PRIMES OF PRESCRIBED CONGRUENCE CLASS IN SHORT INTERVALS

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Received: 7/11/12, Accepted: 9/3/12, Published: 10/19/12

Abstract
Suppose $k \leq 72$ is a positive integer and a is an integer coprime to k. We show that for $x \geq 106706$, the interval $(x, 1.048x]$ contains a prime congruent to a modulo k.

1. The Result

We are often in need of primes that fit specific requirements. Sometimes we want them to be of a certain size. The prototype of a result ensuring the existence of primes of the right size is the Bertrand Postulate, a classical result first proved in 1850 by Chebyshev; it states that for $x \geq 1$, the interval $(x, 2x]$ contains a prime. For a wonderful proof, see [1]. Refinements of this result show that for each $\delta > 0$, there is a constant $B(\delta)$ such that for $x \geq B(\delta)$, $(x, x + \delta x]$ contains a prime.

For some purposes, it is not the *size* of the prime that matters so much as its *remainder* upon division by some fixed integer. In this optic, the celebrated theorem of Dirichlet ensures that for co-prime integers a, k, the congruence class a mod k contains infinitely many primes.

For some applications in number theory, one needs an amalgam of both types of results: we often need primes of the right size *and* of the right congruence class. Estimating the size of the smallest prime in the congruence class a mod k for arbitrary a and k is a very subtle problem, intimately related to the Riemann Hypothesis. In this short note, we focus on a less demanding problem, namely,

> *What can one say about the existence of primes of prescribed congruence classes in short intervals?*

We use standard techniques from analytic number theory (See *e.g.* [2], [5]) and an easily manageable computation to show how to obtain effective existence results of this type for small moduli k. For example, we prove that for $x \geq 7$, the interval $(x, 2x]$ contains a prime congruent to $1 \bmod 4$ as well as a prime congruent to $3 \bmod 4$.

Theorem 1. *Suppose* $1 \leq k \leq 72$*, and* a *is any integer coprime to* k*. If* $x \geq 106706$*, or more precisely if* $x \geq N(k)$ *where* $N(k)$ *is given in Table 1, then the interval* $(x, 1.048x]$ *contains a prime congruent to* $a \bmod k$*.*

Proof. Fix a positive integer $k \leq 72$. We define

$$\theta(x; k, a) = \sum_{\substack{p \equiv a(k) \\ p \leq x}} \log p,$$

where the sum is over the primes p not exceeding x in the congruence class $a \bmod k$. The interval $(x, y]$ contains a prime in the congruence class $a \bmod k$ if and only if $\theta(y; k, a) - \theta(x; k, a) > 0$.

Our proof relies on the explicit estimates of Ramaré-Rumely [5], in which two uniform bounds are given for $\theta(x; k, a)$ in the ranges $x \geq 10^{10}$ and $x < 10^{10}$. We begin by assuming $x \geq 10^{10}$. In [5, Thm. 1], the authors obtain the bound

$$\max_{1 \leq y \leq x} \left| \theta(y; k, a) - \frac{y}{\varphi(k)} \right| \leq \epsilon \frac{x}{\varphi(k)} \tag{1}$$

for $x \geq 10^{10}$ where $\epsilon = 0.023269$ and φ is Euler's phi function (see [5, Table 1]). Applying (1) twice, once with parameter $(1 + \delta)x$ and then again with parameter x, we find

$$\theta(x(1 + \delta); k, a) - \theta(x; k, a) \geq ((1 - \epsilon)(1 + \delta) - (1 + \epsilon)) \frac{x}{\varphi(k)}.$$

To show the left hand side is positive, it suffices to ensure that $\delta - 2\epsilon - \epsilon\delta > 0$, i.e.

$$\delta > \frac{2\epsilon}{1 - \epsilon}.$$

The latter holds as long as $\delta > 0.04765$. In particular, we have shown that for $x \geq 10^{10}$, $(x, 1.048x]$ contains a prime congruent to $a \bmod k$.

For $x \leq 10^{10}$, we can appeal to [5, Thm. 2]:

$$\max_{1 \leq y \leq x} \left| \theta(y; k, a) - \frac{y}{\varphi(k)} \right| \leq 2.072\sqrt{x}, \qquad 1 \leq x \leq 10^{10}. \tag{2}$$

Applying (2) with parameter $1.048x$ as well as with x, we find

$$\theta(1.048x; k, a) - \theta(x; k, a) \geq \frac{.048x}{\varphi(k)} - 2.072(\sqrt{1.048x} + \sqrt{x}).$$

The right hand side is positive as long as

$$\sqrt{x} > \frac{2.072(1+\sqrt{1.048})\varphi(k)}{.048}.$$

Thus, we have shown that

$$\left(\frac{259(5+\sqrt{131/5})\varphi(k)}{30}\right)^2 < x \le 10^{10} \Longrightarrow$$
$$\theta(1.048x;k,a) - \theta(x;k,a) > 0.$$

Since $k \le 72$, we have $\varphi(k) \le 70$, hence $\theta(1.048x;k,a) - \theta(x;k,a) > 0$ as soon as $x \ge 37393267$. The rest of the proof is a finite computation which we carried out in PARI/GP [4]. Namely, we verify that for each $k \le 72$, and each integer x in the range

$$N(k) \le x \le \left(\frac{259(5+\sqrt{131/5})\varphi(k)}{30}\right)^2,$$

the interval $(x, 1.048x]$ contains primes of every eligible congruence class modulo k. Here, $N(k)$ is the optimal lower bound for each modulus k, as listed in Table 1. The largest $N(k)$ for $k \le 72$ occurs for $k = 71$ and has value $N(71) = 106706$, completing the proof of the theorem. Our program in GP/PARI used the tables of primes incorporated into the package and proceeded by reducing all primes in the stated interval modulo k to ensure that all eligible residue classes modulo k were represented. □

1.1. Remarks

1. We note that Ramaré and Rumely [5, Thm. 1] provide effective estimates for quite a few other moduli k, including, for example all composite integers $k \in [73, 112]$. They also give better values of ϵ for larger lower bounds on x, which allow one to obtain similar statements about existence of primes of given residue class in $(x, x+\delta x]$ for smaller values of δ. For instance, if $k \le 72$, one obtains the result that $(x, 1.0175x]$ contains a prime in any congruence class a mod k with a co-prime to k, as long as $x \ge 10^{100}$.

2. In [3], Kadiri shows how to obtain a bound $N(k)$ as above for any k which is "non-exceptional," meaning for which one can prove an appropriate zero-free region for all Dirichlet L-functions of conductor k. Kadiri gives tables of $N(k)$ only for large k, viz. $k \ge 5 \cdot 10^4$. By contrast, our emphasis here is on small moduli k.

3. Table 1 below can be considered a refinement of the table computed by Harborth and Kemnitz in [2].

k	$N(k)$	k	$N(k)$	k	$N(k)$	k	$N(k)$
1	–	19	18246	37	32049	55	49236
2	213	20	5061	38	18246	56	24437
3	532	21	8559	39	22398	57	47421
4	887	22	6156	40	11272	58	24229
5	1793	23	23503	41	44330	59	70736
6	532	24	4859	42	8559	60	19902
7	3732	25	18538	43	45475	61	75246
8	2169	26	7856	44	20498	62	43683
9	3103	27	13962	45	23542	63	53072
10	1793	28	10364	46	23503	64	44320
11	6156	29	24229	47	73003	65	56097
12	1792	30	6429	48	13883	66	18534
13	7856	31	30271	49	60715	67	80335
14	3732	32	16501	50	18538	68	30194
15	6429	33	18534	51	30648	69	46621
16	5589	34	11593	52	27454	70	32040
17	11593	35	32040	53	68864	71	106706
18	3103	36	7013	54	13963	72	26463

Table 1

References

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