# INVERSES OF MOTZKIN AND SCHRÖDER PATHS

**Heinrich Niederhausen**
*Dept. of Mathematical Sciences, Florida Atlantic University, Boca Raton, Florida*
niederha@fau.edu

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**Abstract**

The connection between weighted counts of Motzkin paths and moments of orthogonal polynomials is well known. We look at the inverse generating function of Motzkin paths with weighted horizontal steps, and relate it to Chebyshev polynomials of the second kind. The inverse can be used to express the number of paths ending at a certain height in terms of those ending at height 0. Paths of a more general horizontal step length $w$ are also investigated. We suggest three applications for the inverse. First, we use the inverse Motzkin matrix to express some Hankel determinants of Motzkin paths. Next, we count the paths inside a horizontal band using a ratio of inverses. Finally, for Schröder paths ($w = 2$) we write the number of paths inside the same band that end on the top side of the band in terms of those ending at height 0, with the help of the inverse Schröder matrix.

## 1. Introduction

A Motzkin path takes steps $\nearrow$, $\searrow$, and $\longrightarrow$ (NE, SE, and East), does not cross the $x$-axis, and starts and ends on the $x$-axis. We will denote by $M_n$ the number of Motzkin paths ending at $(n, 0)$. Motzkin [9] studied these paths in 1946. A Dyck path is a Motzkin path without the East steps. In his paper "On permutations with restricted patterns and Dyck paths," C. Krattenthaler [6] made a case for the paradigm:

> Whenever you encounter generating functions which can be expressed in terms of continued fractions or Chebyshev polynomials, then expect that Dyck or Motzkin paths are at the heart of your problem, and will help to solve it.

Here we show that the paradigm cannot only be carried over to *generalized* Motzkin paths, i.e. paths where the East step has length $w$, but also applies to the matrix of inverse numbers when $w = 1$ and $w = 2$ (the Schröder paths). We find the inverse useful in some applications (see also A. Ralston and P. Rabinowitz, 1978 [10, p. 256]). For example, the *bounded* Motzkin numbers $M_n^{(k)}$ are the number of Motzkin paths staying strictly below the parallel to the $x$-axis at height $k$. They have a generating function expressed by the inverse $(m_{i,j})$ through the *inverse Motzkin polynomial* $m_k(t) = \sum_{i=0}^{k} m_{k,i} t^{k-i}$:

$$\sum_{n\geq 0} M_n^{(k)} t^n = \frac{m_{k-1}(t)}{m_k(t)}$$

(see (10)). That makes us wonder if paths with different horizontal step lengths $w$ have similar properties. In the case of $w = 2$ (*Schröder paths*), the generating function identity described in Theorem 5 holds:

$$t^{-k} \widehat{\mathcal{S}}^{(k)}(t)\, \hat{s}_{k-1}(t) = t^{-k} \widehat{\mathcal{S}}^{(k)}(t, k-1)$$

(equal as power series), where $\widehat{\mathcal{S}}^{(k)}(t, k-1)$ is the generating function of the bounded (compressed) Schröder paths ending on $y = k-1$, just below the upper boundary. Similarly, $\widehat{\mathcal{S}}^{(k)}(t)$ is the generating function for those paths ending on the $x$-axis.

We give a quick overview of Motzkin paths in the next section. This is followed by a section on the inverse, and then applications of the inverse matrix. Next, we investigate the case of a general East step length $w$, and its beautiful connection to Chebyshev polynomials. Finally, we return to $w = 2$, the Schröder numbers.

## 2. Weighted Motzkin Numbers

The *weighted Motzkin numbers*, $M_{n;\beta}$, are the number of paths taking steps from the set $\{\nearrow, \searrow, \longrightarrow\}$ (NE, SE, and East), starting at the origin, staying weakly above the $x$-axis, and ending on the $x$-axis in $(n, 0)$. The horizontal steps get weight $\beta$, which is also called the number of colors. We denote by $M(n, k; \beta)$ the Motzkin paths that end at $(n, k)$, staying weakly above the $x$-axis. Hence $M_{n;\beta} = M(n, 0; \beta)$. In general, the NE steps are colored with $\alpha$, and the SE steps with $\gamma$ colors (see Merlini and Sprugnoli (2010) [8]). However, this does not increase the generality of the problem. If we call $M_{n,k}^{[\alpha,\beta,\gamma]}$ the Motzkin numbers as colored by Merlini and Sprugnoli, then $M_{n,k}^{[\alpha,\beta,\gamma]} = \alpha^k (\alpha\gamma)^{(n-k)/2} M\left(n, k; \beta/\sqrt{\alpha\gamma}\right)$.

A weighted Motzkin path is counted by the recursion

$$M(n, m; \beta) = M(n-1, m+1; \beta) + \beta M(n-1, m; \beta) + M(n-1, m-1; \beta)$$

for $m \geq 0$, and $M(n, m; \beta) = 0$ if $m < 0$. These numbers (with weight $\beta = 1$) were studied by T. Motzkin in 1946 [9].

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $m$ $\uparrow$ | | | | | | | |
| 5 | | | | | | | 1 |
| 4 | | | | | | 1 | $5\beta$ |
| 3 | | | | 1 | | $4\beta$ | $4+10\beta^2$ |
| 2 | | | 1 | $3\beta$ | $\searrow$ | $3+6\beta^2$ | $15\beta+10\beta^3$ |
| 1 | | 1 | $2\beta$ | $2+3\beta^2$ | $\rightarrow$ | $8\beta+4\beta^3$ | $5+20\beta^2+5\beta^4$ |
| 0 | 1 | $\beta$ | $1+\beta^2$ | $3\beta+\beta^3$ | $\nearrow$ | $2+6\beta^2+\beta^4$ | $10\beta+10\beta^3+\beta^5$ |
| $n\rightarrow$ | 0 | 1 | 2 | 3 | | 4 | 5 |

$M(n,m;\beta)$ $\qquad$ $M_{n;\beta}$ is given in row 0.

The above table shows that for $\beta=1$ the original Motzkin numbers are $1,1,2,4,9,21,51,127,\ldots$ (sequence A001006 in the On-Line Encyclopedia of Integer Sequences (OEIS )). The matrix $\left(M(n,m;\beta)_{n,m=0,1,2,\ldots}\right)$ is a Riordan matrix.

It is well known that the general $\beta$-weighted Motzkin numbers have the generating function

$$\mu(t;j,\beta):=\sum_{n\geq 0} M(n+j,j;\beta)\,t^n=\left(\frac{1-\beta t-\sqrt{(1-\beta t)^2-4t^2}}{2t^2}\right)^{j+1},$$

thus

$$\mu(t):=\sum_{n\geq 0} M_{n;\beta}t^n=\sum_{n\geq 0} M(n,0;\beta)\,t^n=\frac{1-\beta t-\sqrt{(1-\beta t)^2-4t^2}}{2t^2} \tag{1}$$

is the generating function of the Motzkin numbers, satisfying the quadratic equation [1]

$$\mu(t)=1+\beta t\mu(t)+t^2\mu(t)^2. \tag{2}$$

Hence

$$M_{n+2;\beta}-\beta M_{n+1;\beta}=\sum_{i=0}^{n} M_{i;\beta}M_{n-i;\beta},$$

a well-known identity that is combinatorially shown by using the *"First Return Decomposition."* The generating function (in $t^2$) of the Catalan numbers $C_n$ is easily obtained by setting $\beta=0$ in (1), but it also follows from $\beta=2$ (bicolored Motzkin paths) that

$$\frac{1-2t-\sqrt{(1-2t)^2-4t^2}}{2t^2}=\frac{1-2t-\sqrt{1-4t}}{2t^2}=\sum_{n\geq 1} C_n t^{n-1}$$

(in $t$). We can choose $\beta = 1$ and get

$$(1+t)\sum_{n\geq 1} C_n \left(\frac{t}{1+t}\right)^{n-1} = \sum_{n\geq 0} M_{n;1} t^n$$
$$M_{n;1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} C_{k+1}.$$

For general $\beta$, the explicit expression

$$M_{n;\beta} = \sum_{k=0}^{n/2} \binom{n}{2k} \frac{\beta^{n-2k}}{2k+1} \tbinom{2k+1}{k}$$

follows from (1).

## 3. The Inverse

Define $\phi(t)$ such that $t/\phi(t)$ is the compositional inverse of $t\mu(t)$. Thus,

$$\phi(t\mu(t)) = \mu(t) = 1 + \beta t\mu(t) + t^2 \mu(t)^2$$

by (2), and therefore

$$\phi(t) = 1 + \beta t + t^2.$$

This simple form of the inverse is the reason for many special results for Motzkin numbers. Note that

$$1/\phi(t) = \left(1 + \beta t + t^2\right)^{-1} = \sum_{n\geq 0} U_n(-\beta/2)\, t^n =: U(t; -\beta/2)$$

is the generating function of the *Chebyshev polynomials*

$$U_n(-\beta/2) = \sum_{k=0}^{n/2} \binom{n-k}{k} (-1)^{n-k} \beta^{n-2k}$$

of the second kind.

Because of the inverse relationship between $t\mu(t)$ and $t/\phi(t)$ we have that the matrix inverse of $(M(i,j;\beta))_{n\times n}$ equals $(m_{i,j})_{n\times n}$, where

$$\sum_{i\geq 0} m_{i,j} t^i = t^j \phi(t)^{-j-1}.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 4 & 5 & 3 & 1 & 0 \\ 9 & 12 & 9 & 4 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 1 & 1 & -3 & 1 & 0 \\ -1 & 2 & 3 & -4 & 1 \end{pmatrix} = (m_{i,j})_{4\times 4}$$

Inverse Motzkin matrix when $\beta = 1$

Note that $(m_{i,j})$ is also a *Riordan matrix*. The above generating function for $m_{i,j}$ implies that

$$\begin{aligned} m_{i,j} &= \left[t^i\right] \frac{1}{1+\beta t+t^2}\left(\frac{t}{1+\beta t+t^2}\right)^j = \left[t^{i-j}\right]\left(1+\beta t+t^2\right)^{-j-1} \qquad (3) \\ &= C_{i-j}^{j+1}\left(-\beta/2\right). \end{aligned}$$

The polynomials $C_n^{\lambda}(x) = \sum_{k=0}^{n/2} \binom{n-k+\lambda-1}{n-k}\binom{n-k}{n-2k}(-1)^k (2x)^{n-2k}$ are the *Gegenbauer polynomials*, and therefore

$$m_{i,j} = \sum_{l=0}^{(i-j)/2} \binom{i-l}{i-j-l}\binom{i-j-l}{l}(-1)^l(-\beta)^{i-j-2l}. \qquad (4)$$

The recurrence relation for the (orthogonal) Gegenbauer polynomials,

$$2x(n+\lambda)C_n^{\lambda}(x) = (n+2\lambda-1)C_{n-1}^{\lambda} + (n+1)C_{n+1}^{\lambda}(x),$$

immediately gives us a recurrence for the inverse numbers $m_{i,j}$, $0 \le j \le i-1$,

$$(i-j)\,m_{i,j} = -\beta i m_{i-1,j} - (i+j)\,m_{i-2,j}$$

with initial values $m_{i,j} = \delta_{i,j}$ for $j \ge i$.

We later need in the paper the following inverse Motzkin polynomial $m_k(t) :=$

$$\begin{aligned} \sum_{j=0}^{k} m_{k,j} t^{k-j} &= \sum_{j=0}^{k} C_j^{(k-j+1)}\left(-\beta/2\right) t^j \\ &= \sum_{l=0}^{k/2}\sum_{j=0}^{k-2l} \binom{k-l}{k-j-l}\binom{k-j-l}{k-j-2l}(-1)^l(-\beta)^{k-j-2l} t^{k-j} \\ &= \sum_{l=0}^{k/2}\binom{k-l}{l}(-1)^l t^{2l}(1-\beta t)^{k-2l} = t^k U_k\left(\frac{1-\beta t}{2t}\right). \qquad (5) \end{aligned}$$

We see from this form of the Motzkin polynomial that if $\beta = 1/t$ then

$$\sum_{j=0}^{k} m_{k,j}\beta^j = U_k(0) = \begin{cases} (-1)^{k/2} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

From

$$\left((M(i,j;\beta))_{0\le i,j\le n}\right)^{-1} = (m_{i,j})_{0\le i,j\le n}$$

follows

$$\sum_{k=0}^{n} M(i,k;\beta)\, m_{k,j} = \delta_{i,j}.$$

However, in the case of Motzkin matrices more than this simple linear algebra result holds, showing how the number of paths ending at $(i,j)$ can be obtained from those ending on the axis.

**Theorem 1** *For all nonnegative integers $i$ and $j$, the following holds:*

$$M(i,j;\beta) = \sum_{k=0}^{j} m_{j,k} M_{i+k;\beta}$$

*and*

$$m_{i,j} = \sum_{k=0}^{i-j} m_{i+1,j+1+k} M_{k;\beta}.$$

The proof can be done via generating functions. Note that

$$\sum_{n\ge 0}\sum_{j\ge 0} x^j t^n M(n,j;\beta) = \frac{\mu(t)}{1-xt\mu(t)} = \frac{1}{1+\beta x + x^2 - x/t}\left(\mu(t) - \frac{x}{t}\right)$$

and

$$\sum_{j\ge 0} x^j \sum_{i\ge j} m_{i,j} t^i = \sum_{j\ge 0} x^j t^j \phi(t)^{j+1} = \frac{\phi(t)}{1-xt\phi(t)} = \frac{1}{1/\phi(t) - xt} = U\left(t, \frac{\beta - x}{2}\right).$$

Replace $t$ by $x$ and $x$ by $1/t$ in the above generating function for the inverse $m_{i,j}$ to get the Laurent series

$$\sum_{j\ge 0} t^{-j} \sum_{i\ge j} m_{i,j} x^i = \frac{1}{1+\beta x + x^2 - x/t} = U(x, (\beta - 1/t)/2).$$

Hence

$$\sum_{n\ge 0}\sum_{j\ge 0} x^j t^n M(n,j;\beta) = \left(\mu(t) - \frac{x}{t}\right)\sum_{j\ge 0} t^{-j} \sum_{i\ge j} m_{i,j} x^i.$$

Now both sides must be power series in $x$ and $t$. This condition gives the theorem. The theorem also has the following corollary, since $M(i,j;\beta) = \delta_{i,j}$ for all $0 \le i \le j$.

**Corollary 2** *We have*

$$\sum_{k=0}^{j} m_{j,k} M_{i+k,\beta} = \delta_{i,j} \text{ for } 0 \le i \le j. \tag{6}$$

## 4. Two Applications of the Inverse Motzkin Matrix

Hankel determinants of combinatorial number sequences are a topic that pose many challenges; Hankel determinants for Motzkin numbers were thoroughly explored in [3]. Theorem 1 says that

$$(m_{i,j})_{0\le i,j\le n}\,(M_{i+j;\beta})_{0\le i,j\le n} = (M\,(i,j;\beta))_{0\le i,j\le n}\,,$$

which gives a direct way of calculating the first *Hankel determinant* of Motzkin numbers

$$\det\,(M_{i+j;\beta})_{0\le i,j\le n} = \frac{1}{\det\,(m_{i,j})}\det\,(M\,(i,j;\beta)) = 1. \tag{7}$$

However, subsequent Hankel determinants are more complicated, especially determinants of linear combinations of Hankel matrices; we want to show how to calculate a determinant of a linear combination proposed by Cameron and Yip [2]. It gives the second Hankel determinant as a special case (see (8)).

As a second application of inverses we look at Motzkin paths that cannot exceed a certain height. We will see how those numbers can be expressed as a convolution of inverses.

### 4.1. The Hankel Determinant $|aM_{i+j;\beta}+bM_{i+j+1;\beta}|_{0\le i,j\le n-1}$

The matrix $(aM_{i+j;\beta}+bM_{i+j+1;\beta})_{0\le i,j\le n-1}$ can be factored as

$$(aM_{i+j;\beta}+bM_{i+j+1;\beta})_{0\le i,j\le n-1} = (M_{i+j;\beta})_{0\le i,j\le n-1}\begin{pmatrix} a & 0 & 0 & \dots & x_0 \\ b & a & 0 & & x_1 \\ 0 & b & a & & x_2 \\ & & & \vdots & \\ 0 & 0 & 0 & a & x_{n-2} \\ 0 & 0 & 0 & b & x_{n-1}\end{pmatrix},$$

where we have to determine $x_0, x_1, \ldots, x_{n-1}$. When multiplied by the $i$-th row of $(M_{i+j;\beta})_{0\le i,j\le n-1}$, the last column on the right must give $aM_{i+n-1;\beta}+bM_{i+n;\beta}$ when $n>0$. But from Corollary 2 it follows that $\sum_{k=0}^{n-1} m_{n,k}M_{i+k,\beta} = \delta_{i,n} - M_{i+n,\beta}$ for all $i=0,\ldots,n-1$, and thus $aM_{i+n-1;\beta}+bM_{i+n;\beta} = aM_{i+n-1;\beta} - b\sum_{k=0}^{n-1} m_{n,k}M_{i+k,\beta}$ for all $i=0,\ldots,n-1$. Hence we can choose $x_k=-bm_{n,k}$ for $k=0,\ldots,n-2$, and $x_{n-1}=a-bm_{n,n-1}$. Therefore, the Hankel determinant of $(aM_{i+j;\beta}+bM_{i+j+1;\beta})_{0\le i,j\le n-1}$ equals

$$\left|(M_{i+j;1})_{0\le i,j\le n-1}\right|\begin{vmatrix} a & 0 & 0 & \dots & -bm_{n,0} \\ b & a & 0 & & -bm_{n,1} \\ 0 & b & a & & -bm_{n,2} \\ & & & \vdots & \\ 0 & 0 & 0 & a & -bm_{n,n-2} \\ 0 & 0 & 0 & b & a-bm_{n,n-1}\end{vmatrix}.$$

The determinant of the left factor is 1 and the determinant of the right factor can be evaluated by elementary column operations. Thus,

$$\det\left(\left(aM_{i+j;\beta}+bM_{i+j+1;\beta}\right)_{0\leq i,j\leq n-1}\right) = a^n - \sum_{i=0}^{n-1}(-1)^{n-1-i}\,b^{n-i}a^i m_{n,i}$$
$$= \sum_{i=0}^{n}(-1)^{n-i}\,b^{n-i}a^i C_{n-i}^{i+1}\left(-\beta/2\right)$$

(see (3)). Note that $\sum_{i=0}^n t^i C_{n-i}^{i+1}\left(-\beta/2\right) = U_n\left(\left(-\beta+t\right)/2\right)$ (see (5)). Therefore, $\det\left(\left(aM_{i+j;\beta}+bM_{i+j+1;\beta}\right)_{0\leq i,j\leq n-1}\right)$

$$\begin{aligned}
&= (-b)^n\,U_n\left(\frac{-a/b-\beta}{2}\right)\\
&= \sum_{k=0}^{n/2}\binom{n-k}{k}(-1)^k\,b^{2k}\left(a+b\beta\right)^{n-2k}\\
&= \frac{2^{-n-1}}{\sqrt{\left(a+b\beta\right)^2-4b^2}}\\
&\quad\times\left(\left(\sqrt{\left(a+b\beta\right)^2-4b^2}+a+b\beta\right)^{n+1}+\left(\sqrt{\left(a+b\beta\right)^2-4b^2}-a-b\beta\right)^{n+1}\right).
\end{aligned}$$

The generating function of the Hankel determinants equals

$$\sum_{n=1}^{\infty}\det\left(\left(aM_{i+j;\beta}+bM_{i+j+1;\beta}\right)_{0\leq i,j\leq n-1}\right)t^n$$
$$= \sum_{n=1}^{\infty}(-tb)^n\,U_n\left(\frac{-a/b-\beta}{2}\right) = \frac{1}{1-\left(a+b\beta\right)t+\left(bt\right)^2}.$$

Let us look at few interesting cases.

1. If $a+\beta b=0$ and $b\neq 0$, then $-a/b-\beta=0$. Hence, $\det\left(\left(aM_{i+j;\beta}-\frac{a}{\beta}M_{i+j+1;\beta}\right)_{0\leq i,j\leq n-1}\right) = (a/\beta)^n\,U_n\left(0\right) = (-1)^{n/2}\left(a/\beta\right)^n$ if $n$ is even, and 0 else.

2. If $a=\sin^2\theta$ and $b=\cos\theta-1\neq 0$ and $\beta=1-\cos\theta$, then $-a/b-\beta = \frac{\sin^2\theta}{1-\cos\theta}+\left(-1+\cos\theta\right) = \frac{\sin^2\theta-(1-\cos\theta)^2}{1-\cos\theta} = 2\cos\theta$. Hence, $\det\left(\left(\sin^2\theta M_{i+j;(1-\cos\theta)/2}-\left(2\cos\theta-2\right)M_{i+j+1;(1-\cos\theta)/2}\right)_{0\leq i,j\leq n-1}\right)$ $= \left(2-2\cos\theta\right)^n U_n\left(\cos\theta\right) = 2^n\left(1-\cos\theta\right)^n\frac{\sin(n+1)\theta}{\sin\theta}$.

3. If $a=b=1$, then $\det\left((M_{i+j;\beta}+M_{i+j+1;\beta})_{0\le i,j\le n-1}\right)=$

$$\frac{1}{2^{n+1}\sqrt{(\beta+1)^2-4}}$$
$$\times\left(\left(1+\beta+\sqrt{(\beta+1)^2-4}\right)^{n+1}-\left(1+\beta-\sqrt{(\beta+1)^2-4}\right)^{n+1}\right)$$
$$=\sum_{k=0}^{n}(-1)^{n-k}\binom{k}{n-k}(\beta+1)^{2k-n},$$

which approaches $n+1$ if $\beta\to 1$. In the case of a Dyck path, we obtain $\delta_{0,n}$ for this determinant of the sum of matrices.

4. If $b=1$ and $a=0$, then the second Hankel determinant of the Motzkin numbers is

$$\det\left((M_{i+j+1;\beta})_{0\le i,j\le n-1}\right)=\sum_{k=0}^{n/2}\binom{n-k}{k}(-1)^k\beta^{n-2k}$$
$$=\frac{\left(\beta+\sqrt{\beta^2-4}\right)^{n+1}-\left(\beta-\sqrt{\beta^2-4}\right)^{n+1}}{2^{n+1}\sqrt{\beta^2-4}} \quad (8)$$

for $\beta\ne 2$. If $\beta=2$, then $\det\left((M_{i+j+1;2})_{0\le i,j\le n-1}\right)=n+1$.

5. Sulanke and Xin [13, Proposition 2.2] showed that the Fibonacci-like recursion

$$\det\left((M_{i+j+1;\beta})_{0\le i,j\le n-1}\right)$$
$$=\beta\det\left((M_{i+j+1;\beta})_{0\le i,j\le n-2}\right)-\det\left((M_{i+j+1;\beta})_{0\le i,j\le n-3}\right)$$

holds.

6. If $a=1$ and $b=0$, then $\det\left(M_{i+j;\beta}\right)_{0\le i,j\le n-1}=1$, independent of $\beta$ (see (7).

7. The same approach also shows the recursion

$$|M_{i+j+2;\beta}|_{0\le i,j\le n-1}=|M_{i+j+2;\beta}|_{0\le i,j\le n-2}+|M_{i+j+1;\beta}|^2_{0\le i,j\le n-1}.$$

### 4.2. Motzkin in a Band

The number of Motzkin paths staying strictly below the line $y=k$ for $k>0$ is known to have the generating function [4, Proposition 12]

$$\sum_{n\ge 0}M_n^{(k)}t^n=\mu(t)\,\frac{1-(t\mu(t))^{2k}}{1-(t\mu(t))^{2(k+1)}}=\frac{1}{t}\,\frac{\left(\frac{1}{t\mu}\right)^k-(t\mu)^k}{\left(\frac{1}{t\mu}\right)^{k+1}-(t\mu)^{k+1}}.$$

| $m \uparrow$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| k=4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | | | | 1 | 4 | 14 | 44 | 133 | 392 | 1140 |
| 2 | | | 1 | 3 | 9 | 25 | 69 | 189 | 518 | 1422 |
| 1 | | 1 | 2 | 5 | 12 | 30 | 76 | 196 | 512 | 1353 |
| 0 | 1 | 1 | 2 | 4 | 9 | 21 | 51 | 127 | 323 | 835 |
| -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $n \rightarrow$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

$M_n^{(4)}$ is given in row 0.

From $\mu(t)(1-\beta t)-1 = t^2\mu(t)^2$ (see (2)) it follows that

$$\mu_{1,2}(t) = \left(1-\beta t \pm \sqrt{(1-\beta t)^2 - 4t^2}\right) / \left(2t^2\right),$$

and thus

$$\mu_1 + \mu_2 = (1-\beta t)/t^2 \text{ and } \mu_1\mu_2 = 1/t^2.$$

Hence

$$\begin{aligned}
\sum_{n\geq 0} M_{n;\beta}^{(k)} t^n &= \frac{1}{t}\frac{(t\mu_1)^k - (t\mu_2)^k}{(t\mu_1)^{k+1} - (t\mu_2)^{k+1}} = \frac{1}{t}\frac{(t\mu_2)^{-k} - (t\mu_1)^{-k}}{(t\mu_2)^{-k-1} - (t\mu_1)^{-k-1}} \\
&= \frac{\sum_{j=0}^{(k-1)/2} (-1)^j \binom{k-1-j}{j} t^{2j} (1-\beta t)^{k-1-2j}}{\sum_{j=0}^{k/2} (-1)^j \binom{k-j}{j} t^{2j} (1-\beta t)^{k-2j}} \\
&= \frac{\sum_{i=0}^{k-1} m_{k-1,i} t^{k-1-i}}{\sum_{i=0}^{k} m_{k,i} t^{k-i}} = \frac{U_{k-1}\left(\frac{1-\beta t}{2t}\right)}{tU_k\left(\frac{1-\beta t}{2t}\right)}
\end{aligned} \tag{9}$$

(see (5)). We have shown the following:

**Theorem 3** *We have* $\sum_{n\geq 0} M_{n;\beta}^{(k)} t^n = \frac{m_{k-1}(t)}{m_k(t)} = \frac{U_{k-1}\left(\frac{1-\beta t}{2t}\right)}{tU_k\left(\frac{1-\beta t}{2t}\right)}$.

The OEIS [14] lists many special cases for $k$. Here are a few for $\beta = 1$:

1. $\sum_{n\geq 0} M_{n;1}^{(1)} t^n = \frac{1}{1-t} \iff 1, 1, 1, 1, \ldots$.
2. $\sum_{n\geq 0} M_{n;1}^{(2)} t^n = \frac{1-t}{(1-t)^2 - t^2} = 1 + t + 2t^2 + 4t^3 + 8t^4 + 16t^5 \ldots$, thus $1, 1, 2, 4, 8, 16, 32, 64, \ldots$, the powers of 2.
3. $\sum_{n\geq 0} M_{n;1}^{(3)} t^n = \frac{2t-1}{(1-t)(t^2+2t-1)}$, thus $1, 1, 2, 4, 9, 21, 50, 120, \ldots$ (A171842).
4. $\sum_{n\geq 0} M_{n;1}^{(4)} t^n = \left(1 - 3t + t^2 + t^3\right) / \left(1 - 4t + 3t^2 + 2t^3 - t^4\right)$, thus $1, 1, 2, 4, 9, 21, 51, 127, 322, 826, \ldots$ (A005207); generating function by Alois P. Heinz.

The special form of the generating function

$$\sum_{n\geq 0} M_{n;\beta}^{(k)} t^n = \frac{U_{k-1}\left(\frac{1-\beta t}{2t}\right)}{tU_k\left(\frac{1-\beta t}{2t}\right)} = \frac{m_{k-1}(t)}{m_k(t)} \tag{10}$$

holds with weight $\beta$, for all $k = 1, 2, \ldots$. It is equivalent to the recursion $\sum_{j=0}^{k} M_{n-j;\beta}^{(k)} m_{k,k-j} = 0$ for $M_{n;\beta}^{(k)}$, for all $n \geq k$, with initial values $\sum_{j=0}^{n} M_{n-j;\beta}^{(k)} m_{k,k-j} = m_{k-1,k-1-n}$ for all $n = 0, \ldots, k-1$.

## 5. Horizontal Steps of Length $w$

Let us consider the step set $\{\nearrow, \searrow, \longrightarrow^w\}$, where $\rightarrow^w =: (w, 0)$, for any positive integer $w$. Denote the number of paths from $(0,0)$ to $(n,j)$ by $W(n,j;\beta)$, where the horizontal steps (of length $w$) are weighted by $\beta$. This is an obvious generalization of Motzkin paths. We would like to see similar results as Theorem 3 in such cases. However, we only have a result for the case $w = 2$, the Schröder paths, which we will discuss later.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $m\uparrow$ | | | | | | | | | 1 | 0 |
| 7 | | | | | | | | 1 | 0 | 8 |
| 6 | | | | | | | 1 | 0 | 7 | $7\beta$ |
| 5 | | | | | | 1 | 0 | 6 | $6\beta$ | 27 |
| 4 | | | | | 1 | 0 | 5 | $5\beta$ | 20 | $35\beta$ |
| 3 | | | | 1 | 0 | 4 | $4\beta$ | 14 | $24\beta$ | $48+10\beta^2$ |
| 2 | | | 1 | 0 | 3 | $3\beta$ | 9 | $15\beta$ | $28+6\beta^2$ | $63\beta$ |
| 1 | | 1 | 0 | 2 | $2\beta$ | 5 | $8\beta$ | $14+3\beta^2$ | $30\beta$ | $42+20\beta^2$ |
| 0 | 1 | 0 | 1 | $\beta$ | 2 | $3\beta$ | $5+\beta^2$ | $10\beta$ | $14+6\beta^2$ | $35\beta+\beta^3$ |
| $n\rightarrow$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

$w = 3$ ($W_n$ is given in row 0)

The case $w = 3$ and its Hankel determinants were investigated in detail by Sulanke and Xin [13].

### 5.1. The Recursion for $W$

We get the recursion

$$\begin{aligned} W(n,j;\beta) &= W(n-1,j+1;\beta) + W(n-1,j-1;\beta) + \beta W(n-w,j;\beta) \\ W(n,j;\beta) &= 0 \text{ for } j > n \\ W_{n;\beta} &= W(n,0;\beta). \end{aligned}$$

The following generating function is well known:

$$\sum_{n\geq 0} W_{n;\beta} t^n = \left(z - \sqrt{z^2-1}\right)/t \text{ if } z = \frac{1-\beta t^w}{2t} \tag{11}$$
$$=: \mu_w(t;\beta),$$

which says that $\mu_w(t;\beta)$ solves the quadratic equation $\mu_w(t;\beta) = 1+\beta t^w \mu_w(t;\beta) + t^2 \mu_w(t;\beta)^2$ [13]. For a general right step length $w$ it is difficult to find the inverse power series of $t\mu_w(t;\beta)$. However, we look at the case $w = 2$ in the last two sections.

Because $W(n,j;\beta)$ is again a Riordan matrix, we have that

$$\sum_{n\geq 0} W(n+j,j;\beta)\, t^{n+j+1} = \left(z - \sqrt{z^2-1}\right)^{j+1} \tag{12}$$

if $z = \frac{1-\beta t^w}{2t}$. The astonishing result is that the dependence on the weight $\beta$ and the step length $w$ can all be packed into the variable $z$.

We want to find another form for the generating function that also gives us a result for general Motzkin paths in a band. These numbers are no longer a Riordan matrix. The recursion can be reformulated as

$$W(n,j;\beta) = W(n+1,j-1;\beta) - W(n,j-2;\beta) - \beta W(n+1-w,j-1;\beta)$$

for $j \leq n+2$. We find the generating function identity $\sum_{i\geq 0} W(i,j;\beta)\, t^i =$

$$\begin{aligned}
&\sum_{i\geq 0} W(i+1,j-1;\beta)\, t^i - \sum_{i\geq 0} W(i,j-2;\beta)\, t^i - \beta \sum_{i\geq w-1} W(i+1-w,j-1;\beta)\, t^i \\
&= \sum_{i\geq 0} W(i+1,j-1;\beta)\, t^i - \beta \left( \sum_{i\geq -1} W(i+1,j-1;\beta)\, t^{i+1+w-1} \right) \\
&\qquad - \sum_{i\geq 0} W(i,j-2;\beta)\, t^i \\
&= t^{-1} \sum_{i\geq 0} W(i+1,j-1;\beta)\, t^{i+1} - \beta t^{w-1} \left( \sum_{i\geq -1} W(i+1,j-1;\beta)\, t^{i+1} \right) \\
&\qquad - \sum_{i\geq 0} W(i,j-2;\beta)\, t^i \\
&= \left(t^{-1} - \beta t^{w-1}\right) \left( \sum_{i\geq 0} W(i,j-1;\beta)\, t^i - \delta_{j,1} \right) - \sum_{i\geq 0} W(i,j-2;\beta)\, t^i.
\end{aligned}$$

Let $\mathcal{W}(t,j;\beta) = \sum_{i\geq 0} W(i,j;\beta)\, t^i$. In this notation,

$$\begin{aligned}
\mathcal{W}(t,j;\beta) &= 2z\mathcal{W}(t,j-1;\beta) - \mathcal{W}(t,j-2;\beta) \;\text{ for } j>1 \\
\mathcal{W}(t,1;\beta) &= 2z\mathcal{W}(t,0;\beta) - 1/t,
\end{aligned} \tag{13}$$

where $z$ is given in (11). For example,

$\mathcal{W}(t,2;\beta) = \frac{(1-\beta t^w)}{t}\mathcal{W}(t,1;\beta) - \mathcal{W}(t,0;\beta)$
$= \frac{(1-\beta t^w)}{t}\frac{1}{t}\left((1-\beta t^w)\mathcal{W}(t,0;\beta)-1\right) - \mathcal{W}(t,0;\beta)$
$= \left(\frac{(1-\beta t^w)^2}{t^2}-1\right)\mathcal{W}(t,0;\beta) - \frac{(1-\beta t^w)}{t^2}$, and $\mathcal{W}(t,0;\beta) = \mu_w(t;\beta)$ (see (11).

$\mathcal{W}(t,3;\beta) = \frac{(1-\beta t^w)}{t}\mathcal{W}(t,2;\beta) - \mathcal{W}(t,1;\beta)$
$= \frac{(1-\beta t^w)}{t}\left(\left(\frac{(1-\beta t^w)^2}{t^2}-1\right)\mathcal{W}(t,0;\beta) - \frac{(1-\beta t^w)}{t^2}\right) - \frac{1}{t}\left((1-\beta t^w)\mathcal{W}(t,0;\beta)-1\right)$
$= \left(\frac{(1-\beta t^w)^2}{t^2}-2\right)\left(\frac{(1-\beta t^w)}{t}\mu_w(t;\beta)\right) + \frac{1}{t} - \frac{(1-\beta t^w)^2}{t^3}$.

We find an explicit expression for $\mathcal{W}(t,j;\beta)$ in the next section, an alternative to formula (12).

### 5.2. Solution to the Recursion for $\mathcal{W}$ and $\mathcal{W}^{(k)}$

The linear recursion (13) is called Fibonacci-like. It is of the form

$$\sigma_n = u\sigma_{n-1} + v\sigma_{n-2},$$

with $u = 2z$ and $v = -1$, for $n > 1$. We know the initial values $\sigma_0$ and $\sigma_1 = u\sigma_0 - 1/t$.

Hence $\sigma_n = [\tau^n]\frac{\sigma_0+(\sigma_1-u\sigma_0)\tau}{1-u\tau-v\tau^2} = [\tau^n]\frac{\sigma_0-\tau/t}{1-u\tau+\tau^2}$ in this case, or $\sigma_n =$

$$[\tau^n]\left(\sigma_0 - \frac{\tau}{t}\right)\sum_{i=0}^{\infty}\binom{i}{j}(-1)^j u^{i-j}\tau^{i-j+2j} \tag{14}$$
$$= \sigma_0 U_n(z) - \frac{1}{t}U_{n-1}(z),$$

if $z = \frac{1-\beta t^w}{2t}$ as in the previous section.

**Lemma 4** *We have*

$$\mathcal{W}(t,j;\beta) = \frac{z-\sqrt{z^2-1}}{t}U_j(z) - \frac{1}{t}U_{j-1}(z) \tag{15}$$
$$\sum_{j\geq 0}\sum_{i\geq 0} W(i,j;\beta)\,t^i x^j = \frac{z-\sqrt{z^2-1}-x}{t(1-2zx+x^2)}$$
$$= \frac{1-\beta t^w - 2xt^2 - \sqrt{(1-\beta t^w)^2 - 4t^2}}{2t\left(x^2t - x + x\beta t^w + t\right)},$$

*where $U_j = 0$ for all $j < 0$.*

The explanation for the two forms for $\sum_{j\geq 0}\sum_{i\geq 0} W(i,j;\beta)\,t^i$, namely

$$\sum_{j\geq 0}\sum_{n\geq 0} W(n,j;\beta)\,t^n x^j = \frac{1}{t}\frac{z-\sqrt{z^2-1}}{1-x\left(z-\sqrt{z^2-1}\right)} = \frac{z-\sqrt{z^2-1}-x}{t(1-2zx+x^2)}$$

(see (12)), is the identity

$$\left(z-\sqrt{z^2-1}\right)\left(1-2zx+x^2\right)=\left(z-\sqrt{z^2-1}-x\right)\left(1-x\left(z-\sqrt{z^2-1}\right)\right).$$

The generating function $\mathcal{W}^{(k)}(t,j;\beta)=\sum_{n\geq 0}W^{(k)}(n,j;\beta)\,t^n$ is generating the case where the lattice paths stay strictly below $y=k$. The numbers $W^{(k)}(n,j;\beta)$ are the number of paths with $\beta$-weighted horizontal steps of length $w$, and diagonal up (NE) and down (SE) steps, that do not reach the line $y=k$, and stay above the $x$-axis. That means, $0\leq j<k$. We also know $\mathcal{W}^{(k)}(t,0;\beta)$

$$\begin{aligned}
&=\sum_{n\geq 0}W_{n;\beta}^{(k)}t^n=\mu_w(t;\beta)\,\frac{1-(t\mu_w(t;\beta))^{2k}}{1-(t\mu_w(t;\beta))^{2(k+1)}}\\
&=\frac{\left(z-\sqrt{z^2-1}\right)}{t}\frac{1-\left(z-\sqrt{z^2-1}\right)^{2k}}{1-\left(z-\sqrt{z^2-1}\right)^{2(k+1)}}\\
&=\frac{1}{t}\frac{\sum_{i=0}^{(k-1)/2}\binom{k-1-i}{k-1-2i}(-1)^i(2z)^{k-1-2i}}{\sum_{i=0}^{k/2}\binom{k-i}{k-2i}(-1)^i(2z)^{k-2i}}\\
&=\frac{U_{k-1}(z)}{tU_k(z)},
\end{aligned}\tag{16}$$

which is the same formula as for $\sum_{n\geq 0}M_{n;\beta}^{(k)}t^n$ in (9), only $z$ depends on the horizontal step length $w$.

The recursion for $\mathcal{W}^{(k)}(t,j;\beta)$ is the same as for $\mathcal{W}(t,j;\beta)$, only the initial values have changed (see $\mathcal{W}^{(k)}(t,0;\beta)$ above).

We get

$$\mathcal{W}^{(k)}(t,j;\beta)=\frac{U_{k-1}(z)}{tU_k(z)}U_j(z)-\frac{U_{j-1}(z)}{t}=\frac{U_{k-1}(z)\,U_j(z)-U_k(z)\,U_{j-1}(z)}{tU_k(z)}.\tag{17}$$

This form of $\mathcal{W}^{(k)}(t,j;\beta)$ shows clearly that $\mathcal{W}^{(k)}(t,k;\beta)=0$. This is the only form for $\mathcal{W}^{(k)}(t,j;\beta)$, because $W^{(k)}(n,j;\beta)$ is not a Riordan matrix.

## 6. Schröder Numbers

If $w=2$, then every horizontal steps gains two units. We denote the number of paths to $(n,j)$ by $S(n,j;\beta)$, which are the weighted (large) Schröder numbers.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $j\uparrow$ | | | | | $1$ | $0$ | $5+5\beta$ |
| $3$ | | | | $1$ | $0$ | $4+4\beta$ | $0$ |
| $2$ | | | $1$ | $0$ | $3+3\beta$ | $0$ | $9+15\beta+6\beta^2$ |
| $1$ | | $1$ | $0$ | $2+\beta$ | $0$ | $5+8\beta+3\beta^2$ | $0$ |
| $0$ | $1$ | $0$ | $1+\beta$ | $0$ | $2+3\beta+\beta^2$ | $0$ | $5+10\beta+6\beta^2+\beta^3$ |
| $n\rightarrow$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ |

The weighted Schröder numbers $S(n,j;\beta)$. The numbers $S_{n;\beta}$ are in row 0.

The following matrix contains the "*compressed*" Schröder numbers by removing the zeroes and shifting all entries into the empty places to the left. This is the same effect as replacing $t^2$ in $\sum_{n=0}^{\infty} S(n,j;\beta)\,t^j$ by $t$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 6 & 4 & 1 & 0 & 0 \\ 22 & 16 & 6 & 1 & 0 \\ 90 & 68 & 30 & 8 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 2 & -4 & 1 & 0 & 0 \\ -2 & 8 & -6 & 1 & 0 \\ 2 & -12 & 18 & -8 & 1 \end{pmatrix}$$

Compressed Schröder numbers ($\beta=1$) Inverse compressed Schröder numbers

The power series $\sum_{n=0}^{\infty} S(n,j;\beta)\,t^j$ is given in (11) and Lemma 4. For the compressed Schröder numbers $\hat{S}(n,j;\beta)$ this equation says (with $\hat{z}=(1-\beta t)/(2\sqrt{t})$)

$$\begin{aligned}\widehat{\mathcal{S}}(t,j;\beta) &:= \sum_{n\geq 0}\hat{S}(n,j;\beta)\,t^j = \frac{\hat{z}-\sqrt{\hat{z}^2-1}}{\sqrt{t}}U_j(\hat{z}) - U_{j-1}(\hat{z})/\sqrt{t} \\ &= \left(\frac{\hat{z}-\sqrt{\hat{z}^2-1}}{\sqrt{t}}\right)^{j+1}, \text{ and thus}\end{aligned}$$

$$\sum_{n\geq 0}\sum_{j\geq 0}\widehat{\mathcal{S}}(n,j;\beta)\,t^j x^n = \frac{1-\beta t-2xt-\sqrt{(1-\beta t)^2-4t}}{2\left(x^2t-x\sqrt{t}+x\beta t+t\right)}.$$

Note that

$$\widehat{\mathcal{S}}^{(k)}(t;\beta) = \sum_{n\geq 0}\hat{S}_n^{(k)}t^n = \frac{U_{k-1}(\hat{z})}{\sqrt{t}U_k(\hat{z})} \tag{18}$$

by (16). Here $U_n(\hat{z}) = \sum_{k=0}^{n/2}\binom{n-k}{k}(-1)^k\, 2^{n-2k}\hat{z}^{n-2k}$.

### 6.1. Inverse Schröder Numbers

We did not mention the inverse of the $W(n,j;\beta)$ matrix for general horizontal step length $w$. Explicitly finding this inverse in its general form is difficult. We now approach this task for $w=2$, the Schröder numbers. From (11) we see that for the uncompressed Schröder numbers

$$\mu_2(t;\beta)=\left(z-\sqrt{z^2-1}\right)/t=1+\beta t^2\mu_2(t;\beta)+t^2\mu_2(t;\beta)^2$$

holds with $z=\frac{1-\beta t^2}{2t}$. Hence it holds for the inverse $t/\phi_2(t)$ of $t\mu_2(t;\beta)$ that

$$\begin{aligned}\phi_2(t\mu_2(t;\beta))&=\mu_2(t;\beta)=1+\beta t^2\mu_2(t;\beta)+t^2\mu_2(t;\beta)^2\\&=1+\frac{\beta t^2\mu_2(t;\beta)^2}{\mu_2(t;\beta)}+t^2\mu_2(t;\beta)^2=1+\frac{\beta(t\mu_2(t;\beta))^2}{\phi_2(t\mu_2(t;\beta))}+(t\mu_2(t;\beta))^2.\end{aligned}$$

This quadratic equation for $\phi_2(t)$ is easily solved:

$$\phi_2(t)=\frac{1}{2}+\frac{1}{2}t^2+\frac{1}{2}\sqrt{(1+t^2)^2+4t^2\beta},$$

which is also a generating function in $t^2$. However, we cannot simply replace $t^2$ by $t$ in $\phi_2(t)$ to get the inverse generating function of the compressed Stirling numbers, because of the multiplication by $t$ of $\hat{\mu}(t)=1+\beta t\mu_2\left(\sqrt{t};\beta\right)+t\mu_2\left(\sqrt{t};\beta\right)^2$. The compressed generating function

$$t\hat{\mu}(t)=\sqrt{t}\left(\frac{(1-\beta t)}{2\sqrt{t}}-\sqrt{\frac{(1-\beta t)^2}{4t}-1}\right)=\frac{1}{2}\left(1-\beta t-\sqrt{(1-\beta t)^2-4t}\right)$$

has the compositional inverse $t/\hat{\phi}(t)=t\frac{1-t}{1+\beta t}$, as can be easily checked. Thus for $\beta=1$, $\hat{\phi}(t)$ is the generating function of the Mittag-Leffler polynomials.

$$\begin{pmatrix}1&0&0&0\\1+\beta&1&0&0\\2+3\beta+\beta^2&2+2\beta&1&0\\5+10\beta+6\beta^2+\beta^3&5+8\beta+3\beta^2&3+3\beta&1\\\vdots&\vdots&\vdots&\vdots\end{pmatrix}^{-1}$$

$$=\begin{pmatrix}1&0&0&0&0\\-1-\beta&1&0&0&0\\\beta+\beta^2&-2-2\beta&1&0&0\\-\beta^2-\beta^3&1+4\beta+3\beta^2&-3-3\beta&1&0\\\beta^3+\beta^4&-2\beta-6\beta^2-4\beta^3&3+9\beta+6\beta^2&-4-4\beta&1\\\vdots&\vdots&\vdots&\vdots&\vdots\end{pmatrix}$$

The compressed Stirling matrix and its inverse $\left(\hat{s}_{n,k}\right)_{n,k\geq 0}$

For $\beta = 1$, the inverse matrix is A080246 in [14]. Within the same reference we find the generating function of the $k$-th column:

$$\sum_{n\geq 0} \hat{s}_{n+k,k} t^n = \left(\frac{1-t}{1+t}\right)^{k+1} \quad \text{if } \beta = 1.$$

For general $\beta$, the inverse matrix $(\hat{s}_{n,k})$ is a Riordan array, and

$$\sum_{n\geq k} \hat{s}_{n,k} t^n = t^k \left(\frac{1-t}{1+\beta t}\right)^{k+1},$$

the column generating function. The row generating polynomials are

$$\hat{s}_n(t) = \sum_{k=0}^{n} \hat{s}_{n,k} t^{n-k} = \sum_{k=0}^{n} t^k \sum_{j=0}^{k} \binom{n-k+1}{j} (-1)^k (1+\beta)^j \binom{k-1}{k-j} \beta^{k-j}.$$

Hence

$$\begin{aligned}
\hat{s}_{n,n-k} &= \sum_{j=0}^{k} \binom{n-k+1}{j} (1+\beta)^j \binom{k-1}{k-j} \beta^{k-j} \\
&= (-1)^k \sum_{j=0}^{k} \binom{n+1-2j}{k-j} \frac{n-k+1}{n-j+1} \binom{n-j+1}{j} \beta^{k-j},
\end{aligned}$$

and

$$\begin{aligned}
\hat{s}(t;x) &= \sum_{n\geq 0} \hat{s}_n(t) x^n = \sum_{n=0}^{\infty} x^n \left(\frac{1-xt}{1+\beta xt}\right)^{n+1} \\
&= (1-xt)\, U(x\sqrt{t}; \hat{z}) = (1-xt) \sum_{n=0}^{\infty} U_n(\hat{z}) \left(xt^{1/2}\right)^n.
\end{aligned}$$

Thus we can write the row generating polynomial $\hat{s}_n(t)$ as

$$\hat{s}_n(t) = t^{n/2} U_n(\hat{z}) - t^{(n+1)/2} U_{n-1}(\hat{z}). \tag{19}$$

## 7. Schröder Numbers in a Band

The main tool for expressing bounded Schröder numbers are of course the Chebyshev polynomials of the second kind. We will show that the inverse Schröder numbers can also be of (limited) value.

We have seen in (17) that

$$\widehat{\mathcal{S}}^{(k)}(t,j;\beta)=\sum_{n\geq 0}\hat{S}(n,j;\beta)\,t^{n}=\sqrt{t}^{j-1}\frac{U_{k-1}(\hat{z})}{U_{k}(\hat{z})}U_{j}(\hat{z})-\sqrt{t}^{j-1}U_{j-1}(\hat{z})$$
$$=\sqrt{t}^{j-1}\frac{U_{k-1}(\hat{z})\,U_{j}(\hat{z})-U_{k}(\hat{z})\,U_{j-1}(\hat{z})}{U_{k}(\hat{z})},$$

where $\hat{z}=(1-\beta t)/(2\sqrt{t})$, as before. Hence the second term, $-\sqrt{t}^{j-1}U_{j-1}(\hat{z})$, is a polynomial of degree at most $j-1$ in $t$, which effects only the coefficients $\hat{S}(n,j;\beta)$ where $n<j$, i.e., $\hat{S}(n,j;\beta)=0$. Hence we can say that the *power series part* of $t^{-j}\mathcal{S}^{(k)}(t,j;\beta)$ equals $\sqrt{t}^{-j-1}\frac{U_{k-1}(\hat{z})}{U_{k}(\hat{z})}U_{j}(\hat{z})$. In (19) we found

$$\hat{s}_{n}(t)=t^{n/2}U_{n}(\hat{z})-t^{(n+1)/2}U_{n-1}(\hat{z}).$$

Hence $t^{-k}\mathcal{S}^{(k)}(t;\beta)\,\hat{s}_{k-1}(t)=$

$$t^{-k}\left(\frac{U_{k-1}(\hat{z})}{U_{k}(\hat{z})}\left(t^{(k-2)/2}U_{k-1}(\hat{z})-t^{(k-1)/2}U_{k-2}(\hat{z})\right)\right)$$
$$=t^{(-k-2)/2}\frac{U_{k-1}(\hat{z})\,U_{k-1}(\hat{z})}{U_{k}(\hat{z})}-t^{(-k-1)/2}\frac{U_{k-1}(\hat{z})\,U_{k-2}(\hat{z})}{U_{k}(\hat{z})}$$
$$\equiv t^{-k}\left(\mathcal{S}^{(k)}(t,k-1;\beta)-t\mathcal{S}^{(k)}(t,k-2;\beta)\right)$$
$$\equiv t^{-k}\sum_{n\geq 0}\left(\hat{S}(n,k-1;\beta)-\hat{S}(n-1,k-2;\beta)\right)t^{n}$$
$$\equiv t^{-k}\sum_{n\geq 0}\left(\hat{S}(n-1,k-1;\beta)\right)t^{n}$$
$$\equiv t^{-k}\mathcal{S}^{(k)}(t,k-1;\beta),$$

where $f(t)\equiv g(t)$ means the power series parts of $f(t)$ equals the power series part of $g(t)$. Note that the recursion $\hat{S}(n-1,k-1;\beta)=\hat{S}(n,k-1;\beta)-\hat{S}(n-1,k-2;\beta)$ only holds when $j=k-1$, which is the top nonzero row, because $\hat{S}(n,k;\beta)=0$ for all $n$.

| $j\uparrow$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $k=4$ | | | | | 0 | 0 | 0 | 0 | 0 |
| 3 | | | | 1 | 7 | 36 | 168 | 756 | 3353 |
| 2 | | | 1 | 6 | 29 | 132 | 588 | 2597 | 11 430 |
| 1 | | 1 | 4 | 16 | 67 | 288 | 1253 | 5480 | 24 020 |
| 0 | 1 | 2 | 6 | 22 | 89 | 377 | 1630 | 7110 | 31130 |
| $n\rightarrow$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

The compressed bounded ($k=4$) Schröder numbers ($\beta=1$)

We have proven the following theorem:

**Theorem 5** *The power series part of* $t^{-k}\mathcal{S}^{(k)}(t;\beta)\,\hat{s}_{k-1}(t)$ *is* $t^{-k}\mathcal{S}^{(k)}(t,k-1;\beta)$.

**Examples** (a) $t^{-4}\mathcal{S}^{(4)}(t;1)\,\hat{s}_3(t) = \frac{(t-1)\left(2t^3-8t^2+6t-1\right)\left(1-4t+t^2\right)}{(1-7t+13t^2-7t^3+t^4)t^4}$ $= t^{-4}\left(t\frac{U(3,\hat{z})U(3,\hat{z})}{U(4,\hat{z})} - t^{3/2}U(3,\hat{z})\,U(2,\hat{z})\,/U(4,\hat{z})\right) = \left(t^{-4}-4t^{-3}+2t^{-2}\right)+1+7t+36t^2+168t^3+756t^4+3353t^5+O\left(t^6\right)$.

(b) $t^{-4}\mathcal{S}^{(4)}(t,3;1) = \sqrt{t}^{-4}\frac{U(3,\hat{z})}{U(4,\hat{z})}U(3,\hat{z}) = \left(t^{-3}-3t^{-2}+t^{-1}\right)+1+7t+36t^2+168t^3+756t^4+3353t^5+O\left(t^6\right)$.

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