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A RESULT ON RAMANUJAN-LIKE CONGRUENCE PROPERTIES OF THE RESTRICTED PARTITION FUNCTION $p(n, m)$ ACROSS BOTH VARIABLES

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Abstract

By congruentially identifying formal power series with polynomials and then investigating symmetric properties of those polynomials we prove theorems on infinite families of Ramanujan-type congruences for a restricted partition function. The function $p(n, m)$ enumerates the partitions of n into exactly m parts. Our divisibility results are modulo prime powers for $p(n, m)$ and are established for both variables n and m simultaneously.

1. Introduction

Recent papers, [2] and [3], establish several surprising infinite families of Ramanujan-type congruences for the restricted partition function $p(n, m)$. In [3], results modulo prime powers are given while [2] regards consecutive values of $p(n, m)$ each of which is congruent to an odd prime. Here $p(n, m)$ enumerates the partitions of n into exactly m parts.

The primary goal of this paper is to combine the results of [2] and [3]. In doing so we will establish infinite families of Ramanujan-type divisibility properties of $p(n, m)$ across both variables n and m simultaneously. We will conclude this paper with a theorem that presents an infinite family of these congruences in a striking fashion. For example:

Example 1. For $j \geq 0$ and $2 \leq t \leq 5$ we have $p(300j, t) \equiv 0 \pmod{25}$.

We note that when $t = 3$ and 4 the above congruence holds modulo 5^3.

2. Statement of Theorems

A result of Y. H. Kwong from [5] reveals that $\{p(n,m)\}_{n\geq 0}$ is periodic modulo prime powers ℓ^α for certain values of m. We will establish infinite families of congruences that cluster near the beginning and end of the period of $\{p(n,m)\ (mod\ \ell^\alpha)\}_{n\geq 0}$. The clusters consist of consecutive values of $p(n,m)$ and become ever more numerous for larger values of m. Our first theorem comes from congruentially identifying generating functions with polynomials.

We require the following definition.

Definition 2. $\mathrm{lcm}(m)$ is the least common multiple of the numbers from 1 to m.

Theorem 3. *Let ℓ be an odd prime and let integers j, α, t, σ be such that $j \geq 0,\ \alpha \geq 1,\ 2 \leq t < \ell$, and $1 - \frac{t^2-t}{2} \leq \sigma < t$. Then*

$$p(\mathrm{lcm}(\ell-1) \cdot \ell^\alpha j + \sigma, t) \equiv 0 \pmod{\ell^\alpha}. \tag{1}$$

For $2 \leq t < \ell$, Theorem 3 gives us exactly $\frac{t^2+t}{2} - 1$ consecutive values of $p(n,t)$, each of which is divisible by ℓ^α.

Example 4. Set $\ell = 5$ and $\alpha = 2$ in (1) so that for the number of parts t with $2 \leq t < 5$ and $1 - \frac{t^2-t}{2} \leq \sigma < t$:

- For $t = 4$ we have $-5 \leq \sigma < 4$ and so $p(12j \cdot 5^2 + \sigma, 4) \equiv 0 \pmod{5^2}$;
- For $t = 3$ we have $-2 \leq \sigma < 3$ and so $p(12j \cdot 5^2 + \sigma, 3) \equiv 0 \pmod{5^2}$;
- For $t = 2$ we have $0 \leq \sigma < 2$ and so $p(12j \cdot 5^2 + \sigma, 2) \equiv 0 \pmod{5^2}$.

The following corollary is a combination of Theorem 3 with $\sigma = 0$ and a result from [3]. This corollary displays an infinite family of Ramanujan-type congruences for a two variable restricted partition function across both variables simultaneously.

Corollary 5. Let ℓ be an odd prime and let integers j, α, t be such that $j \geq 0,\ \alpha \geq 1,\ 2 \leq t \leq \ell$. Then

$$p(\mathrm{lcm}(\ell-1) \cdot \ell^\alpha j, t) \equiv 0 \pmod{\ell^\alpha}. \tag{2}$$

Example 6. Set $\ell = 7$ and $\alpha = 2$ in (2) so that for $2 \leq t \leq 7$:

$$\begin{aligned}
p(2940j, 7) &\equiv 0 \pmod{49};\\
p(2940j, 6) &\equiv 0 \pmod{49};\\
p(2940j, 5) &\equiv 0 \pmod{49};\\
p(2940j, 4) &\equiv 0 \pmod{49};\\
p(2940j, 3) &\equiv 0 \pmod{49};\\
p(2940j, 2) &\equiv 0 \pmod{49}.
\end{aligned}$$

Some but not all of the above congruences hold modulo higher powers of 7.

3. Background Material

The methods of proof for these theorems come from congruentially identifying generating functions with polynomials and then investigating symmetry properties of these polynomials. A detailed study regarding the congruential identification of generating functions to polynomials and symmetric properties of them can be found in [1] and [3]. Results regarding the periodicity of generating functions can be found in [4], [5], [6], and [7].

The following lemma due to Y. H. Kwong is crucial for our results.

Lemma 7. ([5]) *For a nonnegative integer* t*, the sequence* $\{p(n,t) \pmod{\ell^\alpha}\}_{n\geq 0}$ *is periodic with minimum period* $\mathrm{lcm}(\ell-1)\ell^\alpha$ *so long as*

$$\sum_{\delta\geq 0} \phi(\ell^\delta)\left\lfloor \frac{t}{\ell^\delta}\right\rfloor \leq \ell,$$

where ϕ *is Euler's totient function.*

Remark 8. [3] Given a nonnegative integer t, let d be any of the natural numbers that are multiples of $\mathrm{lcm}(t)$ and let ℓ^α be a primary factor of d. Set $K(\ell,t,d) = \sum_{\delta\geq 0}\phi(\ell^\delta)\lfloor\frac{t}{\ell^\delta}\rfloor$. Whenever $K(\ell,t,d)\leq \ell$ we will say that Kwong's Criterion is satisfied. Moreover, when t is such that $2\leq t<\ell$ and $d=\mathrm{lcm}(\ell-1)\ell^\alpha$ we have

$$K(\ell,t,d) = \ell-1<\ell, \tag{3}$$

and Kwong's Criterion is satisfied.

Lemma 7 tells us that modulo 5 the coefficients of the generating function for $p(n,4)$ are periodic with period $\mathrm{lcm}(4)\cdot 5 = 60$. However, in congruentially identifying generating functions with polynomials it may be the case that the number of terms in the polynomial is less than what is expected. For example:

Example 9. We have

$$\sum_{n=0}^{\infty}\left[p(n,4)-p(n-60,4)\right]q^n = \frac{q^4(1-q^{60})}{(q;q)_4} \equiv \frac{q^4(1-q^{12})^5}{(q;q)_4}$$

$$\equiv q^4(1+q+\cdots+q^{11})(1+q^2+\cdots+q^{10})(1+q^3+q^6+q^9)(1+q^4+q^8)(1-q^{12}) \pmod 5$$

The period of 60 gives us information on 60 terms in this sequence, yet the polynomial in the far right side of Example 9 is of degree 54 and has exactly 51 terms. The 'absent' terms from the polynomial correspond to terms of the sequence, equivalently, coefficients of the generating function, that are zero modulo 5.

So that the partition theoretic information found in the generating function does not vanish when re-expressed as a polynomial, we will embellish our polynomial by including additional terms, each of which has coefficient zero. We will call these additional terms *periodic terms*, we will call this embellished polynomial a *periodic polynomial,* and we will take the periodic terms into consideration when determining the degree of the periodic polynomial. It is these periodic terms that give us our results.

Theorem 10 establishes how we congruentially identify generating functions with polynomials without any loss of desired partition theoretic information. The statement of the following theorem has been slightly modified from its appearance in [3] to streamline our procedures.

Theorem 10. ([3]) *Given an odd prime ℓ and t such that $2 \leq t < \ell$, let $d = \operatorname{lcm}(\ell-1)\ell^{\alpha}$. Then the generating function for the difference between the number of partitions of n into exactly t parts and the number of partitions of $n-d$ into exactly t parts is congruent modulo ℓ^{α} to a periodic polynomial, $A(q;\ell,t,d)$, of degree d with exactly d terms, including periodic terms:*

$$\sum_{n=0}^{\infty} \left(p(n,t) - p(n-d,t)\right) q^n = \frac{q^t(1-q^d)}{(q;q)_t} \equiv A(q;\ell,t,d) \pmod{\ell^{\alpha}}. \tag{4}$$

Example 11. We have

$$\sum_{n=0}^{\infty} \left[p(n,4) - p(n-60,4)\right] q^n = \frac{q^4(1-q^{60})}{(q;q)_4} \equiv A(q,5,4,60)$$

$$\equiv 0q + 0q^2 + 0q^3 + [q^4(1+q+\cdots+q^{11})(1+q^2+\cdots+q^{10})(1+q^3+q^6+q^9)$$
$$(1+q^4+q^8)(1-q^{12})] + 0q^{55} + 0q^{56} + 0q^{57} + 0q^{58} + 0q^{59} + 0q^{60} \pmod{5}.$$

Remark 12. Compare Example 11 to Example 9.

4. Proof of Theorem 3

Proof. Let $d = \operatorname{lcm}(\ell-1)\ell^{\alpha}$ be as in Remark 8 so that Kwong's Criterion is satisfied. Lemma 7 tells us that the sequence $\{p(n,t) \pmod{\ell^{\alpha}}\}_{n\geq 0}$ is periodic with minimum period $\operatorname{lcm}(\ell-1)\ell^{\alpha}$. However, the rational function $q^t(1-q^d)/(q;q)_t$ identifies modulo ℓ^{α} with a standard polynomial of degree $d - \frac{t^2-t}{2}$ with $d - \frac{t^2+t}{2} + 1$ terms:

$$\frac{q^t(1-q^d)}{(q;q)_t} \equiv q^t + q^{t+1} + 2q^{t+2} + \cdots \pm q^{d-\frac{t^2-t}{2}-1} + \pm q^{d-\frac{t^2-t}{2}} \pmod{\ell^{\alpha}}. \tag{5}$$

From the right-hand side of (4) in Theorem 10 we have that $A(q;\ell,t,d)$ is a periodic polynomial and has d terms, including those that are periodic. The difference in number of terms between $A(q;\ell,t,d)$ and the left-hand side of 6 is simply $\frac{t^2+t}{2}-1$. The first $t-1$ and last $\frac{t^2-t}{2}$ terms of $A(q;\ell,t,d)$ are periodic terms:

$$A(q;\ell,t,d) = 0q + 0q^2 + \cdots + 0q^{t-1} + q^t + q^{t+1} + \cdots \tag{6}$$

$$\pm q^{d-\frac{t^2-t}{2}-1} \pm q^{d-\frac{t^2-t}{2}} + 0q^{d-\frac{t^2-t}{2}+1} + \cdots + 0q^{d-1} + 0q^d.$$

Since $\{p(n,t) \pmod{\ell^\alpha}\}_{n\geq 0}$ is periodic with period d for $2 \leq t < \ell$, we have, for $1-\frac{t^2-t}{2} \leq \sigma < t$, that $p(\mathrm{lcm}(\ell-1)\ell^\alpha j + \sigma, t) \equiv 0 \pmod{\ell^\alpha}$. □

Example 13 is a re-expression of Example 4.

Example 13. Set $\ell = 5$ and $\alpha = 2$, with $2 \leq t \leq 4$ to obtain the following;

$$\sum_{n=0}^{\infty} [p(n,4) - p(n-300,4)]\, q^n = \frac{q^4(1-q^{300})}{(q;q)_4}$$

$$\equiv A(q;5,4,300) \equiv 0q + 0q^2 + 0q^3 + q^4 + \cdots$$

$$\cdots - q^{294} + 0q^{295} + 0q^{296} + 0q^{297} + 0q^{298} + 0q^{299} + 0q^{300} \pmod{25},$$

where the coefficient a_r on q^r corresponds to $p(300j+r,4)$ and so for $-5 \leq \sigma \leq 3$ we have that $p(300j+\sigma,4) \equiv 0 \pmod{25}$;

$$\sum_{n=0}^{\infty} [p(n,3) - p(n-300,3)]\, q^n = \frac{q^3(1-q^{300})}{(q;q)_3}$$

$$\equiv A(q;5,3,300) \equiv 0q + 0q^2 + q^3 + \cdots + q^{297} + 0q^{298} + 0q^{299} + 0q^{300} \pmod{25},$$

where the coefficient a_r on q^r corresponds to $p(300j+r,3)$ and so for $-2 \leq \sigma \leq 2$ we have that $p(300j+\sigma,3) \equiv 0 \pmod{25}$;

$$\sum_{n=0}^{\infty} [p(n,2) - p(n-300,2)]\, q^n = \frac{q^2(1-q^{300})}{(q;q)_2}$$

$$\equiv A(q;5,2,300) \equiv 0q + q^2 + \cdots - q^{298} - q^{299} + 0q^{300} \pmod{25},$$

where the coefficient a_r on q^r corresponds to $p(300j+r,2)$ and so for $0 \leq \sigma \leq 1$ we have that $p(300j+\sigma,2) \equiv 0 \pmod{25}$.

5. Proof of Corollary 5

We need the following result to complete the proof of Corollary 5.

Theorem 14. *[3] For ℓ an odd prime, $j \geq 0$, and $0 \leq k \leq \frac{\ell-3}{2}$ and $\alpha \geq 1$,*

$$p(\mathrm{lcm}(\ell-1)\ell^{\alpha} j - k\ell, \ell) \equiv 0 \pmod{\ell^{\alpha}}. \tag{7}$$

Theorem 14 gives us $\frac{\ell-1}{2}$ Ramanujan-type congruences for each ℓ.

Proof. Corollary 5 follows immediately from Theorem 3 by fixing $\sigma = 0$ for $2 \leq t < \ell$ and setting $k = 0$ in Theorem 14. □

6. Concluding Remarks

There are many more results regarding Ramanujan-type congruences for $p(n, m)$ and other integer partition functions to be gained from this line of inquiry. Continued study of the symmetry properties of the polynomial $A(q; \ell, t, d)$ will yield additional Ramanujan-type congruences for $p(n, m)$ which cluster in the middle of the periods of $\{p(n, t) \pmod{\ell^{\alpha}}\}_{n \geq 0}$. No combinatorial proofs for any of these theorems are yet known.

References

[1] B. Kronholm, On Congruence Properties of p(n,m), PAMS **133** (2005), 2891-2895.

[2] B. Kronholm, On Congruence Properties of Consecutive Values of $p(n, m)$, *Integers* **7** (2007), #A16.

[3] B. Kronholm, Generalized Congruence Properties of the Restricted Partition Function $p(n, m)$, *Ramanujan J.*, to appear.

[4] Y. H. Kwong, Minimum periods of binomial coefficients modulo M, *Fibonacci Quar.* **27** (1989), 348-351.

[5] Y. H. Kwong, Minimum periods of partition functions modulo M, *Utilitas Math.* **35** (1989), 3-8.

[6] Y. H. Kwong, Periodicities of a class of infinite integer sequences modulo M, *J. Number Theory* **31** (1989), 64-79.

[7] A. Nijenhuis and H. S. Wilf, Periodicities of partition functions and Stirling numbers modulo p, *J. Number Theory*, **25** (1987), 308-312.