CONVOLUTION IDENTITIES FOR STIRLING NUMBERS OF THE FIRST KIND VIA INVOLUTION

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Abstract
We provide bijective proofs of some recent convolution identities for the Stirling numbers of the first kind, which were proven earlier using algebraic methods, by defining appropriate sign-changing involutions.

1. Introduction

The Stirling numbers of the first kind, which we'll denote by $s(n,k)$, may be defined by the generating function

$$x(x-1)\cdots(x-n+1)=\sum_{k=0}^{n}s(n,k)x^k.$$

Thus, they are the connection constants between two of the most fundamental bases of the vector space of polynomials in one variable. They are also given, equivalently, by the triangular recurrence relation

$$s(n+1,k)=s(n,k-1)-ns(n,k), \qquad 1\leqslant k\leqslant n+1,$$

with initial values $s(n,0)=\delta_{n,0}$ and $s(0,k)=\delta_{0,k}$ for all non-negative integers n and k. Note further that $s(n,k)=0$ if $k\geqslant 0$ and $n<k$ or if $n\geqslant 0>k$. Numerous properties of the $s(n,k)$ can be found, for example, in the books [1, Ch. 24], [5], and [6] as well as at [8, A008275]. We use the notation $s(n,k)$ for consistency with [2], though there are some advantages to the bracket notation used in [6] (see also [7]).

The following convolution identities for $s(n,k)$ are the main results in [2] and were established there by algebraic methods using recurrences:

$$\sum_{j=0}^{m}\binom{m}{j}\frac{s(n-j,k+m)}{(n-j)!}=\frac{1}{n!}\sum_{r=0}^{m}(-1)^r s(n-m,k+r)s(m+1,m+1-r), \quad (1)$$

$$\sum_{j=0}^{n-k+1}\binom{k-1+j}{j}m^j s(n,k-1+j)$$
$$=\sum_{r=0}^{k}(-1)^{m+1-k+r}s(m+1,k-r)s(n-m,r), \quad (2)$$

$$\sum_{j=k+m}^{n}\binom{j}{k+m}m^{j-k-m}s(n,j)=n!\sum_{r=0}^{m}\binom{m}{r}\frac{s(n-r,k+m)}{(n-r)!}. \quad (3)$$

They hold for all non-negative integers n, k, and m with $n \geqslant m$ and occur as Theorems 1, 3, and 4, respectively, in [2]. Here, we provide bijective proofs of these identities in the style of [3] and [4] by defining appropriate sign-changing involutions. We will use the main combinatorial interpretation of $|s(n,k)|$ as the number of ways in which to arrange n distinct objects in k cycles. In particular, recall that $c(n,k) := (-1)^{n-k}s(n,k) = |s(n,k)|$ defines the *signless* Stirling number of the first kind, which gives the cardinality of the set consisting of the permutations of $[n] = \{1,2,\ldots,n\}$ having k cycles (see, e.g., [9, p. 18]).

We will use the convention that empty sums take the value zero and empty products the value one. If $n \geqslant 0$, then the binomial coefficient $\binom{n}{k}$ is given by $\frac{n!}{k!(n-k)!}$ if $0 \leqslant k \leqslant n$ and is zero if $k < 0$ or if $k > n$.

2. Combinatorial Proofs

In this section, we provide bijective proofs of identities (1)–(3). Using $s(n,k) = (-1)^{n-k}c(n,k)$, the left-hand side of each one of the identities (1)–(3) may be interpreted as a signed sum of cardinalities of certain classes of combinatorial objects. We then define an involution in each case on the union of these classes which reverses the sign. The set of survivors of the involution (i.e., the set of unmatched objects) will comprise a subset of the objects whose (signed) cardinality is given by the right-hand side of the identity, which establishes it.

2.1. Identity (3)

We first prove identity (3), rewritten slightly in the form

$$\sum_{j=k+m}^{n}\binom{j}{k+m}m^{j-k-m}s(n,j)=\sum_{r=0}^{m}r!\binom{m}{r}\binom{n}{r}s(n-r,k+m). \quad (4)$$

Proof. We will assume further that $n \geqslant 1$ and $0 \leqslant k \leqslant n-m$, for otherwise the identity is trivial. If $k+m \leqslant j \leqslant n$, then let $\mathcal{A}_j$ denote the set of "painted" permutations of $[n]$ having j cycles wherein exactly $k+m$ cycles are unpainted and each of the remaining $j-k-m$ is painted with one of m possible colors, labeled $1, 2, \ldots, m$. Define the sign of $\lambda \in \mathcal{A}_j$ by $sgn(\lambda) = (-1)^{n-j}$ and let $\mathcal{A} = \cup_{j=k+m}^{n} \mathcal{A}_j$. Then the left-hand side of (4) gives the total signed weight of all of the members of $\mathcal{A}$ according to the number of cycles j.

Let $\mathcal{A}^* \subseteq \mathcal{A}$ consist of those permutations in which all painted cycles are singletons and no two cycles are painted the same color. Then the right-hand side of (4) gives the total signed weight of all of the members of $\mathcal{A}^*$ according to the number of colors used to paint cycles. To see this, suppose that exactly r colors are used to paint cycles in $\lambda \in \mathcal{A}^*$. Then there are $\binom{m}{r}$ choices for the colors, $\binom{n}{r}$ choices for the elements of $[n]$ occurring in painted cycles, and $r!$ ways to assign the colors to the cycles. The remaining $n-r$ elements of $[n]$ are then arranged in $k+m$ unpainted cycles, which accounts for the $s(n-r, k+m)$ factor. Note that the sign of λ is $(-1)^{n-(k+m)-r}$.

To complete the proof of (4), it suffices to identify a sign-changing involution of $\mathcal{A} - \mathcal{A}^*$. To do so, suppose $\alpha \in \mathcal{A} - \mathcal{A}^*$ and that i_o is the *smallest* index $i \in [m]$ such that there are at least two elements of $[n]$ belonging to cycles painted by color i. Let $a < b$ denote the two smallest elements of $[n]$ occurring within the cycles of α that are painted with color i_o. Let us assume further that the smallest element is written first within each cycle. If the cycle $(a \cdots b \cdots)$ occurs in α, then replace it with the two shorter cycles $(a \cdots), (b \cdots)$ (both painted color i_o), and if the cycles $(a \cdots)$ and $(b \cdots)$ occur in α, then merge them into one large cycle $(a \cdots b \cdots)$ (of the same color), leaving the rest of α undisturbed. If α' denotes the resulting permutation, then α and α' have opposite sign (as the number of! cycles differs by one) and the mapping $\alpha \mapsto \alpha'$ is an involution of $\mathcal{A} - \mathcal{A}^*$. $\square$

To illustrate the involution, suppose $n = 20$, $k = 1$, $m = 4$, and $j = 11$ and let $\alpha \in \mathcal{A}_{11}$ be given by

$$\begin{aligned}\alpha =&(1,3)^4,\ (2,15)^3,\ (4,5,11,7),\ (6)^4,\ (8,12)^3,\ (9)^3,\ (10),\ (13,18,16),\\ &(14),\ (17)^1, (19,20),\end{aligned}$$

where the color of each painted cycle is denoted by a superscript and the remaining cycles are unpainted. Then $i_o = 3$ and $\alpha' \in \mathcal{A}_{10}$ is given by

$$\begin{aligned}\alpha' =&(1,3)^4,\ (2,15,8,12)^3,\ (4,5,11,7),\ (6)^4,\ (9)^3,\ (10),\ (13,18,16),\ (14),\\ &(17)^1,\ (19,20).\end{aligned}$$

2.2. Identity (1)

We next prove identity (1), rewritten as

$$\sum_{j=0}^{m} \frac{n!}{(n-j)!}\binom{m}{j} s(n-j,k+m) = \sum_{j=0}^{m} s(n-m,k+j)c(m+1,m+1-j). \quad (5)$$

Proof. We will assume further that $n \geqslant 1$ and $0 \leqslant k < n-m$, for otherwise the identity is trivial. Suppose j is fixed, $0 \leqslant j \leqslant m$. Let (α, β) denote an ordered pair, where $\alpha = ((a_1, b_1), (a_2, b_2), \ldots, (a_j, b_j))$ is itself a *sequence* of j ordered pairs in which $a_1, a_2, \ldots, a_j$ are *distinct* elements of $[n]$ (in any order) and each b_i belongs to $[m]$ with $b_1 < b_2 < \cdots < b_j$, and β is a permutation of the set $[n] - \{a_1, a_2, \ldots, a_j\}$ having $k+m$ cycles. Let $\mathcal{B}_j$ denote the set consisting of all such possible ordered pairs (α, β). Note that $|\mathcal{B}_j| = \frac{n!}{(n-j)!}\binom{m}{j}c(n-j, k+m)$. Let members of $\mathcal{B}_j$ have sign $(-1)^{n-j-k-m}$ and let $\mathcal{B} = \cup_{j=0}^{m}\mathcal{B}_j$. Then the sum on the left-hand side of (5) gives the total signed weight of all the members of $\mathcal{B}$.

We describe the members of $\mathcal{B}_j$ more closely. Suppose $(\alpha, \beta) \in \mathcal{B}_j$ is as given above. Let $S = \{b_1, b_2, \ldots, b_j\}$ and $S^c = [m] - S = \{c_1, c_2, \ldots, c_{m-j}\}$, where $c_1 < c_2 < \cdots < c_{m-j}$. For convenience, let $c_0 = 0$ and $c_{m+1-j} = m+1$. Note that for each $i \in [j]$, we have $c_t < b_i < c_{t+1}$ for some uniquely determined index $t \in \{0, 1, \ldots, m-j\}$. Furthermore, we assume that the cycles of the permutation β are written so that the smallest element is first within each cycle, with the cycles arranged from left to right in increasing order according to first elements.

Now let $\mathcal{B}_j^* \subseteq \mathcal{B}_j$ consist of those ordered pairs (α, β) satisfying the following properties:

(i) the first $m-j$ cycles of β from left to right are singletons;

(ii) if $i \in [j]$ and $(a_i, b_i) \in \alpha$, then a_i is less than the first element of the $(t+1)$-st cycle of β, where $c_t < b_i < c_{t+1}$.

Let $\mathcal{B}^* = \cup_{j=0}^{m}\mathcal{B}_j^*$. In Lemma 2.1 below, we show that

$$|\mathcal{B}_j^*| = c(n-m, k+j)c(m+1, m+1-j), \qquad 0 \leqslant j \leqslant m,$$

which implies that the signed weight of all the members of $\mathcal{B}^*$ is given by the right-hand side of (5).

To complete the proof of (5), it is then enough to define a sign-changing involution of $\mathcal{B} - \mathcal{B}^*$. Suppose $\lambda = (\alpha, \beta) \in \mathcal{B} - \mathcal{B}^*$. Let ℓ_o be the *smallest* index $\ell \in [m]$ such that one of the following conditions holds:

(i) $\ell = c_r$ for some $r \in [m-j]$, with the r-th cycle of β containing at least two elements;

(ii) $\ell = b_i$ for some $i \in [j]$, with $c_t < b_i < c_{t+1}$ and a_i larger than the first element of the $(t+1)$-st cycle of β.

If condition (i) holds and $\ell_o = c_{r_o}$, then remove the second element w of the r_o-th cycle and insert the ordered pair (w, c_{r_o}) into α (which can only be done in one way since the sequence of second entries in α increases). If (ii) holds and $\ell_o = b_{i_o}$, then remove the ordered pair (a_{i_o}, b_{i_o}) from α and add a_{i_o} to β just after the first element of the (t_o+1)-st cycle, where $c_{t_o} < b_{i_o} < c_{t_o+1}$. Let λ' denote the member of $\mathcal{B}$ which results from performing either of the above procedures. Then λ and λ' have opposite parity since j, the length of α, changes by one and the mapping $\lambda \mapsto \lambda'$ is seen to be an involution of $\mathcal{B} - \mathcal{B}^*$. □

To illustrate the involution, suppose $n = 24$, $m = 10$, and $k = 1$. We define $\lambda = (\alpha, \beta) \in \mathcal{B}$ as follows, using the notation in the preceding proof. Let $j = 6$ and $S = \{1, 3, 4, 7, 8, 10\}$ so that $b_1 = 1$, $b_2 = 3$, etc., and $c_1 = 2$, $c_2 = 5$, $c_3 = 6$, and $c_4 = 9$. Let α be the *sequence* of ordered pairs (a_i, b_i) given by

$$\alpha = ((2,1),(1,3),(4,4),(5,7),(7,8),(9,10)),$$

and let β be the permutation of $[24] - \{1, 2, 4, 5, 7, 9\}$ given by

$$\beta = (3),(6),(8,15,10),(11,17),(12),(13,16),(14,18),(19),(20),(21,24),(22,23).$$

Then $\lambda \in \mathcal{B} - \mathcal{B}^*$, with $\ell_o = 6$, since it is the smallest index $\ell \in [10]$ for which either condition (i) or (ii) above holds. Note that (i) holds and $r_o = 3$ since $c_3 = 6$. Applying the involution, we obtain $\lambda' = (\alpha', \beta') \in \mathcal{B} - \mathcal{B}^*$ in which

$$\alpha' = ((2,1),(1,3),(4,4),(15,6),(5,7),(7,8),(9,10))$$

and

$$\beta' = (3),(6),(8,10),(11,17),(12),(13,16),(14,18),(19),(20),(21,24),(22,23).$$

Note that λ and λ' have opposite parity since $j = 7$ in λ' and that $\ell_o = 6$ in λ' with condition (ii) holding.

We now prove our claim concerning $|\mathcal{B}_j^*|$. In the proof that follows, we use $[m, n]$ to denote the set $\{m, m+1, \ldots, n\}$ if $m \leqslant n$ are non-negative integers, with $[m, n] = \varnothing$ if $m > n$.

Lemma 2.1 *If $\mathcal{B}_j^*$ is as defined above, then*

$$|\mathcal{B}_j^*| = c(n-m, k+j)c(m+1, m+1-j), \qquad 0 \leqslant j \leqslant m. \tag{6}$$

Proof. We may further assume $j \leqslant n - m - k$, for otherwise $\mathcal{B}_j^* = \varnothing$ and the result is clear. We'll use the same notation as in the proof above when discussing $(\alpha, \beta) \in \mathcal{B}_j^*$. Note that β must have at least $m + 1 - j$ cycles since $m < n$ and suppose that r is the smallest member of the $(m+1-j)$-th cycle of β. Since each

a_i is less than r, as are the elements comprising the first $m-j$ cycles of β, we have $r \geqslant m+1$. On the other hand, we also have $r \leqslant m+1$ since r is the smallest element of the permutation obtained by taking away from β its first $m-j$ cycles; note that this permutation is of size $(n-j)-(m-j) = n-m$, with its elements belonging to $[n]$. Thus, we have $r = m+1$, which implies that the elements of $\{a_1, a_2, \ldots, a_j\}$, taken together with the elements belonging to the first $m-j$ cycles of β, comprise the set $[m]$. Therefore, the remaining cycles of β compr! ise a permutation of the set $[m+1, n]$ having $(k+m)-(m-j) = k+j$ cycles, whence there are $c(n-m, k+j)$ choices for these cycles.

So to complete the proof of (6), we need to show that there are $c(m+1, m+1-j)$ choices for α and the first $m-j$ cycles of β. If $1 \leqslant i \leqslant m+1-j$, then let $J_i = [c_{i-1}+1, c_i - 1]$; note that $J_i = \varnothing$ if $c_i = c_{i-1}+1$. Let $K_i = \{a_s : s \in [j] \text{ with } b_s \in J_i\}$, construed as a sequence in the obvious way; note that K_i is the empty sequence if $J_i = \varnothing$. Let $(d_i), (d_2), \ldots, (d_{m-j})$ denote the first $m-j$ cycles of β from left to right. Let us now form the permutation of $[m+1]$ having $m+1-j$ cycles whose i-th cycle comprises the sequence $d_i K_i$ for each $i \in [m+1-j]$, where $d_{m+1-j} = m+1$ and $d_i K_i$ connotes the sequence obtained by writing d_i before the sequence K_i so that the (new) first element is d_i. Note that here the largest element is written first within the cycles of the resulting permutation. Since the process just described of forming a permutation of $[m+1]$ havin! g $m+1-j$ cycles starting from α and the elements d_i is easily seen to be reversible, there are then $c(m+1, m+1-j)$ choices regarding α and the d_i, which completes the proof. □

We illustrate the correspondence described in the second paragraph of the last proof. Let $m = 11$ and $j = 7$. Suppose $\alpha = ((a_i, b_i))_{1 \leq i \leq 7}$ is given by

$$\alpha = ((6,2), (2,3), (4,4), (9,7), (3,8), (1,10), (11,11)),$$

whence $c_1 = 1$, $c_2 = 5$, $c_3 = 6$, $c_4 = 9$ and $d_1 = 5$, $d_2 = 7$, $d_3 = 8$, $d_4 = 10$. Then the resulting permutation σ of $[12]$ having 5 cycles is given by

$$\sigma = (5), (7,6,2,4), (8), (10,9,3), (12,1,11);$$

note that K_1 and K_3 are both empty.

2.3. Identity (2)

Combining the arguments for (1) and (3) above will give a combinatorial explanation of (2), rewritten as

$$\sum_{j=0}^{n-k+1} \binom{k-1+j}{j} m^j s(n, k-1+j) = \sum_{j=0}^{k} c(m+1, k-j) s(n-m, j). \qquad (7)$$

Proof. We will assume further that $1 \leqslant k \leqslant n+1$, for otherwise the identity is trivial. If $0 \leqslant j \leqslant n-k+1$, then let $\mathcal{C}_j$ denote the set of permutations of $[n]$ having $k-1+j$ cycles wherein j cycles are each painted with one of m possible colors and the remaining $k-1$ cycles are unpainted. Applying the involution used to show (4) above to $\mathcal{C} = \cup_{j=0}^{n-k+1}\mathcal{C}_j$ yields

$$\sum_{j=0}^{n-k+1}\binom{k-1+j}{j}m^j s(n,k-1+j) = \sum_{j=0}^{m} j!\binom{m}{j}\binom{n}{j}s(n-j,k-1).$$

Note that the right-hand side of this identity gives the total signed weight of the set of survivors $\mathcal{C}^*$ of the involution, which consists of those members of $\mathcal{C}$ whose painted cycles are all singletons with no two cycles painted the same color. Applying now a slightly modified version of the involution used to show (5) above to $\mathcal{C}^*$ yields

$$\sum_{j=0}^{m} j!\binom{m}{j}\binom{n}{j}s(n-j,k-1) = \sum_{j=0}^{k} c(m+1,k-j)s(n-m,j).$$

Combining the two involutions then gives (7). □

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