-1}}\left(x_p^{k+j}\prod_{i=1}^{p-1}\left(x_i+1\right)^{r_{i+1}}\right)$$
$$= \frac{1}{k!}\sum_{j=0}^{r_{p+1}}\binom{r_{p+1}}{j}\frac{(k+j)!\exp\left(r_1 t\right)}{(k+j-r_p)!}\times$$
$$D^{r_1}_{x_1=\exp(t)-1}D^{r_2}_{x_2=x_1}\cdots D^{r_{p-1}}_{x_{p-1}=x_{p-2}}\left(x_{p-1}^{k+j-r_p}\prod_{i=1}^{p-1}\left(x_i+1\right)^{r_{i+1}}\right).$$

By the induction hypothesis, we have

$$\frac{1}{(k+j-r_p)!}D^{r_1}_{x_1=\exp(t)-1}D^{r_2}_{x_2=x_1}\cdots D^{r_{p-1}}_{x_{p-1}=x_{p-2}}\left(x_{p-1}^{k+j-r_p}\prod_{i=1}^{p-1}\left(x_i+1\right)^{r_{i+1}}\right)$$
$$= \sum_{n\geq 0}\left\{ {n+|\mathbf{r}_p| \atop k+j-r_p+r_p} \right\}_{\mathbf{r}_p}\frac{t^n}{n!} = \sum_{n\geq 0}\left\{ {n+|\mathbf{r}_p| \atop k+j} \right\}_{\mathbf{r}_p}\frac{t^n}{n!}.$$

Then

$$A = \sum_{j=0}^{r_{p+1}}\binom{r_{p+1}}{j}\frac{(k+j)!}{k!}\sum_{n\geq 0}\left\{ {n+|\mathbf{r}_p| \atop k+j} \right\}_{\mathbf{r}_p}\frac{t^n}{n!}$$
$$= \sum_{n\geq 0}\frac{t^n}{n!}\sum_{j=0}^{r_{p+1}}\binom{r_{p+1}}{j}\left\{ {n+|\mathbf{r}_p| \atop k+j} \right\}_{\mathbf{r}_p}\frac{(k+j)!}{k!}$$
$$= \sum_{n\geq 0}\left\{ {n+|\mathbf{r}_{p+1}| \atop k+r_{p+1}} \right\}_{\mathbf{r}_{p+1}}\frac{t^n}{n!}.$$

The last equality is justified by Theorem 3. □

The Dobiński and Stirling formulas can be written to the $\mathbf{r}_p$-case as follows:

Theorem 8. *Let*

$$B_n\left(z;\mathbf{r}_p\right) := \sum_{k=0}^{n+|\mathbf{r}_{p-1}|}\left\{ {n+|\mathbf{r}_p| \atop k+r_p} \right\}_{\mathbf{r}_p} z^k, \quad n\geq 0.$$

For $r_1 \leq \cdots \leq r_p$ we have

$$B_n(z;\mathbf{r}_p) = \exp(-z) \sum_{k\geq 0} P_n(k;\mathbf{r}_p) \frac{z^k}{k!},$$

$$\left\{ \begin{matrix} n+|\mathbf{r}_p| \\ k+r_p \end{matrix} \right\}_{\mathbf{r}_p} = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} P_n(j;\mathbf{r}_p).$$

Proof. Use Theorem 8 to get

$$\sum_{n\geq 0} B_n(z;\mathbf{r}_p) \frac{t^n}{n!} = \exp(r_1 t - z) D^{r_1}_{x_1=\exp(t)-1} D^{r_2}_{x_2=x_1} \cdots D^{r_{p-1}}_{x_{p-1}=x_{p-2}} \times \left(\exp(z(x_{p-1}+1)) \prod_{i=1}^{p-1} (x_i+1)^{r_{i+1}} \right).$$

The expansion of $\exp(z(x_{p-1}+1))$ and differentiation with respect to x_{p-1} give

$$\begin{aligned} \sum_{n\geq 0} B_n(z;\mathbf{r}_p) \frac{t^n}{n!} &= \sum_{j\geq 0} \exp(r_1 t - z) D^{r_1}_{x_1=\exp(t)-1} D^{r_2}_{x_2=x_1} \cdots D^{r_{p-1}}_{x_{p-1}=x_{p-2}} \\ &\quad \times \left(\frac{z^j}{j!} (x_{p-1}+1)^j \prod_{i=1}^{p-1} (x_i+1)^{r_{i+1}} \right) \\ &= \exp(r_1 t - z) D^{r_1}_{x_1=\exp(t)-1} D^{r_2}_{x_2=x_1} \cdots D^{r_{p-2}}_{x_{p-2}=x_{p-3}} \\ &\quad \times \left(\sum_{j\geq 0} (j+r_p)^{\underline{r_{p-1}}} (x_{p-2}+1)^{j+r_p} \frac{z^j}{j!} \prod_{i=1}^{p-3} (x_i+1)^{r_{i+1}} \right) \end{aligned}$$

and by successive differentiation we obtain

$$\begin{aligned} \sum_{n\geq 0} B_n(z;\mathbf{r}_p) \frac{t^n}{n!} &= \exp(r_1 t - z) \sum_{j\geq 0} P_0(j;\mathbf{r}_p) (x_1+1)^{j+r_p-r_1} \frac{z^j}{j!} \Bigg|_{x_1=\exp(t)-1} \\ &= \exp(-z) \sum_{j\geq 0} P_0(j;\mathbf{r}_p) \frac{z^j}{j!} \exp((j+r_p)t) \\ &= \exp(-z) \sum_{n\geq 0} \frac{t^n}{n!} \sum_{j\geq 0} P_n(j;\mathbf{r}_p) \frac{z^j}{j!}. \end{aligned}$$

Then, by identification, the first identity of the theorem results. The second identity of the theorem results upon using the expansion:

$$B_n(z;\mathbf{r}_p) = \sum_{i,j\geq 0} (-1)^i P_n(j;\mathbf{r}_p) \frac{z^{i+j}}{i!j!} = \sum_{k\geq 0} \frac{z^k}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} P_n(j;\mathbf{r}_p). \qquad \square$$

From Theorem 8 we may state that:

Corollary 9. *For $r_1 \leq \cdots \leq r_p$ we have*

$$\sum_{n\geq 0} \left\{ {n+|\mathbf{r}_p| \atop k+r_p} \right\}_{\mathbf{r}_p} \frac{t^n}{n!} = \frac{\exp(r_p t)}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} P_0(j;\mathbf{r}_p) \exp(jt),$$

$$\sum_{n,k\geq 0} \left\{ {n+|\mathbf{r}_p| \atop k+r_p} \right\}_{\mathbf{r}_p} z^k \frac{t^n}{n!} = \exp(r_p t - z) \sum_{j\geq 0} P_0(j;\mathbf{r}_p) \frac{(z\exp(t))^j}{j!}.$$

2.3. Identities and Consequences

A combinatorial interpretation of the r-Stirling numbers of the coefficient of $z^{\underline{k}}$ in the polynomial $(z+r)^n$ is given in [3] by

$$(z+r)^n = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r z^{\underline{k}}.$$

In Theorem 10 given below, we generalize this result on giving a combinatorial interpretation by the $\mathbf{r}_p$-Stirling numbers of the coefficient of $z^{\underline{k}}$ in the polynomial $P_n(z;\mathbf{r}_p)$. In other words, we write the polynomial $P_n(z;\mathbf{r}_p)$ as a linear combination of falling factorials, proving that the $\mathbf{r}_p$-Stirling numbers can be interpreted as connection constants, see for instance [7].

Theorem 10. *For $r_1 \leq \cdots \leq r_p$, we have*

$$P_n(z;\mathbf{r}_p) = \sum_{k=0}^{n+|\mathbf{r}_{p-1}|} \left\{ {n+|\mathbf{r}_p| \atop k+r_p} \right\}_{\mathbf{r}_p} z^{\underline{k}}.$$

Proof. Uponn using Theorem 8, the two expressions of $B_n(z;\mathbf{r}_p)$ give

$$D_{z=0}^{m}(\exp(z) B_n(z;\mathbf{r}_p)) = \sum_{l=0}^{m} \binom{m}{l} D_{z=0}^{l}(B_n(z;\mathbf{r}_p)) = \sum_{k=0}^{m} \left\{ {n+|\mathbf{r}_p| \atop k+r_p} \right\}_{\mathbf{r}_p} m^{\underline{k}},$$

$$D_{z=0}^{m}(\exp(z) B_n(z;\mathbf{r}_p)) = (m+r_p)^{\underline{r_1}} \cdots (m+r_p)^{\underline{r_{p-1}}} (m+r_p)^n = P_n(m;\mathbf{r}_p).$$

These imply that

$$P_n(m;\mathbf{r}_p) = \sum_{k=0}^{m} \left\{ {n+|\mathbf{r}_p| \atop k+r_p} \right\}_{\mathbf{r}_p} m^{\underline{k}}.$$

Then, the polynomial $P_n(z;\mathbf{r}_p) - \sum_{k=0}^{m} \left\{ {n+|\mathbf{r}_p| \atop k+r_p} \right\}_{\mathbf{r}_p} z^{\underline{k}}$ vanishes for all non-negative integer $z = m$. It results that $P_n(z;\mathbf{r}_p) = \sum_{k=0}^{n+|\mathbf{r}_{p-1}|} \left\{ {n+|\mathbf{r}_p| \atop k+r_p} \right\}_{\mathbf{r}_p} z^{\underline{k}}$. □

The three corollaries given below present consequences of Theorems 8 and 10. The first one gives an expression of $\left\{ {n+|\mathbf{r}_p| \atop k+r_p} \right\}_{\mathbf{r}_p}$ in terms of $\left\{ {|\mathbf{r}_p| \atop k+r_p} \right\}_{\mathbf{r}_p}$ and the r-Stirling numbers.

Corollary 11. *For $r_1 \leq \cdots \leq r_p$ we have*

$$\left\{ {n+|\mathbf{r}_p| \atop k+r_p} \right\}_{\mathbf{r}_p} = \sum_{j=0}^{k} \left\{ {|\mathbf{r}_p| \atop j+r_p} \right\}_{\mathbf{r}_p} \left\{ {n+j+r_p \atop k+r_p} \right\}_{j+r_p}, \quad 0 \leq k \leq n+|\mathbf{r}_{p-1}|.$$

In particular, for $p=2$ and $r \leq s$, we obtain

$$\left\{ {n+r+s \atop k+s} \right\}_{r,s} = \sum_{j=0}^{\min(k,r)} \binom{s}{j}\binom{r}{j} \left\{ {n+r+s-j \atop k+s} \right\}_{r+s-j} j!, \quad 0 \leq k \leq n+r.$$

Proof. From Theorems 8 and 10 we have

$$\begin{aligned} B_n(z;\mathbf{r}_p) &= \exp(-z) \sum_{k\geq 0} (k+r_p)^n P_0(k;\mathbf{r}_p) \frac{z^k}{k!} \\ &= \exp(-z) \sum_{k\geq 0} (k+r_p)^n z^k \left(\sum_{j=0}^{k} \left\{ {|\mathbf{r}_p| \atop j+r_p} \right\}_{\mathbf{r}_p} \frac{1}{(k-j)!} \right) \\ &= \exp(-z) \sum_{j=0}^{|\mathbf{r}_{p-1}|} \left\{ {|\mathbf{r}_p| \atop j+r_p} \right\}_{\mathbf{r}_p} z^j \sum_{k\geq 0} \frac{(k+j+r_p)^n}{k!} z^k \\ &= \sum_{j=0}^{|\mathbf{r}_{p-1}|} \left\{ {|\mathbf{r}_p| \atop j+r_p} \right\}_{\mathbf{r}_p} z^j \sum_{i=0}^{n} \left\{ {n+j+r_p \atop i+j+r_p} \right\}_{j+r_p} z^i \\ &= \sum_{j=0}^{n+|\mathbf{r}_{p-1}|} z^k \sum_{j=0}^{k} \left\{ {|\mathbf{r}_p| \atop j+r_p} \right\}_{\mathbf{r}_p} \left\{ {n+j+r_p \atop k+r_p} \right\}_{j+r_p}. \end{aligned}$$

The corollary follows from the definition of the polynomial $B_n(z;\mathbf{r}_p)$. □

Theorem 8 implies the following corollary:

Corollary 12. *For $r_1 \leq \cdots \leq r_p$ we have*

$$\begin{aligned} z\frac{d}{dz}\left(z^{r_p}\exp(z) B_n(z;\mathbf{r}_p)\right) &= z^{r_p}\exp(z) B_{n+1}(z;\mathbf{r}_p), \\ \frac{d}{dz}\left(\exp(z) B_n(z;\mathbf{r}_p)\right) &= \exp(z) B_n(z;\mathbf{r}_p+\mathbf{e}_p), \\ B_{n+1}(z;\mathbf{r}_p) &= zB_n(z;\mathbf{r}_p+\mathbf{e}_p) + r_p B_n(z;\mathbf{r}_p). \end{aligned}$$

Corollary 13. *For $r_1 \leq \cdots \leq r_p$ we have*

$$\left\{ {n+|\mathbf{r}_p|+k \atop r_p+j+k} \right\}_{\mathbf{r}_p+k\mathbf{e}_p} = \sum_{i=0}^{k} \binom{k}{i} \left\{ {n+|\mathbf{r}_p| \atop r_p+j+i} \right\}_{\mathbf{r}_p} (j+i)^{\underline{i}}.$$

Proof. Use Corollary 12 and the Leibnitz rule to get

$$\exp(-z)\frac{d^k}{dz^k}\left(\exp(z)B_n(z;\mathbf{r}_p)\right)=B_n(z;\mathbf{r}_p+k\mathbf{e}_p)=\sum_{j=0}^{k}\binom{k}{j}\frac{d^j}{dz^j}\left(B_n(z;\mathbf{r}_p)\right).$$

The corollary follows from the definitions of the polynomials $B_n(z;\mathbf{r}_p+k\mathbf{e}_p)$ and $B_n(z;\mathbf{r}_p)$ in the last identity. ☐

Theorem 8 can be used to generalize the discrete Poisson distribution and the inequalities given by Bouroubi [2] on the single variable Bell polynomials as follows:

Proposition 14. *Let t be a real number, α,β be positive real numbers with $\frac{1}{\alpha}+\frac{1}{\beta}=1$ and for $r_1\le\cdots\le r_p$ let*

$$B_t(\lambda;\mathbf{r}_p):=\exp(-\lambda)\sum_{k\ge 0}P_t(k;\mathbf{r}_p)\frac{\lambda^k}{k!},\quad t\in\mathbb{R},\ r_p\ge 1.$$

For $\lambda>0$, let X be a random variable defined by its discrete probability

$$P(X=k)=\frac{P_t(k;\mathbf{r}_p)}{B_t(\lambda;\mathbf{r}_p)}\exp(-\lambda)\frac{\lambda^k}{k!},\quad k=0,1,2,\ldots.$$

Then

$$\mathrm{E}(X+r_p)^x=\frac{B_{t+x}(\lambda;\mathbf{r}_p)}{B_t(\lambda;\mathbf{r}_p)},\quad x\in\mathbb{R}$$

and

$$B_{t+x+y}(\lambda;\mathbf{r}_p)B_t(\lambda;\mathbf{r}_p)\ge B_{t+x}(\lambda;\mathbf{r}_p)B_{t+y}(\lambda;\mathbf{r}_p),\quad x,y\ge 0,$$
$$B_{t+x+y}(\lambda;\mathbf{r}_p)\le\left(B_{t+\alpha x}(\lambda;\mathbf{r}_p)\right)^{1/\alpha}\left(B_{t+\beta y}(\lambda;\mathbf{r}_p)\right)^{1/\beta},\quad x,y\in\mathbb{R},$$
$$\left(B_{t+x}(\lambda;\mathbf{r}_p)\right)^{1/x}\le\left(B_{t+y}(\lambda;\mathbf{r}_p)\right)^{1/y}\left(B_t(\lambda;\mathbf{r}_p)\right)^{1/x-1/y},\quad 0<x\le y,$$
$$\left(B_{t+y}(\lambda;\mathbf{r}_p)\right)^2\le B_{t+y-x}(\lambda;\mathbf{r}_p)B_{t+y+x}(\lambda;\mathbf{r}_p),\quad 0\le x\le y.$$

Proof. The expectation's equality is evident. The first inequality follows from the inequality

$$\mathrm{E}(X+s)^{x+y}\ge\mathrm{E}(X+s)^x\,\mathrm{E}(X+s)^y,\quad x,y\ge 0$$

and to obtain the second inequality use Hölder's inequality

$$\mathrm{E}(X+s)^{x+y}\le\left(\mathrm{E}(X+s)^{\alpha x}\right)^{1/\alpha}\left(\mathrm{E}(X+s)^{\beta y}\right)^{1/\beta},\quad x,y\in\mathbb{R}.$$

The third inequality follows from Lyapunov's inequality

$$\left(\mathrm{E}(X+s)^x\right)^{1/x}\le\left(\mathrm{E}(X+s)^y\right)^{1/y},\quad 0<x\le y$$

and the fourth inequality follows from Schwarz's inequality

$$\left(\mathrm{E}(X+s)^y\right)^2\le\mathrm{E}(X+s)^{y-x}\,\mathrm{E}(X+s)^{y+x},\quad 0\le x\le y.$$

For these inequalities you can see [1]. ☐

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