ON FINITE SUMS OF GOOD AND SHAR THAT INVOLVE RECIPROCALS OF FIBONACCI NUMBERS

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Abstract
Recently Nathaniel Shar presented a finite sum, involving the Fibonacci numbers, that generalizes a classical result first considered by I. J. Good and others. In this paper we provide a generalization of Shar's sum. Furthermore, we give an analogue for the Lucas numbers. Finally we note that our generalization of Shar's sum and its analogue for the Lucas numbers carry over to certain one parameter generalizations of the Fibonacci and Lucas numbers.

1. Introduction

The Fibonacci numbers are defined, for all integers n, by

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, \quad F_1 = 1.$$

The Lucas numbers are defined, for all integers n, by

$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, \quad L_1 = 1.$$

Set $\alpha = \left(1+\sqrt{5}\right)/2$, which is the positive root of $x^2 - x - 1 = 0$. Then the closed forms (the Binet forms) for F_n and L_n are

$$F_n = \left(\alpha^n + (-1)^{n+1}\alpha^{-n}\right)/\sqrt{5}, \;\; L_n = \alpha^n + (-1)^n \alpha^{-n}.$$

The finite sum

$$\sum_{i=1}^{n-1} \frac{1}{F_{2^{i+1}}} = 1 - \frac{F_{2^n-1}}{F_{2^n}}, \; n > 1, \tag{1}$$

was first obtained by Good[1], and appears as identity (90) in Vajda[6, page 182]. The more general sum

$$\sum_{i=1}^{n-1} \frac{1}{F_{k2^{i+1}}} = \frac{F_{k2^n-2k}}{F_{2k}F_{k2^n}}, \; n > 1, \tag{2}$$

in which k is a positive integer, was considered by Hoggatt and Bicknell[4], and later by Greig[2]. However, these authors expressed the right side of (2) in a manner that is not as succinct as that given here.

Recently Shar[5] proved the following theorem.

Theorem 1. *Let $\{a_i\}$, $i \geq 1$, be an increasing sequence of even positive integers with $a_1 = 2$. Then*

$$\sum_{i=1}^{n-1} \frac{F_{a_{i+1}-a_i}}{F_{a_i}F_{a_{i+1}}} = 1 - \frac{F_{a_n-1}}{F_{a_n}},\ n > 1. \tag{3}$$

Shar's goal was to provide a so called bijective proof of (1), and in the process extended his proof to establish (3). Clearly (3) is a significant generalization of (1). Note however that the sum (3) does not generalize (2).

In Section 2 we generalize Shar's result by relaxing the restrictions on the sequence $\{a_i\}$. In Section 3 we give an analogue of Shar's result for the Lucas numbers. For the Lucas analogue the sequence $\{a_i\}$ is replaced by an arithmetic sequence of integers in which the common difference is even. Finally, in Section 4, we note that our generalization of Shar's sum and its analogue for the Lucas numbers carry over to certain one parameter generalizations of the Fibonacci and Lucas numbers.

2. The Main Result

Let m and n be integers. We require the identities

$$F_{-n} = (-1)^{n+1} F_n, \tag{4}$$

and

$$F_{n+m+1} = F_{n+1}F_{m+1} + F_nF_m. \tag{5}$$

Identities (4) and (5) occur on pages 28 and 59, respectively, in [3]. In (5), upon replacing m with $-m-1$ we obtain

$$F_{n-m} = (-1)^m \left(F_nF_{m+1} - F_{n+1}F_m\right). \tag{6}$$

Our main result is contained in the theorem that follows, where the terms of the sequence $\{a_i\}$ may be positive or negative.

Theorem 2. *Let $\{a_i\}$, $i \geq 1$, be any sequence of integers where no term is zero, and where all terms have the same parity. Then*

$$\sum_{i=1}^{n-1} \frac{F_{a_{i+1}-a_i}}{F_{a_i}F_{a_{i+1}}} = \frac{F_{a_n-a_1}}{F_{a_1}F_{a_n}},\ n > 1. \tag{7}$$

Proof. With the use of (6) we have

$$\begin{aligned}
\sum_{i=1}^{n-1} \frac{F_{a_{i+1}-a_i}}{F_{a_i}F_{a_{i+1}}} &= \sum_{i=1}^{n-1} \frac{(-1)^{a_i}\left(F_{a_{i+1}}F_{a_i+1} - F_{a_{i+1}+1}F_{a_i}\right)}{F_{a_i}F_{a_{i+1}}} \\
&= \sum_{i=1}^{n-1} (-1)^{a_1}\left(\frac{F_{a_i+1}}{F_{a_i}} - \frac{F_{a_{i+1}+1}}{F_{a_{i+1}}}\right) \\
&= (-1)^{a_1}\left(\frac{F_{a_1+1}}{F_{a_1}} - \frac{F_{a_n+1}}{F_{a_n}}\right) \\
&= \frac{(-1)^{a_1}\left(F_{a_n}F_{a_1+1} - F_{a_n+1}F_{a_1}\right)}{F_{a_1}F_{a_n}} \\
&= \frac{F_{a_n-a_1}}{F_{a_1}F_{a_n}}.
\end{aligned}$$

This establishes the theorem. □

Note that in the second line of the proof we were able to replace $(-1)^{a_i}$ by $(-1)^{a_1}$ because all the terms of $\{a_i\}$ have the same parity.

When $a_1 = 2$ we see from the recurrence for the Fibonacci numbers that the right side of (7) reduces to the right side of (3). When $a_i = k2^i$, k a positive integer, (7) reduces to (2).

As Shar pointed out, attractive sums arise from (3) when $\{a_i\}$ is an arithmetic sequence, since then the numerator of the summand is a constant. The same is true for (7). For instance, let $b > 0$ be an even integer, let $c > 0$ be any integer, and define $a_i = bi + c, i \geq 1$. Then (7) becomes

$$\sum_{i=1}^{n-1} \frac{1}{F_{bi+c}F_{b(i+1)+c}} = \frac{F_{b(n-1)}}{F_bF_{b+c}F_{bn+c}},\ n > 1. \tag{8}$$

By induction it can be shown that $\alpha^n = \alpha F_n + F_{n-1}$ for all integers n. With this in mind, and with the use of the closed form for F_n, we let $n \to \infty$ in (8) to obtain

$$\sum_{i=1}^{\infty} \frac{1}{F_{bi+c}F_{b(i+1)+c}} = \frac{1}{F_bF_{b+c}\alpha^{b+c}} = \frac{1}{F_bF_{b+c}\left(\alpha F_{b+c} + F_{b+c-1}\right)}. \tag{9}$$

3. An Analogue of the Main Result for Lucas Numbers

Next we give an analogue of Theorem 2 for the Lucas sequence. Here the sequence $\{a_i\}$ is replaced by any arithmetic sequence of integers where the common difference $b \neq 0$ is even. This is the most general situation that we could find. Note that negative terms are allowed, and the arithmetic sequence may vanish since L_n is never zero. With a common difference that is even, the terms of an arithmetic sequence of integers have the same parity. This emphasizes the analogy with Theorems 1 and 2. We require the following identity, which holds for any integers b, c, and n.

$$L_{bn+c}F_{bn} - L_{b(n+1)+c}F_{b(n-1)} = (-1)^{b(n+1)} F_b L_{b+c}. \tag{10}$$

Identity (10) can be proved with the use of the Binet forms, an exercise that we leave for the reader.

Theorem 3. Let $b \neq 0$ be an even integer, and let c be any integer. Then

$$\sum_{i=1}^{n-1} \frac{1}{L_{bi+c}L_{b(i+1)+c}} = \frac{F_{b(n-1)}}{F_b L_{b+c} L_{bn+c}},\ n > 1. \tag{11}$$

Proof. For $n > 1$ denote the right side of (11) by $r(n, b, c)$ and the left side by $l(n, b, c)$. With the use of (10), we have

$$r(n+1, b, c) - r(n, b, c) = \frac{1}{L_{bn+c}L_{b(n+1)+c}} = l(n+1, b, c) - l(n, b, c). \tag{12}$$

Furthermore,

$$r(2, b, c) = \frac{1}{L_{b+c}L_{2b+c}} = l(2, b, c). \tag{13}$$

Then (12) and (13) show that (11) is true, and this proves Theorem 3. □

To highlight the similarity in form of (7) and (11), set $a_i = bi + c$ where b and c are as in the statement of Theorem 3. Then (11) can be written as

$$\sum_{i=1}^{n-1} \frac{F_{a_{i+1}-a_i}}{L_{a_i}L_{a_{i+1}}} = \frac{F_{a_{n-1}-a_0}}{L_{a_1}L_{a_n}},\ n > 1.$$

In (11), assuming that $b > 0$ and letting $n \to \infty$, we obtain

$$\sum_{i=1}^{\infty} \frac{1}{L_{bi+c}L_{b(i+1)+c}} = \frac{1}{\sqrt{5}F_b L_{b+c}\alpha^{b+c}} = \frac{1}{\sqrt{5}F_b L_{b+c}\left(\alpha F_{b+c} + F_{b+c-1}\right)}. \tag{14}$$

We leave it to the reader to evaluate this infinite sum when $b < 0$.

4. Concluding Comments

For a more general setting let p be a positive integer. Define the sequences $\{U_n\}$ and $\{V_n\}$, for all integers n, by

$$U_n = pU_{n-1} + U_{n-2},\quad U_0 = 0,\quad U_1 = 1,$$

and

$$V_n = pV_{n-1} + V_{n-2}, \quad V_0 = 2, \quad V_1 = p.$$

Clearly $\{U_n\}$ generalizes $\{F_n\}$, and $\{V_n\}$ generalizes $\{L_n\}$. Then Theorem 2 and Theorem 3 remain true if F is replaced by U and L is replaced by V. We leave the simple task of verifying this to the interested reader.

References

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