LONG MINIMAL ZERO-SUM SEQUENCES IN THE GROUP $C_2 \oplus C_{2k}$

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Abstract

A sequence in an additively written abelian group is called a minimal zero-sum sequence if its sum is the zero element of the group and none of its proper subsequences has sum zero. The structure of the longest minimal zero-sum sequences in the group $C_2 \oplus C_{2k}$ is known. Their length is equal to $2k+1$. We characterize the minimal zero-sum sequences in $C_2 \oplus C_{2k}$ $(k \geq 3)$ with lengths at least $2\lfloor k/2 \rfloor + 4$. In particular the characterization theorem covers sequences whose lengths are just a bit greater than half of the maximum possible one.

The characterization cannot be extended in the same form to shorter sequences. The argument is based on structural results about minimal zero-sum sequences in cyclic groups.

1. Introduction

A non-empty sequence in an additively written abelian group G is called a *minimal zero-sum sequence* if its terms add up to the zero element of G and all of its proper subsequences have nonzero sums. The *direct Davenport problem* for a finite abelian group G is to find the maximum length $\mathsf{D}(G)$ of a minimal zero-sum sequence in G. The number $\mathsf{D}(G)$ is the *Davenport constant* of the group. The associated *inverse Davenport problem* is to describe the minimal zero-sum sequences of length $\mathsf{D}(G)$. A related inverse zero-sum problem is to characterize all sufficiently long minimal zero-sum sequences over G. The group structure should suggest a reasonable meaning of the words "sufficiently long." This generalized inverse problem is solved for cyclic groups in a sense to be explained below. In this article, we address the same question for the rank-2 group $C_2 \oplus C_{2k}$. A complete solution is available in this probably simplest non-cyclic case.

A characterization and its range depend on the demands imposed on the "long" sequences in $C_2 \oplus C_{2k}$. The characterization theorem for cyclic groups provides well-reasoned suggestions.

Theorem 1. ([6],[10]) *Each minimal zero-sum sequence α of length $\ell \geq \lfloor k/2 \rfloor + 2$ in the cyclic group C_k, $k \geq 3$, has the form $x_1 g, \ldots, x_\ell g$, where g is a term of α that generates C_k and $x_1, \ldots, x_\ell$ are positive integers with sum k.*

The theorem found a number of meaningful and diverse applications; for example see [7], [3], [4]. In search of an analogue for non-cyclic groups we focus attention on the following features. There is a coordinate-form representation $x_1 g, \ldots, x_\ell g$ in Theorem 1, with "coordinates" x_j adding up to the group exponent k. Also α contains a generator g of C_k, and hence it has a term of maximum order. It appears desirable to have analogous attributes for the "long" minimal zero-sum sequences in $C_2 \oplus C_{2k}$. Length $2\lfloor k/2 \rfloor + 4$ is sufficient to ensure them. We chose one of the equivalent ways to state such a result.

A Characterization Theorem for the Group $\boldsymbol{C_2 \oplus C_{2k}}$. *Let $G = C_2 \oplus C_{2k}$ where $k \geq 3$, and let α be a sequence in G with length $\ell \geq 2\lfloor k/2 \rfloor + 4$. Then α is a minimal zero-sum sequence if and only if there exist a term a of α with order $2k$, a basis $\{e, a\}$ of G containing a, with $\operatorname{ord}(e) = 2$, and a representation $(y_1 e + z_1 a), \ldots, (y_\ell e + z_\ell a)$ of α with $y_j \in \{0, 1\}$, $z_j \in \mathbb{Z}$, so that:*

(i) $\sum_{j=1}^{\ell} y_j$ *is even;*

(ii) $0 < z_j < k$ *if* $y_j = 0$ *($1 \leq j \leq \ell$) and*
$0 < z_i + z_j \leq k$ *if* $y_i = y_j = 1$ *($1 \leq i < j \leq \ell$);*

(iii) $\sum_{j=1}^{\ell} z_j = 2k$.

The threshold length is $2\lfloor k/2 \rfloor + 4$, i.e., $k + 4$ if k is even and $k + 3$ if k is odd. Thus the characterization includes sequences with lengths just a little greater than half of the Davenport constant $\mathsf{D}(C_2 \oplus C_{2k})$, which is known to be equal to $2k + 1$.

We refer to Theorem 1 not only for the spirit of the characterization in $C_2 \oplus C_{2k}$ but also for a proof. The general idea is to reduce the essential part of the argument to a problem in the subgroup $2G = \{2x \mid x \in G\}$, which is cyclic of order k. Divide α into blocks with sums in $2G$ and replace each block by its sum. The division yields a minimal zero-sum sequence β in $2G \cong C_k$. We aim to apply Theorem 1 to such sequences β, and then draw conclusions about the original α. To obtain β long enough we need a division with blocks as short as possible. Such divisions exist because the factor group $G/2G$ is an elementary 2-group, namely $C_2 \oplus C_2$. So every two terms of α from the same proper $2G$-coset have sum in $2G$, and we can ensure that almost all nontrivial blocks in β have length 2.

In principle the outlined approach is widely popular. We mention only its related recent applications in [8] and [1], where the main point of interest are the longest minimal zero-sum sequences in a certain group. The respective sequences β are long enough to be analyzed directly. In contrast, our goal is a complete characterization in a wide length range, and some of the involved sequences β are rather short. So not only the full strength of Theorem 1 is required but also additional considerations for the limit cases.

The proof would have been significantly shorter and more transparent if only lengths $\ell \geq k+5$ were considered. Broadly speaking, Theorem 1 suffices for a neatly shaped argument in this reduced "essentially optimal" length range. However most of the substance is to be found in the limit cases $\ell = k+4$ with k even and especially $\ell = k+3$ with k odd. They are the subtle ones. We observe one more time that a difference in length of 1 or 2 may essentially mean a rather substantial difference in zero-sum settings.

The idea is developed in Section 3, which contains the core of the argument. Along with Theorem 1 we need related properties of long minimal zero-sum sequences in cyclic groups. They are presented in Section 2. Section 4 contains a proof of the characterization theorem for $C_2 \oplus C_{2k}$ and examples to the effect that the range of characterization is optimal.

We regard sequences as multisets because the ordering of terms is irrelevant to our purpose. The notation is multiplicative; for example $(e+a)^2a^3e$ is the sequence with terms $e+a, e+a,\ a, a, a, e$. The length, the sum and the sumset of a sequence α are denoted by $|\alpha|$, $S(\alpha)$ and $\Sigma(\alpha)$ respectively. We write $\beta \mid \alpha$ for a subsequence β of α, and $\alpha\beta^{-1}$ is the complementary subsequence of β.

2. Basis of a Sequence in a Cyclic Group

We start with two lemmas for positive integer sequences.

Lemma 2. *Let γ be a sequence with positive integer terms, sum $k \geq 3$ and length $\ell \geq \lfloor k/2 \rfloor + 2$.*

a) If μ is the multiplicity of the term 1 in γ then $\mu \geq 2\ell - k \geq 3$.

b) If $t \in \gamma$ is an arbitrary term then $t \leq \mu - 1$.

Proof. a) The inequalities $\mu \geq 2\ell - k \geq 3$ follows from the estimate $k \geq \mu + 2(\ell - \mu)$ and the length condition, which we use in the form $2\ell \geq k+3$.

b) For $t = 1$ we have $t < \mu - 1$ as $\mu \geq 3$ by a). If $t \neq 1$ then $k \geq \mu + t + 2(\ell - \mu - 1)$, which rewrites as $\mu \geq t + (2\ell - k) - 2$. Now $2\ell \geq k+3$ yields $t \leq \mu - 1$. □

A notion closely related to Theorem 1 was introduced in [6]. It is relevant here too. A positive integer sequence with sum S is *behaving* if its sumset is $\{1, \ldots, S\}$.

Lemma 3. *A nonempty positive integer sequence γ that satisfies $2|\gamma| \geq S(\gamma)$ is behaving or has one of the forms $\gamma = 2^\ell$ and $\gamma = 1^{\ell-1}(\ell+1)$ with $\ell = |\gamma|$. In particular γ is behaving whenever $2|\gamma| > S(\gamma)$.*

Proof. Let $\gamma = \prod_{j=1}^{\ell} x_j$ where $1 \leq x_1 \leq \cdots \leq x_\ell$. Define $x_0 = 0$, $\gamma_j = x_0 x_1 \ldots x_j$, $0 \leq j \leq \ell$, and denote $S_j = S(\gamma_j) = \sum_{i=0}^{j} x_i$, $\Sigma_j = \Sigma(\gamma_j)$. If γ is not behaving then $\Sigma(\gamma) \neq \{1, \ldots, S(\gamma)\}$ and so $\Sigma_\ell = \Sigma(\gamma_\ell) = \Sigma(\gamma) \cup \{0\} \neq \{0, 1, 2, \ldots, S_\ell\}$. Hence there are indices $j > 0$ such that $\Sigma_j \neq \{0, 1, 2, \ldots, S_j\}$. Choose j to be the least one of them to ensure $\Sigma_{j-1} = \{0, 1, 2, \ldots, S_{j-1}\}$. Because $\Sigma_j = \Sigma_{j-1} \cup \{x_j\} \cup (x_j + \Sigma_{j-1})$, it follows that $\Sigma_j = \{0, 1, 2, \ldots, S_{j-1}\} \cup \{x_j, x_j + 1, \ldots, x_j + S_{j-1}\}$. In addition $x_j + S_{j-1} = S_j > S_{j-1}$, and hence $\Sigma_j \neq \{0, 1, 2, \ldots, S_j\}$ only if $x_j \geq 2 + S_{j-1}$. So if $i \geq j$ then $x_i \geq x_j \geq 2 + S_{j-1} \geq 2 + (j-1) = j+1$. Use $2|\gamma| \geq S(\gamma)$, i. e. $2\ell \geq S_\ell$, to obtain the estimate

$$2\ell \geq S_\ell = \sum_{i=0}^{j-1} x_i + \sum_{i=j}^{\ell} x_i \geq (j-1) + (\ell - j + 1)(j+1).$$

The inequality is equivalent to $(j-1)(j-\ell) \geq 0$ and yields $j = 1$ or $j = \ell$. Since $x_j \geq j+1$, we see that $j = 1$ implies $x_1 \geq 2$; likewise $j = \ell$ implies $x_\ell \geq \ell + 1$. It follows easily that $\gamma = 2^\ell$ if $j = 1$ and $\gamma = 1^{\ell-1}(\ell+1)$ if $j = \ell$. In both cases $S(\gamma) = 2\ell$, and hence γ is behaving if $2|\gamma| > S(\gamma)$. □

Now we introduce terminology and notation convenient for our main purpose.

Definition 4. Let g be a generator of the cyclic group C_k.

- The *g-coordinate* of an element $a \in C_k$ is the unique integer $x_g(a) \in [1, k]$ such that $a = x_g(a)g$.
- The singleton $\{g\}$ is a *basis* of a sequence β in C_k if $\sum_{t \in \beta} x_g(t) = k$.

Sequences with a basis are clearly minimal zero-sum sequences (and they can have any length not exceeding k). Naturally the converse is not true. A "short" minimal zero-sum sequence does not have a basis in general, or it may not contain the basis element as a term if a basis exists.

However for length $\geq \lfloor k/2 \rfloor + 2$ the notions of a minimal zero-sum sequence and a sequence with a basis are equivalent. Moreover the basis of a long minimal zero-sum sequence is unique, and the sequence has terms equal to its basis element. For completeness and convenience of the exposition we include the uniqueness of the basis in the next lemma, which is a more detailed restatement of Theorem 1.

Lemma 5. *Each minimal zero-sum sequence β of length $\ell \geq \lfloor k/2 \rfloor + 2$ in C_k, $k \geq 3$, has a unique basis $\{g\}$ with basis element g a term of β. In addition:*

a) If μ is the multiplicity of the basis element g in β then $\mu \geq 2\ell - k \geq 3$.

b) If $t \in \beta$ is an arbitrary term then $x_g(t) \leq \mu - 1$.

Proof. The existence part is the nontrivial one. The sequence has a basis $\{g\}$ by Theorem 1, with $g \in \beta$. Since $\sum_{t\in\beta} x_g(t) = k$ by definition, parts a) and b) follow from their analogues in Lemma 2. There remains the uniqueness.

Let $\{h\}$ be any basis of β. Since g and h generate C_k, there is an integer $s \in (0, k)$ coprime to k such that $g = sh$. For all $a \in C_k$ we have $a = x_g(a)g = (sx_g(a))h$, meaning that the h-coordinate $x_h(a)$ of a is the least positive remainder of $sx_g(a)$ modulo k. In particular $x_h(g) = sx_g(g) = s$ for each of the μ terms equal to g, due to $0 < s < k$. Hence $s\mu \le k$ as $\sum_{t\in\beta} x_h(t) = k$. Now let $t \ne g$ be any term. Because $x_g(t) \le \mu - 1$ by b), we obtain $0 < sx_g(t) \le s(\mu - 1) < s\mu \le k$; thus $x_h(t) = sx_g(t)$. So the last equality holds for all terms, implying $\sum_{t\in\beta} x_h(t) = s\sum_{t\in\beta} x_g(t)$. Both sums are equal to k, and hence $s = 1$, i.e., $h = g$. $\square$

Remark 6. The proof of Theorem 1 in [6] yields a little more than stated. If an abelian group G with order $k \ge 3$ contains a minimal zero-sum sequence α of length $\ge \lfloor k/2 \rfloor + 2$ then G is isomorphic to the cyclic group C_k and α has a basis $\{g\}$ with basis element g a term of α.

It proves essential that the basis remains unchanged under slight modifications of the sequence. With our approach, this observation is indispensable.

Lemma 7. *Let β be a minimal zero-sum sequence with length $\ell \ge \lfloor k/2 \rfloor + 2$ and basis $\{g\}$ in the cyclic group C_k, $k \ge 3$. Two of its terms are replaced by two or more group elements so that the obtained sequence β' is a minimal zero-sum sequence. Then $\{g\}$ is a basis of β'.*

Proof. Let γ be the sequence of g-coordinates of the terms that are not replaced. The minimality of β' implies $k - x_g(u) \notin \Sigma(\gamma)$ for every newly-added term u. Otherwise there is a subsequence of β' with sum zero in which only u is a newly-added term. However u is not the unique new term, so β' is not minimal contrary to the assumption.

We have $|\gamma| = \ell - 2 \ge \lfloor k/2 \rfloor$; thus $\gamma \ne \emptyset$ as $k \ge 3$. Next, $S(\gamma) \le k - 2$ since $\sum_{t\in\beta} x_g(t) = k$ and the 2 deleted g-coordinates are positive. Observe that $2|\gamma| > S(\gamma)$ because $2\lfloor k/2 \rfloor > k - 2$ for $k \in \mathbb{N}$. So γ is behaving by Lemma 3, with sumset $\Sigma(\gamma) = \{1, \ldots, S(\gamma)\}$. Now the conclusion $k - x_g(u) \notin \Sigma(\gamma)$ for a new term u takes the form $k - x_g(u) > S(\gamma)$. It suffices to note that $k - x_g(u) \ne 0$, by the minimality of β' again.

If the replaced terms are both equal to g then $S(\gamma) = k-2$, and $k-x_g(u) > S(\gamma)$ implies that all new terms are also equal to g. Then the change clearly yields the original sequence and the claim is trivial. So suppose that at most one of the replaced terms is equal to g.

Let β' have basis $\{h\}$; the basis exists by Lemma 5 as $|\beta'| \ge \ell \ge \lfloor k/2 \rfloor + 2$. One has $g = sh$ where $s \in (0, k)$ and $(s, k) = 1$. The multiplicity μ of g in β satisfies $\mu \ge 2\ell - k$ (Lemma 5a). Since the replaced terms are not two g's, β' has at least

$\mu - 1$ terms equal to g. For each one of them we obtain $x_h(g) = s$ like in the proof of Lemma 5, and hence $s(\mu - 1) < k$ in view of $\sum_{t \in \beta'} x_h(t) = k$. The inequality is strict because β' has other terms apart from these $\mu - 1$.

Let $p \geq 2$ group elements replace the 2 removed terms of β. Then $|\beta'| = \ell + p - 2$, so the multiplicity of h in β' is at least $2(\ell + p - 2) - k = 2p + (2\ell - k - 4) \geq 2p - 1$. As $2p - 1 > p$ for $p \geq 2$, β has a term h that is not replaced. We have $x_g(h) \leq \mu - 1$ (Lemma 5b). Then $sx_g(h) \leq s(\mu - 1) < k$ by the previous paragraph. Therefore $x_h(h) = sx_g(h)$, and now $x_h(h) = 1$ implies $h = g$. $\square$

One cannot relax the condition $\ell \geq \lfloor k/2 \rfloor + 2$ in Lemma 7. For odd $k > 5$ set $\beta = g^{(k-3)/2}(2g)\left(\frac{k-1}{2}g\right)$. This is a minimal zero-sum sequence with $|\beta| = \lfloor k/2 \rfloor + 1$ and basis $\{g\}$ (there is no other basis by direct inspection). Replace $(2g)$ and $\left(\frac{k-1}{2}g\right)$ by $\left(\frac{k+1}{2}g\right)^3$ to obtain $\beta' = g^{(k-3)/2}\left(\frac{k+1}{2}g\right)^3$, which is a minimal zero-sum sequence with length $\lfloor k/2 \rfloor + 2$. However β' has basis $\{\frac{k+1}{2}g\}$, not $\{g\}$. It is also essential that exactly two terms of β are changed. Let $\beta = g^{(k-3)/2}(x_1g)(x_2g)(x_3g)$ where $k > 5$ is odd and $x_i > 0$ satisfy $x_1 + x_2 + x_3 = \frac{k+3}{2}$. Here $|\beta| = \lfloor k/2 \rfloor + 2$ and β has basis $\{g\}$. Replacing $(x_1g)(x_2g)(x_3g)$ by $\left(\frac{k+1}{2}g\right)^3$ yields $\beta' = g^{(k-3)/2}\left(\frac{k+1}{2}g\right)^3$ again. We saw that β' has basis $\{\frac{k+1}{2}g\} \neq \{g\}$ although $|\beta'| = |\beta| = \lfloor k/2 \rfloor + 2$.

Lemma 7 is sufficient for the main argument in $C_2 \oplus C_{2k}$ if we deal only with sequence lengths $\ell \geq k + 5$. However the limit cases $\ell = k + 4$ with k even and $\ell = k + 3$ with k odd need additional care.

Lemma 8. *Let β be a minimal zero-sum sequence with length $\ell \geq \lfloor k/2 \rfloor + 2$ and basis $\{g\}$ in the cyclic group C_k, $k \geq 3$. Three of its terms are replaced by two group elements u and v so that the obtained sequence β' is a minimal zero-sum sequence. Suppose that $\{g\}$ is* not *a basis of β'. Then the next conditions are satisfied:*

$$k > 3; \quad k \text{ is odd}; \quad \ell = (k+3)/2; \quad \text{the replaced terms are all equal to } g. \tag{1}$$

Furthermore one of the following holds true for β, u and v:

a) $\beta = g^3(2g)^{(k-3)/2}$ and $x_g(u), x_g(v)$ are even integers greater than 3, with $x_g(u) + x_g(v) = k + 3$;

b) $\beta = g^{(k+1)/2}\left(\frac{k-1}{2}g\right)$ and $x_g(u) = x_g(v) = \frac{k+3}{2}$.

Proof. If $k = 3$ then $|\beta| = 3$ and β' is a 2-term minimal zero-sum sequence in C_3. Such a sequence has basis $\{g\}$ where g is any nonzero element of C_3. Hence $k > 3$.

Let γ be the sequence of g-coordinates of the unchanged terms. Like in the proof of Lemma 7 the minimality of β' implies $k - x_g(u) \notin \Sigma(\gamma)$ and $k - x_g(v) \notin \Sigma(\gamma)$. Next, $S(\gamma) \leq k - 3$ and $|\gamma| = \ell - 3 \geq \lfloor k/2 \rfloor - 1$; thus $\gamma \neq \emptyset$ as $k > 3$. Observe that $2|\gamma| \geq S(\gamma)$ because $2(\lfloor k/2 \rfloor - 1) \geq k - 3$ for $k \in \mathbb{N}$. By Lemma 3 γ is behaving or has one of the forms $\gamma = 2^s$, $\gamma = 1^{s-1}(s+1)$ where $s = |\gamma| = \ell - 3$.

In every case $\sum_{t\in\beta'} x_g(t) = S(\gamma) + x_g(u) + x_g(v)$ is divisible by k as $S(\beta') = 0$. Now $S(\gamma) \in (0, k)$ and $x_g(u), x_g(v) \in (0, k]$ yield $\sum_{t\in\beta'} x_g(t) \in (0, 3k)$. Because $\sum_{t\in\beta'} x_g(t) \neq k$ (since $\{g\}$ is not a basis of β'), it follows that $\sum_{t\in\beta'} x_g(t) = 2k$.

If γ is behaving then $\Sigma(\gamma) = \{1, 2, \ldots, S(\gamma)\}$. Like in the proof of Lemma 7 $k - x_g(u) \notin \Sigma(\gamma)$ yields $k - x_g(u) > S(\gamma)$. We use $x_g(u) \neq k$ here, which is due to the minimality of β'; likewise $x_g(v) \neq k$. Now $k - x_g(u) > S(\gamma)$ and $0 < x_g(v) < k$ yield $0 < S(\gamma) + x_g(u) + x_g(v) = \sum_{t\in\beta'} x_g(t) < 2k$, which is false.

Hence $2|\gamma| = S(\gamma)$ as $2|\gamma| > S(\gamma)$ implies that γ is behaving (Lemma 3). So all inequalities in the chain $2|\gamma| = 2(\ell - 3) \geq 2(\lfloor k/2\rfloor - 1) \geq k - 3 \geq S(\gamma)$ are equalities. This leads to all conditions (1) except the first one, which was already established. Next, $s = \ell - 3 = (k-3)/2$, so $\gamma = 2^{(k-3)/2}$ or $\gamma = 1^{(k-5)/2}\left(\frac{k-1}{2}\right)$. The equalities $\sum_{t\in\beta'} x_g(t) = S(\gamma) + x_g(u) + x_g(v) = 2k$ and $S(\gamma) = k - 3$ give $x_g(u) + x_g(v) = k + 3$. Hence $x_g(u), x_g(v) > 3$ due to $x_g(u), x_g(v) \in (0, k)$.

If $\gamma = 2^{(k-3)/2}$ then $\beta = g^3(2g)^{(k-3)/2}$ as the 3 replaced terms are equal to g. Also $\Sigma(\gamma) = \{2x \mid 1 \leq x \leq \frac{k-3}{2}\}$, so the conditions $k - x_g(u), k - x_g(v) \notin \Sigma(\gamma)$ and $x_g(u), x_g(v) > 3$ imply that $x_g(u), x_g(v)$ are even integers in $[4, k-1]$, with sum $k + 3$. The conclusions lead to case a).

Likewise if $\gamma = 1^{(k-5)/2}\left(\frac{k-1}{2}\right)$ we obtain $\beta = g^{(k+1)/2}\left(\frac{k-1}{2}g\right)$, and in addition $\Sigma(\gamma) = \{1, \ldots, \frac{k-5}{2}\} \cup \{\frac{k-1}{2}, \ldots, k-3\}$. Therefore $k - x_g(u), k - x_g(v) \notin \Sigma(\gamma)$ yield $x_g(u), x_g(v) \in \{1, 2, \frac{k+3}{2}\}$. So $x_g(u) = x_g(v) = \frac{k+3}{2}$ in view of $x_g(u), x_g(v) > 3$, and the outcome is case b).

The sequences β' obtained in a) and b) are indeed minimal zero-sum sequences, and $\{g\}$ is not a basis of either one. $\square$

Lemma 8 is stated with the general length condition $\ell \geq \lfloor k/2\rfloor + 2$, but it is present only for the sake of the limit cases $\ell = \lfloor k/2\rfloor + 2$, with k even and odd. Its part concerning lengths $\ell > \lfloor k/2\rfloor + 2$ is a trivial consequence of Lemma 7. Indeed then $|\beta'| \geq \lfloor k/2\rfloor + 2$, so β' has a unique basis by Lemma 5. On the other hand β can be obtained from β' by removing 2 terms and adding 3. Hence Lemma 7 shows that β and β' have the same basis.

3. Factorizations

Let α be a minimal zero-sum sequence in a group G and H a subgroup of G. Since $S(\alpha) = 0 \in H$, one can partition α into subsequences with sums in H, H-blocks. Replacing each block by its sum gives a minimal zero-sum sequence over H. Call the new sequence an *H-factorization* of α if its blocks are *minimal* in the sense that their projections onto the factor group G/H under the natural homomorphism are minimal zero-sum sequences. The terms of a factorization are the sums of its blocks, but for flexibility of speech we sometimes call terms the blocks themselves.

Factorizations are meaningful if they provide information about the original sequence. In the case of the group $G = C_2 \oplus C_{2k}$ an appropriate choice for H is the subgroup $2G = \{2x : x \in G\}$. The key circumstance is that $2G$ is cyclic of order k and the factor group $G/2G$ is isomorphic to $C_2 \oplus C_2$. Henceforth let U, V, W denote the three proper cosets of $2G$. Then $G/2G = \{2G, U, V, W\}$ and the relations $2U = 2V = 2W = 2G$, $U + V = W$, $V + W = U$, $W + U = V$, $U + V + W = 2G$ hold. There are three kinds of minimal zero-sum sequences in $G/2G$, of lengths $1, 2$ or 3. They consist of: the zero element $2G$; two equal nonzero elements; all three nonzero elements. Hence the $2G$-factorizations of a minimal zero-sum sequence α in G have terms of three kinds: one term of α from $2G$; a sum of two terms from the same proper $2G$-coset, a *pair*; a sum of three terms from the three proper $2G$-cosets, a *triple*. The terms of α in $2G$ are terms of every $2G$-factorization, its *trivial* terms. The *nontrivial* terms of a $2G$-factorization are its pairs and triples; call them also *blocks*. For brevity let us sometimes write *factorization* instead of *$2G$-factorization*.

We consider mostly $2G$-factorizations with a maximum number of pairs and call them *standard*. It follows from the properties of $C_2 \oplus C_2$ that a zero-sum sequence α in $G = C_2 \oplus C_{2k}$ either has an even number of terms in all three proper $2G$-cosets or an odd number of terms in each one of them. In the even case the standard factorizations are obtained by dividing completely into pairs the terms outside $2G$; there are no triples. In the odd case a standard factorization has exactly one triple, the remainder consists of pairs and terms of α from $2G$. Hence the number t of triples in a standard $2G$-factorization of α is 0 or 1, and t depends only on α, not on a particular standard factorization.

> *For the rest of the section fix a minimal zero-sum sequence α with length $|\alpha| \geq 2\lfloor k/2 \rfloor + 4$ in $G = C_2 \oplus C_{2k}$, $k \geq 3$. Let α have d terms in $2G$, and let $t \in \{0, 1\}$ be the number of triples in a standard $2G$-factorization of α.*

The length condition means $|\alpha| \geq k + 4$ if k is even and $|\alpha| \geq k + 3$ if k is odd.

The next lemma establishes a foundation for everything that follows.

Lemma 9. *Every standard $2G$-factorization of α has length $\geq \lfloor k/2 \rfloor + 2$, a unique basis and at least $3 - t$ nontrivial terms equal to its basis element.*

Proof. Let β be a standard $2G$-factorization. Exactly $d + 3t$ terms of α are not in pairs of β. Hence $|\beta| = d + t + \frac{1}{2}(|\alpha| - d - 3t)$ and so

$$2|\beta| = |\alpha| + d - t. \tag{2}$$

Observe that $2|\beta| - k \geq 3$. Otherwise $|\alpha| \geq k + 3$, $d \geq 0$ and $t \leq 1$ imply $|\alpha| = k + 3$, $d = 0$ and $t = 1$. However $|\alpha| = k + 3$ can hold only for k odd (by the length condition), in which case $|\alpha|$ is even; but then (2) is not true with $d = 0$, $t = 1$. Hence $2|\beta| - k \geq 3$ indeed, and so $|\beta| \geq \lfloor k/2 \rfloor + 2$. Because β is a minimal zero-sum sequence in the cyclic group $2G$ of order k, the conclusions of Lemma 5 apply.

Therefore β has a unique basis $\{g\}$ with g a generator of $2G$ and at least $2|\beta| - k$ terms equal to g. Let p of these be trivial and q nontrivial. Then $p + q \geq 2|\beta| - k$ and (2) yield $q \geq 2|\beta| - k - p = (|\alpha| - k) + (d - p) - t$. So the length condition and $d \geq p$ lead to $q \geq 3 - t$. □

In what follows we deal repeatedly with *overlapping blocks*. These are two different blocks of a standard factorization that contain terms of α from the same proper $2G$-coset. We say that such blocks overlap. They can be two pairs of terms of α from the same proper $2G$-coset or a pair and a triple.

Our approach rests entirely on the fact that "slightly different" factorizations share the same basis. More exactly it is crucial that changes of the following two kinds leave intact the basis of a standard factorization β.

A. Two overlapping blocks B_1, B_2 of β are replaced in a natural way by two new blocks to yield another standard factorization. More specifically let B_1 be a pair and B_2 a triple, say $B_1 = u_1u_2$ and $B_2 = uvw$ where $u_1, u_2 \in U$ and $u \in U$, $v \in V, w \in W$. Replacing B_1 and B_2 by the pair uu_2 and the triple u_1vw produces a new standard factorization β'. In the other case B_1 and B_2 are pairs, say $B_1 = u_1u_2$, $B_2 = u_1'u_2'$ with $u_1, u_2, u_1', u_2' \in U$. Then a new standard factorization β' is obtained by removing B_1 and B_2 and adding the pairs u_1u_1' and u_2u_2'.

B. Three pairs u_1u_2, v_1v_2, w_1w_2 of β with terms of α from the three proper $2G$-cosets are replaced by the triples $u_1v_1w_1$ and $u_2v_2w_2$. We use this change only in the case $t = 0$. The new factorization β' is not standard since it has 2 triples; call it *non-standard*. Since β' is obtained from β by removing 3 terms and adding 2 new ones, Lemma 9 implies $|\beta'| = |\beta| - 1 \geq \lfloor k/2 \rfloor + 1$.

For both changes **A** and **B**, the $2G$-factorizations β and β' are minimal zero-sum sequences in the cyclic group $2G \cong C_k$, $k \geq 3$. Furthermore $|\beta| \geq \lfloor k/2 \rfloor + 2$ (Lemma 9). A change **A** removes 2 terms of β and adds 2 new ones to yield β'. By Lemma 7 then the basis $\{g\}$ of β is also a basis of β'.

The same conclusion is needed for changes **B**. Here we rely on Lemma 8 as β' is obtained from β by removing 3 terms and adding 2 new ones. The lemma does serve the purpose whenever at least one of conditions (1) is not satisfied. In particular $|\alpha| \geq k + 4$ readily gives $2|\beta| > k + 3$ in view of (2) and $t = 0$. Hence the third condition (1) does not hold for β, implying that $\{g\}$ is a basis of β'.

So for $|\alpha| \geq k+4$ certain knowledge about cyclic groups is enough to ensure that the basis of β is a basis of β', under both changes **A** and **B**. We mean Lemma 7 for $|\alpha| \geq k + 5$ (see the remark after the proof of Lemma 8) and Lemma 8 for the limit case $|\alpha| = k + 4$ with k even. But the issue about change **B** remains unresolved for $|\alpha| = k + 3$ with k odd, due to the exceptions in Lemma 8. Still, changes **B** do not affect the basis even in this more subtle limit case. However proving so requires also "non-cyclic" considerations. We present them in the next lemma. For uniformity it is stated for an arbitrary change **B**.

Lemma 10. *Suppose that $t = 0$. Let β be a standard $2G$-factorization with basis $\{g\}$, and let u_1u_2, v_1v_2, w_1w_2 be pairs of β with $u_i \in U$, $v_i \in V$, $w_i \in W$, $i = 1, 2$. Replace them by the triples $u_1v_1w_1$ and $u_2v_2w_2$ to obtain a non-standard $2G$-factorization β'. Then $\{g\}$ is a basis of β'.*

Proof. Suppose on the contrary that $\{g\}$ is not a basis of β', which is obtained from β by removing 3 terms and adding 2 new ones. Both β and β' are minimal zero-sum sequences in $2G \cong C_k$, $k \geq 3$, and β has length $\geq \lfloor k/2 \rfloor + 2$ (Lemma 9). Hence Lemma 8 applies. First, conditions (1) are satisfied, namely:

$$k > 3; \quad k \text{ is odd}; \quad |\beta| = (k+3)/2; \quad u_1 + u_2 = v_1 + v_2 = w_1 + w_2 = g.$$

In addition one of the next alternatives holds for β, $x_g(u_1v_1w_1)$ and $x_g(u_2v_2w_2)$:

a) $\beta = g^3(2g)^{(k-3)/2}$ and $x_g(u_1v_1w_1) > 3$, $x_g(u_2v_2w_2) > 3$;

b) $\beta = g^{(k+1)/2}\left(\frac{k-1}{2}g\right)$ and $x_g(u_1v_1w_1) = x_g(u_2v_2w_2) = \frac{k+3}{2}$.

Henceforth we usually write $x_g(u_1u_2)$, $x_g(uvw)$ instead of $x_g(u_1+u_2)$, $x_g(u+v+w)$.

With $t = 0$ identity (2) turns into $2|\beta| = |\alpha| + d$. Since $|\alpha| \geq k + 3$ and $d \geq 0$, the equality $|\beta| = (k+3)/2$ holds only if $d = 0$ (and $|\alpha| = k+3$ with k odd). So β has only nontrivial terms (as $d = 0$), and all of them are pairs (as $t = 0$). Since $k > 3$, in each of the cases a), b) β has a term $\neq g$. In other words a pair of β has sum $\neq g$. Let $w_1'w_2'$ be such a pair and we may assume $w_1', w_2' \in W$. Clearly $w_1'w_2'$ is different from the pairs u_1u_2, v_1v_2, w_1w_2. In a) the only term of β different from g is $2g$, in b) such a term is only $\frac{k-1}{2}g$. So denote $x_g(w_1'w_2') = p$ where $p = 2$ in the first case and $p = \frac{k-1}{2}$ in the second. Since $w_1' + w_2' \neq g = w_1 + w_2$, one of w_1', w_2' is different from one of w_1, w_2. Let $w_1' \neq w_2$.

Replace the pairs $w_1w_2, w_1'w_2'$ of β by w_1w_1', w_2w_2'. This change **A** gives a standard factorization β'' with basis $\{g\}$ like β. So $\sum_{t\in\beta} x_g(t) = \sum_{t\in\beta''} x_g(t) = k$, implying $x_g(w_1w_1') + x_g(w_2w_2') = x_g(w_1w_2) + x_g(w_1'w_2') = p+1$. Hence $x_g(w_1w_1') \leq p$.

Next, replace the pairs u_1u_2, v_1v_2, w_1w_1' of β'' by the triples $u_1v_1w_1, u_2v_2w_1'$. The result is a non-standard factorization δ obtained from β'' by removing 3 terms and adding 2 (a change **B**). Furthermore $w_1 + w_1' \neq w_1 + w_2 = g$, so the 3 replaced terms are not all equal to g. Hence the last of the conditions (1) in Lemma 8 is not satisfied, implying that δ has basis $\{g\}$ as well as β''. It follows that

$$x_g(u_1v_1w_1) + x_g(u_2v_2w_1') = x_g(u_1u_2) + x_g(v_1v_2) + x_g(w_1w_1') = x_g(w_1w_1') + 2.$$

Because $x_g(w_1w_1') \leq p$, this implies $x_g(u_1v_1w_1) \leq p + 1$.

Since $p = 2$ in a) and $p = \frac{k-1}{2}$ in b), we obtain $x_g(u_1v_1w_1) \leq 3$ in the first case and $x_g(u_1v_1w_1) \leq \frac{k+1}{2}$ in the second. However $x_g(u_1v_1w_1) > 3$ in a) and $x_g(u_1v_1w_1) = \frac{k+3}{2}$ in b). The contradiction ends the proof. $\square$

Once Lemma 10 is available, one can summarize:

> *Let β be a standard $2G$-factorization, and let β' be obtained from β by a change* **A** *or a change* **B**. *Then the basis of β is a basis of β'.*

Now we are ready to approach the main argument.

Lemma 11. *There exist a term a of α and a standard $2G$-factorization β such that $a \notin 2G$, $\{2a\}$ is a basis of β and β contains a pair aa.*

Proof. Let β be a standard $2G$-factorization of α in which the multiplicity of the basis element g is a minimum. Suppose in addition that β has two overlapping blocks each with sum g. Then the conclusion follows directly. Indeed let two different blocks B_1, B_2 of β have sum g each and contain terms of α from the same proper $2G$-coset, say U. One of B_1, B_2 is a pair; let $B_1 = u_1u_2$ with $u_1, u_2 \in U$. As for B_2, it contains a term $u \in U$ and either $B_2 = uu'$ with $u' \in U$ (if B_2 is a pair) or $B_2 = uvw$ with $v \in V, w \in W$ (if B_2 is a triple). We show that u_1 and u_2 are equal. So if $a \in U$ is their common value then $2a = u_1 + u_2 = g$ is a basis element of β. Hence the term a satisfies the requirements together with β.

Let $B_2 = uvw$ be a triple. The case of a pair $B_2 = uu'$ is analogous. Replace the blocks u_1u_2, uvw of β by the blocks u_1u, u_2vw (a change **A**). The new standard factorization has basis $\{g\}$ as well as β. Since $u_1+u_2 = u+v+w = g$, the minimum choice of β implies $u_1+u = u_2+v+w = g$. Hence $u_2 = u$, and $u_1 = u$ by symmetry. Thus $u_1 = u_2$ and the claim follows.

So having two overlapping blocks each with sum g in β is a sufficient condition for the conclusion to hold. This condition can be ensured if $t = 1$. Exactly one nontrivial term of β is a triple uvw, the remaining ones are pairs. By Lemma 9 β has at least $3 - t = 2$ nontrivial terms g. One of them is the sum of a pair u_1u_2, say with $u_1, u_2 \in U$. We may assume $u_1 \neq u_2$. If $u + v + w = g$ then u_1u_2 and uvw overlap and the sufficient condition is satisfied. If $u + v + w \neq g$ let $u \neq u_1$ without loss of generality. Swap u and u_1 to obtain a new standard factorization β', with the triple u_1vw and the pair uu_2 instead of uvw and u_1u_2. This is a change **A**, so β' has basis $\{g\}$. Since $u_1 + u_2 = g$, $u + v + w \neq g$ and $u + u_2 \neq u_1 + u_2 = g$, the minimal choice of β implies $u_1+v+w = g$. Moreover the multiplicity of g in β' is a minimum as g occurs the same number of times in β and β'. Since the triple u_1vw of β' has sum g, we are back to the previous case. This is because β has one more pair P with sum g except u_1u_2, unaffected by the change and also present in β'. Hence u_1vw and P overlap, and we are done with the case $t = 1$.

Let $t = 0$, so that all nontrivial terms of β are pairs. Now Lemma 9 provides at least $3 - t = 3$ nontrivial terms g. Let each of the pairs u_1u_2, v_1v_2, w_1w_2 have sum equal to g. We may assume that they represent all three proper $2G$-cosets or else there are overlapping blocks each with sum g again. So let $u_i \in U$, $v_i \in V$, $w_i \in W$, $i = 1, 2$, and $u_1 + u_2 = v_1 + v_2 = w_1 + w_2 = g$. We show that the terms in one of the three pairs are equal, which is enough to complete the proof.

Replace u_1u_2, v_1v_2, w_1w_2 by the triples $u_iv_jw_m$ and $u_{3-i}v_{3-j}w_{3-m}$, with arbitrary $i, j, m \in \{1, 2\}$. Since $t = 0$, this is a change **B**. By Lemma 10 $\{g\}$ is a basis of the obtained non-standard factorization β'. The same is true for β, therefore

$$x_g(u_iv_jw_m) + x_g(u_{3-i}v_{3-j}w_{3-m}) = x_g(u_1u_2) + x_g(v_1v_2) + x_g(w_1w_2) = 3$$

for all $i, j, m \in \{1, 2\}$. Hence $x_g(u_iv_jw_m) \in \{1, 2\}$ for $i, j, m \in \{1, 2\}$. It follows that in each pair u_1u_2, v_1v_2, w_1w_2 the two terms are equal or differ by g. Let e. g. $u_1 \neq u_2$ and $u_1 - u_2 = g$. Since $u_1 + u_2 = g$, we obtain $2u_2 = 0$. So if the terms in a pair are different then one of them has order 2. Suppose that $u_1 \neq u_2$, $v_1 \neq v_2$, $w_1 \neq w_2$. Then α contains an order-2 element of G from each proper $2G$-coset. However this is impossible. In all $C_2 \oplus C_{2k}$ has 3 elements of order 2, their sum is 0. For k even not every proper $2G$-coset contains such an element. For k odd the conclusion means that all order-2 elements are terms of α, contradicting the minimality of the sequence. ☐

The proof of the characterization theorem is based almost exclusively on the next lemma. In a sense the lemma is an alternative formulation of the main result.

Lemma 12. *There exists a term a of α with the following properties:*

$$a \notin 2G \text{ and } \operatorname{ord}(a) = 2k; \tag{3}$$

$$\alpha \text{ has a } 2G\text{-factorization } \beta \text{ such that } \textstyle\sum_{t\in\beta} x_a(t) = 2k; \tag{4}$$

$$0 < x_a(u) < k \text{ for } u \in \alpha \text{ with } u \in \langle a\rangle; \tag{5}$$

$$0 < x_a(v + w) \leq k \text{ for } v, w \in \alpha \text{ with } v, w \notin \langle a\rangle,\ v \neq w. \tag{6}$$

The assumption $v \neq w$ in (6) means that v and w are distinct terms of α; formally $v \in \alpha w^{-1}$. Otherwise v and w may be equal as group elements.

Proof. Let us remark that the a-coordinate $x_a(v+w)$ in (6) is well defined, provided that $\langle a\rangle$ is an index-2 subgroup of G. The latter will be justified shortly.

Fix a term a of α and a standard $2G$-factorization β with the properties stated in Lemma 11: $a \notin 2G$, $\{2a\}$ is a basis of β and β contains a pair aa. We show that a meets the requirements. Clearly $\operatorname{ord}(a) = 2k$ because $2G \cong C_k$ is a proper subgroup of $\langle a\rangle$, generated by $g = 2a$. Hence a satisfies (3). Let $a \in U$, then $\langle a\rangle = 2G \cup U$ is an index-2 subgroup of G, with proper coset $V \cup W$.

Claim 1. If δ is an arbitrary standard $2G$-factorization of α with basis $\{g\}$ then $\sum_{t\in\delta} x_a(t) = 2k$ and $x_a(t) \in (0, k-1]$ for each term $t \in \delta$.

Claim 2. If δ is an arbitrary non-standard $2G$-factorization of α with basis $\{g\}$ then $\sum_{t\in\delta} x_a(t) = 2k$ and $x_a(t) \in (0, k+1]$ for each term $t \in \delta$.

In both claims $\sum_{t\in\delta} x_g(t) = k$, so the g-coordinate $x_g(t)$ of each $t \in \delta$ satisfies $k \geq x_g(t) + |\delta| - 1$. By Lemma 9 and the subsequent discussion $|\delta| \geq \lfloor k/2\rfloor + 2$ in

Claim 1 and $|\delta| \geq \lfloor k/2 \rfloor + 1$ in Claim 2. This yields $2x_g(t) \in (0, k-1]$ in the first case and $2x_g(t) \in (0, k+1]$ in the second.

Because $t = x_g(t)g = x_g(t)(2a) = (2x_g(t))a$ and $2x_g(t) \in (0, 2k)$ holds in either case, the a-coordinate of t is $x_a(t) = 2x_g(t)$. Hence $\sum_{t\in\delta} x_g(t) = k$ can be written as $\sum_{t\in\delta} x_a(t) = 2k$. Furthermore $x_a(t) \in (0, k-1]$ in Claim 1 and $x_a(t) \in (0, k+1]$ in Claim 2, which completes the justification of the claims. Note that in Claim 2 the equality $x_a(t) = k+1$ holds only if k is odd and $|\delta| = \frac{k+1}{2}$.

Claim 1 applies to the fixed standard factorization β, and hence $\sum_{t\in\beta} x_a(t) = 2k$ and condition (4) holds. For (5) and (6) we apply repeatedly the two claims to suitably chosen factorizations. Most of them are obtained from β through changes **A**, only the last one uses a change **B**.

A term $t \in \alpha$ from $2G \subset \langle a\rangle$ is a term of β, and hence $x_a(t) \in (0, k{-}1]$ by Claim 1. Take a term $u \in U \subset \langle a\rangle$. If $u = a$ then $0 < x_a(u) = 1 < k$. If $u \neq a$ let B be the block of β that contains u. Recall that β has a pair aa. Interchange u from B with an a from such a pair (a change **A**) to obtain a standard factorization δ. It has the same basis $\{g\}$ as β. Hence Claim 1 applies to δ, which contains a pair au; so $x_a(au) \in (0, k{-}1]$. In other words $u{+}a = sa$ with $0 < s < k$, implying $x_a(u) \in (0, k)$ (as $u \neq 0$). Condition (5) is justified.

We pass on to (6). Let v_1, v_2 be terms of α in V. If v_1v_2 is a pair of β we refer to Claim 1 right away to obtain $x_a(v_1v_2) \in (0, k{-}1]$. Let v_1, v_2 belong to the blocks B_1, B_2 of β, $B_1 \neq B_2$. Suppose for instance that B_1, B_2 are the pairs v_1v_1', v_2v_2'. Replace them by the pairs v_1v_2, $v_1'v_2'$ (a change **A**). Now v_1v_2 is a pair in a new standard factorization with the same basis $\{g\}$, so Claim 1 yields $x_a(v_1v_2) \in (0, k{-}1]$ again. We proceed similarly if one of B_1, B_2 is a triple. The case of two terms $w_1, w_2 \in W$ is symmetric.

Now let v, w be terms with $v \in V$, $w \in W$. In the case $t = 1$ there is a triple $T = u'v'w'$ in β. If v is not in T, swap v and v'; do the same with w and w' if needed. Then swap u' and one a from a pair aa (such pairs are unaffected by the previous changes). After these changes **A** now avw is a triple in a standard factorization with the same basis $\{g\}$. Hence $x_a(avw) \in (0, k{-}1]$ by Claim 1. This readily yields $x_a(vw) \in (0, k)$.

Finally we justify (6) in the case $v \in V$, $w \in W$, $t = 0$. Let v and w be in the pairs vv' and ww'. Remove these pairs from β together with a pair aa, then add the triples avw and $av'w'$. This is a change **B**; the new factorization β' is non-standard, and Lemma 10 ensures that $\{g\}$ is a basis of β'. Hence Claim 2 yields $x_a(avw) \in (0, k{+}1]$, which leads to $x_a(vw) \in (0, k]$. $\square$

It is not hard to see that the inequality $x_a(v{+}w) \leq k$ in (6) can turn into equality only if $|\alpha| = k+3$, with k odd and $d = t = 0$. Otherwise (6) can be strengthened to $0 < x_a(v+w) < k$.

4. The Characterization Theorem

For readers' convenience we reproduce the statement of the main result.

Theorem 13. Let $G = C_2 \oplus C_{2k}$ where $k \geq 3$, and let α be a sequence in G with length $\ell \geq 2\lfloor k/2 \rfloor + 4$. Then α is a minimal zero-sum sequence if and only if there exist a term a of α with order $2k$, a basis $\{e, a\}$ of G containing a, with $\operatorname{ord}(e) = 2$, and a representation $\alpha = \prod_{j=1}^{\ell}(y_j e + z_j a)$ of α with $y_j \in \{0, 1\}$, $z_j \in \mathbb{Z}$, so that:

(i) $\sum_{j=1}^{\ell} y_j$ is even;

(ii) $0 < z_j < k$ if $y_j = 0$ $(1 \leq j \leq \ell)$ and
$0 < z_i + z_j \leq k$ if $y_i = y_j = 1$ $(1 \leq i < j \leq \ell)$;

(iii) $\sum_{j=1}^{\ell} z_j = 2k$.

Proof. Sufficiency: By (i) and (iii) $S(\alpha) = \left(\sum_{j=1}^{\ell} y_j\right) e + \left(\sum_{j=1}^{\ell} z_j\right) a = 2ka = 0$, and hence α is a zero-sum sequence. Let $\beta | \alpha$ be a nonempty zero-sum subsequence; without loss of generality let $\beta = \prod_{j=1}^{p}(y_j e + z_j a)$ where $0 < p \leq \ell$. Clearly β has an even number of terms $\notin \langle a \rangle$, i.e., with $y_j = 1$. Let them be $a_1, \ldots, a_{2m}$, $m \geq 0$. Since $a_{2i-1} + a_{2i} = (z_{2i-1} + z_{2i})a$ for $1 \leq i \leq m$ and $a_j = z_j a$ for $2m < j \leq p$, we obtain $0 = S(\beta) = \left(\sum_{j=1}^{p} z_j\right) a$. The sum $\sum_{j=1}^{p} z_j$ is nonempty and can be partitioned into pairs of summands $z_{2i-1} + z_{2i}$ with $1 \leq i \leq m$ and single summands z_j with $2m < j \leq p$. Hence $\sum_{j=1}^{p} z_j$ is positive by condition (ii), which gives $z_{2i-1} + z_{2i} > 0$ and $z_j > 0$ respectively. Since $\sum_{j=1}^{p} z_j$ is divisible by $\operatorname{ord}(a) = 2k$, this implies $\sum_{j=1}^{p} z_j \geq 2k$. Suppose that β is proper; then its complementary subsequence is nonempty and has an even number of terms $\notin \langle a \rangle$ by (i). Hence the same argument yields $\sum_{j=p+1}^{\ell} z_j \geq 2k$; however then (iii) is violated. Therefore α is a minimal zero-sum sequence.

Note that the reasoning uses only the inequalities $z_j > 0$ and $z_i + z_j > 0$ from (ii); $z_j < k$ and $z_i + z_j \leq k$ are not needed.

Necessity: Suppose first that α generates G. Choose a term $a \in \alpha$ with the properties from Lemma 12. By (3) $\langle a \rangle$ is an index-2 subgroup of G. If $a \in U$ then $\langle a \rangle = 2G \cup U$ and the proper $\langle a \rangle$-coset $C = V \cup W$ contains an even number of terms of α. Such terms exist as α generates G. By (6) the sum of every two terms $b, c \in C$ can be expressed as $b + c = pa$ with $0 < p \leq k$. Choose b and c so that p is a minimum. Without loss of generality let $c - b = qa$ with $0 \leq q \leq k$. Then $2c = (p + q)a$ and $0 < p + q \leq 2k$. Note that $p + q$ is even as $2c \in 2G$; thus setting $p + q = 2r$ gives $2c = 2ra$ where $0 < r \leq k$. The group element $e = c - ra \in C$ has order 2, and $b = e + (r - q)a$, $c = e + ra$. Also $\{e, a\}$ is a basis of G. We prove that it serves our purpose.

Denote $\alpha = \prod_{j=1}^{\ell} a_j$ with $a_1 = b$, $a_2 = c$. Define $y_1 = 1$, $z_1 = r - q$; then $y_1 e + z_1 a = b = a_1$. For each $j > 1$ let y_j, z_j be the standard coordinates of a_j

in the basis $\{e, a\}$, i.e., the unique pair of integers y_j, z_j such that $y_j \in \{0, 1\}$, $z_j \in [0, 2k)$ and $a_j = y_j e + z_j a$. In particular $y_2 = 1$, $z_2 = r$. By definition $y_j = 1$ if and only if $a_j \notin \langle a \rangle$. So condition (i) holds as α has an even number of terms in the proper coset C. We proceed to show that:

$$z_j = x_a(a_j) \text{ if } y_j = 0 \ (1 \leq j \leq \ell); \tag{7}$$

$$z_i + z_j = x_a(a_i + a_j) \text{ if } y_i = y_j = 1 \ (1 \leq i < j \leq \ell). \tag{8}$$

Then properties (5) and (6) from Lemma 12 will imply condition (ii).

If $y_j = 0$ then $a_j = z_j a$ and $0 < z_j < 2k$ ($0 \leq z_j < 2k$ by definition and $z_j \neq 0$ since clearly $a_j \neq 0$). Thus (7) holds. Next, $y_i = y_j = 1$ implies $a_i + a_j = (z_i + z_j)a$, and hence the inequality $0 < z_i + z_j \leq 2k$ would suffice to establish (8).

Consider a term a_j with $y_j = 1$ and $j \neq 1, 2$. Then $a_j \in C$ and in addition both sums $a_1 + a_j = (r - q + z_j)a$, $a_2 + a_j = (r + z_j)a$ belong to the progression $\{pa, (p+1)a, \ldots, (k-1)a, ka\} = P$. This is due to (6) and the minimum choice of p. Now $0 \leq z_j < 2k$ gives $r - q \leq r - q + z_j < r - q + 2k$; on the other hand $r - q > -k$ as $r > 0$ and $q \leq k$; also $r - q + 2k < 2k + p$ as $p + q = 2r$ and $r > 0$. Hence $-k < r - q + z_j < p + 2k$, showing that $a_1 + a_j \in P$ only if $p \leq r - q + z_j \leq k$. This implies $2r = p + q \leq r + z_j \leq k + q \leq 2k$, and hence $a_2 + a_j \in P$ only if $p \leq r + z_j \leq k$.

In summary $p \leq r - q + z_j$ and $r + z_j \leq k$ yield $r \leq z_j \leq k - r$ whenever $y_j = 1$, $j \neq 1, 2$. Recalling $z_1 = r - q$, $z_2 = r$, we observe that $z_j \geq r$ for $y_j = 1$, $j \neq 1$. Hence $z_i + z_j \geq (r - q) + r = 2r - q = (p + q) - q = p > 0$ whenever $y_i = y_j = 1$, $i \neq j$. Furthermore $z_1 \leq z_2 = r \leq k$ and $z_j \leq k - r < k$ for $y_j = 1$, $j \neq 1, 2$; therefore $z_i + z_j \leq 2k$ for $y_i = y_j = 1$, $i \neq j$. We proved $0 < z_i + z_j \leq 2k$ for $y_i = y_j = 1$, $i \neq j$, which implies (8) and completes the justification of (ii).

We also use (7) and (8) to verify condition (iii). More exactly, let us show that if β is an arbitrary $2G$-factorization of α then $\sum_{t \in \beta} x_a(t) = \sum_{j=1}^{\ell} z_j$. It suffices to check that for each term $t \in \beta$, which is the sum of $1, 2$ or 3 terms of α, the coordinates z_j of the summands forming t add up to $x_a(t)$.

If $t = a_j$ is a term of α in $2G \subset \langle a \rangle$ then $z_j = x_a(a_j)$ by (7). If $t = a_i + a_j$ with $a_i, a_j \in V$ or $a_i, a_j \in W$, then $x_a(t) = x_a(a_i + a_j) = z_i + z_j$ by (8). If $t = a_i + a_j$ with $a_i, a_j \in U \subset \langle a \rangle$, then (7) gives $x_a(a_i) = z_i$, $x_a(a_j) = z_j$. On the other hand property (5) in Lemma 12 implies $x_a(t) = x_a(a_i) + x_a(a_j)$ (as $0 < x_a(a_i), x_a(a_j) < k$). Hence $x_a(t) = z_i + z_j$. Last, let $t = a_m + a_i + a_j$ with $a_m \in U$, $a_i \in V$, $a_j \in W$. Then $x_a(a_m) = z_m$ and $x_a(a_i + a_j) = z_i + z_j$ by (7) and (8) respectively. On the other hand, properties (5) and (6) from Lemma 12 imply $0 < x_a(a_m) + x_a(a_i + a_j) < 2k$ and so $x_a(t) = x_a(a_m) + x_a(a_i + a_j)$. Therefore $x_a(t) = z_m + z_i + z_j$ and the claim follows.

Finally take a $2G$-factorization β of α such that $\sum_{t \in \beta} x_a(t) = 2k$. Such a factorization exists by property (4), Lemma 12. By the above $\sum_{t \in \beta} x_a(t) = \sum_{j=1}^{\ell} z_j$, and hence $\sum_{j=1}^{\ell} z_j = 2k$ and condition (iii) is established.

We are left with the case where α generates a proper subgroup H of $G = C_2 \oplus C_{2k}$. Then $|H| \leq |G|/2 = 2k$ and H contains all terms of α. Now we use Remark 6. Since $|\alpha| \geq k+3 > |H|/2+2$, the remark implies that H is cyclic and that α has basis $\{a\}$ with basis element a term $a \in \alpha$ that generates H. On one hand $\mathrm{ord}(a) \geq |\alpha| \geq k+3$ because α is minimal and entirely contained in $H = \langle a \rangle$; on the other hand $\mathrm{ord}(a) = |H| \leq 2k$. Since $\mathrm{ord}(a)$ divides the exponent $2k$ and $k < \mathrm{ord}(a) \leq 2k$, we obtain $\mathrm{ord}(a) = 2k$.

If $\alpha = \prod_{j=1}^{\ell} a_j$ then $\sum_{j=1}^{\ell} x_a(a_j) = 2k$ by the definition of a basis. Then $\ell \geq k+3$ implies $x_a(a_j) \in (0,k)$ for all j. Since $a \in G$ has maximum order, there is a basis $\{e, a\}$ of G containing a, with $\mathrm{ord}(e) = 2$. (One can take any order-2 element e that is not in $\langle a \rangle$.) For $j = 1, \ldots, \ell$ set $y_j = 0$, $z_j = x_a(a_j)$. Then $a_j = y_j e + z_j a$, $j = 1, \ldots, \ell$, and (i)–(iii) hold trivially. The proof is complete. □

Easy examples show that none of the conditions (i)–(iii) in Theorem 13 follows from the other two, so each one is essential. Likewise the coordinates z_j cannot be defined simply as standard coordinates with respect to a basis. One must allow "exceptional" z-coordinates like z_1 in the proof. They can be negative or zero; consequently the purpose of condition (ii) is to admit at most one non-positive z-coordinate. By the last remark in Section 3 the inequality $0 < z_i + z_j \leq k$ in (ii) is not strict only because of the limit case $\ell = k+3$, with k odd. For $\ell \geq k+4$ it can be strengthened to $0 < z_i + z_j < k$.

The characterization cannot be extended to shorter sequences over $G = C_2 \oplus C_{2k}$, $k \geq 3$. Let $\{e, b\}$ be a basis of G with $\mathrm{ord}(e) = 2$, $\mathrm{ord}(b) = 2k$. The following sequences are minimal zero-sum sequences in G that do not have a representation with the properties from Theorem 13:

For length $\ell = k+2$ with arbitrary $k \geq 3$: $\quad \alpha = (2b)^{k-1}(-b)e(e+3b); \qquad (9)$

For length $\ell = k+3$ with even $k \geq 4$: $\quad \alpha = (3b)^3(e+2b)^{k-1}(e-7b). \qquad (10)$

The justification uses the (trivial) observation that if $\prod_{j=1}^{\ell}(y_j e + z_j a)$ is any representation of a sequence like in Theorem 13 then the coordinates z_j satisfy relations (7) and (8) that surfaced in the proof.

The next example shows that for length $\ell = k+3$ with odd $k \geq 3$ the inequality $0 < z_i + z_j \leq k$ in (ii) cannot be improved to $0 < z_i + z_j < k$:

$$\alpha = b^{k-1}(e+b)^2 e(e+(k-1)b). \tag{11}$$

Let us remark that in all three examples (9)–(11) the sequence generates G.

Remark 14. Theorem 13 yields as a by-product the least length of a minimal zero-sum sequence in $C_2 \oplus C_{2k}$ ($k \geq 3$) that guarantees a term of maximum order. This length is $k+4$ since there are minimal zero-sum sequences of length $k+3$ without terms of order $2k$, at least for some values of k. For example such is the sequence (10) for k divisible by $2 \cdot 3 \cdot 7 = 42$.

As a straightforward application of the characterization theorem let us consider the direct and inverse Davenport problems for $C_2 \oplus C_{2k}$. Both of them are solved for a long time now. The value of the Davenport constant for all groups of rank 2 was determined independently by Olson [5] and Kruyswijk (see [9]). The inverse problem for $C_2 \oplus C_{2k}$ was solved by Gao and Geroldinger [2]. Needless to say, neither result is used in the proof of Theorem 13.

Corollary 15. *The Davenport constant of $G = C_2 \oplus C_{2k}$ is equal to $2k+1$. Each minimal zero-sum sequence of length $2k+1$ in G has one of the following forms:*

a) $a^{2k-1}uv$ where $a \in G$ has order $2k$ and $u, v \in G$ are such that $u, v \notin \langle a \rangle$ and $u + v = a$;

b) $a^p e(e+a)^{2k-p}$ where $\{e, a\}$ is a basis of $C_2 \oplus C_{2k}$ with $\mathrm{ord}(e) = 2$, $\mathrm{ord}(a) = 2k$ and $p \in [3, 2k-3]$ is an odd integer.

Proof. The cases $k = 1$ and $k = 2$ can be checked directly. Let α be a minimal zero-sum sequence of length $\ell \geq 2\lfloor k/2 \rfloor + 4$ in $G = C_2 \oplus C_{2k}$, $k \geq 3$. Then it satisfies the description in Theorem 13. If all terms of α belong to $\langle a \rangle$ then clearly $\ell \leq 2k$. Otherwise there is an even number of terms not in $\langle a \rangle$, let their z-coordinates be $z_1, \ldots, z_{2m}$ with $z_1 = \min_{1 \leq i \leq 2m} z_i$. Only z_1 can be non-positive among all z-coordinates. Fix an $i \in \{2, \ldots, 2m\}$ and regard the sum $z_1 + z_i > 0$ as a single summand in the equality $\sum_{j=1}^{\ell} z_j = 2k$. Then the sum $\sum_{j=1}^{\ell} z_j$ contains $\ell - 1$ positive integer summands adding up to $2k$, therefore $\ell - 1 \leq 2k$. Because $\ell - 1 = 2k$ is known to be possible, the reasoning yields $\mathsf{D}(C_2 \oplus C_{2k}) = 2k + 1$. Moreover $\ell - 1 = 2k$ if and only if all summands are 1's, and $z_1 + z_i = 1$ in particular. The latter must hold for each $i \in \{2, \ldots, 2m\}$. Now it is not hard to infer that a) and b) are the only outcomes for the inverse Davenport problem. □

5. Concluding Remarks

Let us mention implications of the approach to higher-rank groups of the form $G = C_2^{r-1} \oplus C_{2k}$, with $r > 2$. Here the subgroup $2G$ is cyclic of order k and the factor group $G/2G \cong C_2^r$ is an elementary 2-group again. Hence the same general idea applies. One can derive a certain structural conclusion like with rank 2. There are predictable complications in the proof since the structure of $G/2G \cong C_2^r$ for large r is more involved than the one of C_2^2. For similar reasons (as $G/\langle a \rangle \cong C_2^{r-1}$) the structural result does not provide straightforward consequences with the same ease as Theorem 13. For example the Davenport problems do not obtain immediate solutions like with $r = 2$ above. They are only reduced to other problems of a certain kind. However we believe that the restatement is not trivial. The new problems do need separate treatment, yet there is a cautious hope that they are accessible. Here is an example of what can be achieved for rank 5.

It is a basic fact that, for a general abelian group $G = C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 | \cdots | n_r$, the Davenport constant $\mathsf{D}(G)$ is at least $\sum_{j=1}^{r}(n_j - 1) + 1 = \mathsf{D}^*(G)$. For $G = C_2^{r-1} \oplus C_{2k}$ with $r \in \{2, 3, 4\}$ it is known that the equality $\mathsf{D}(G) = \mathsf{D}^*(G)$ holds. Of particular interest is the case $r = 5$ with k odd, where an example shows that $\mathsf{D}(G) > \mathsf{D}^*(G)$. For $r = 5$ the outlined approach solves the direct Davenport problem. The outcome is:

$$\mathsf{D}(C_2^4 \oplus C_{2k}) = \begin{cases} 2k + 4 = \mathsf{D}^*(G) & \text{if } k \text{ is even;} \\ 2k + 5 = \mathsf{D}^*(G) + 1 & \text{if } k \text{ is odd.} \end{cases}$$

About the inverse problem for $r = 5$, it is curious that several essentially different longest minimal zero-sum sequences turn up in the case k even—but there is a unique one in the case k odd. We hope to address these questions elsewhere.

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