PARTITION OF AN INTEGER INTO DISTINCT BOUNDED PARTS, IDENTITIES AND BOUNDS

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Abstract

The partition function $Q(n)$, which denotes the number of partitions of a positive integer n into distinct parts, has been the subject of a dozen papers. In this paper, we study this kind of partition with the additional constraint that the parts are bounded by a fixed integer. We denote the number of partitions of an integer n into distinct parts, each $\leq k$, by $Q_k(n)$. We find a sharp upper bound for $Q_k(n)$, and more, an infinite series lower bound for the partition function $Q(n)$. In the last section, we exhibit a group of interesting identities involving $Q_k(n)$ that arise from a combinatorial problem.

1. Introduction

Let $Q(n)$ be the number of ways of partitioning a positive integer n into distinct summands. The generating function for this kind of partition is

$$Q(x) = \sum_{n=0}^{\infty} Q(n)x^n = \prod_{j=1}^{\infty}(1 + x^j) \ . \qquad (1)$$

Euler noted that he could easily convert $Q(x)$ to something else, which is in fact another generating function:

$$Q(x) = \prod_{j=1}^{\infty}(1 + x^j) = \frac{\prod_{j=1}^{\infty}(1 - x^{2j})}{\prod_{j=1}^{\infty}(1 - x^j)} = \prod_{j=1}^{\infty} \frac{1}{1 - x^{2j-1}} \ .$$

The last product is the generating function for partitioning an integer into odd summands. Consequently, he concluded that there was a bijection between the set of partitions of a positive integer n into distinct parts, and set of partitions of n

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into odd parts. To read a brief history of Euler work concerning this bijection see, [9].

If $\sigma^o(n)$ denotes the *odd* divisor function, i.e., the sum of odd divisors of n, then the partition function $Q(n)$ satisfies the recurrence equation (see [1], p. 826)

$$Q(n) = \frac{1}{n}\sum_{k=0}^{n-1} Q(k)\sigma^o(n-k),\ n > 0\ . \tag{2}$$

It is easily seen that $\sigma^o(n) = \sigma(n) - \frac{1}{2}\sigma(\frac{n}{2}) = \sigma(n)/(2^{a(n)+1} - 1)$, where $\sigma(n)$ is the sum of divisors of n, and $a(n)$ is the power of 2 in the decomposition of n into prime factors. Therefore, we are able to modify our recurrence equation as follows:

$$Q(n) = \frac{1}{n}\sum_{k=1}^{n} \frac{\sigma(k)}{2^{a(k)+1} - 1} Q(n-k),\ n > 0\ . \tag{3}$$

An investigation in the table of amounts of $Q(n)$ for large numbers demonstrates that it has a considerably slower growth than the unrestricted partition function $P(n)$. To have a comparison with $P(n)$, it is worthwhile to mention Rademacher like series for $Q(n)$ (see [5], [6] and [7]):

$$Q(n) = \frac{1}{2}\sqrt{2}\sum_{k=1}^{\infty} A_{2k-1}(n)\left\{\frac{d}{dn'}\left[J_0\left(\frac{\pi i}{2k-1}, \sqrt{\frac{1}{3}\left(n' + \frac{1}{24}\right)}\right)\right]\right\}_{n=n'} ,$$

where

$$A_k(n) = \sum_{\substack{h=1\\(h,k)=1}}^{k} e^{\pi i[s(h,k)-s(2h,k)]} e^{-2\pi i h n/k}, \quad s(h,k) = \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor\frac{hr}{k}\right\rfloor - \frac{1}{2}\right).$$

Here $s(h,k)$ is a Dedekind sum, and $J_0(x)$ is the zeroth order Bessel function of the first kind. This series representation of $Q(n)$ leads to an asymptotic formula for $Q(n)$:

$$Q(n) \sim \frac{1}{4\cdot 3^{1/4} n^{3/4}} e^{\pi\sqrt{\frac{n}{3}}}, \quad \text{as } n \to \infty\ . \tag{4}$$

Comparing this asymptotic formula with the one for $P(n)$ demonstrates the slower growth of $Q(n)$ (see also [4], pp. 574-580). In Sections 2 and 3, we derive suitable upper and lower bounds for the partition function $Q(n)$.

It is well known that the general partition function $P(n),\ n > 0$, is convex (see [8]). The convexity for the amounts of $Q(n)$ takes place if $n \geq 4$, which means that the inequality $Q(n) \leq \frac{1}{2}\{Q(n+1) + Q(n-1)\}$ holds for $n \geq 4$. A short proof of this fact is presented in Section 3.

In this paper, we are mainly concerned to a restricted form of partition of an integer into distinct parts. Let $Q_k(n)$ denote the number of partitions of a positive integer n into distinct parts, each $\leq k$. This partition function has an interesting combinatorial interpretation. If $X = \{1, 2, 3, \cdots, k\}$, then $Q_k(n)$ is the number of subsets of X for which the sum of the members is n. The partition function $Q_k(n)$ has the generating function

$$Q_k(x) = \prod_{j=1}^{k}(1 + x^j) = \sum_{n=0}^{\theta} Q_k(n)x^n, \quad \theta = \frac{k(k+1)}{2} \ . \tag{5}$$

We know that $Q_k(x)$ is a symmetric unimodal polynomial. It means that its coefficients goes up to somewhere, (for $Q_k(x)$ the climax occurs at $\lfloor \frac{k(k+1)}{4} \rfloor$) then symmetrically goes down. The symmetry of the coefficients is almost evident, but proving the unimodal property of $Q_k(x)$ is difficult. To my knowledge there is not a known combinatorial proof for this fact, but there is a non-elementary proof based on semi-simple Lie Algebras. The interested reader might have a look at [10] to see a proof of the unimodal property for $Q_k(x)$ (For further discussion on the unimodal property and Lie algebras see, [12]). Since $Q(n) = Q_k(n)$ for each $k > n$, the unimodal property of $Q_k(x)$ leads to the monotonicity of the partition function $Q(n)$.

Let $P_k(n)$ denote the number of partitions of an integer n into parts, each $\leq k$, and let $P_k(x)$ be its generating function. The relation between $Q_k(n)$ and $P_k(n)$ can be stated by means of the identity

$$P_k(x) = \frac{1}{\prod_{j=1}^{k}(1 - x^j)} = \frac{\prod_{j=1}^{k}(1 + x^j)}{\prod_{j=1}^{k}(1 - x^{2j})} = P_k(x^2)Q_k(x) \ ,$$

which leads to a recurrence equation relating $P_k(n)$ to $Q_k(n)$:

$$P_k(n) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} Q_k(n - 2i)P_k(i) \ . \tag{6}$$

2. An Elementary Upper Bound for $Q_k(n)$

Pribitkin [11] has introduced a remarkable elementary method to obtain a sharp upper bound for the partition function $P_k(n)$. With modification of his method, we are able to find a sharp upper bound for $Q_k(n)$. As in [11], we employ the dilogarithm function $Li_2(x) = \sum_{m=1}^{\infty} \frac{x^m}{m^2}$, where $|x| < 1$. It is clear that $Li_2(1) = \frac{\pi^2}{6}$. We also will need the simple fact $e^x - e^{-x} > 2x$ for $x > 0$, that has appeared in [11]. The main result of this section is stated in the next theorem.

Theorem 1. *Let k,n be positive integers, $n \leq \lfloor \frac{k(k+1)}{4} \rfloor$. Then we have the following inequality:*

$$Q_k(n) < \frac{A(k,n)}{\sqrt{n}} e^{\pi\sqrt{n/3} - \frac{1}{\pi}\sqrt{3n}Li_2(e^{-\pi\alpha/\sqrt{3n}})} \ ,$$

where $A(k,n) = \frac{2\sqrt{n}}{k^2+k-4n+2} + \frac{\pi}{2\sqrt{3}}$, $\alpha = \lceil k/2 \rceil$.

Proof. If $0 < x < 1$, we have

$$Q_k(x) = \prod_{j=1}^{k}(1+x^j) < \frac{1}{(1-x)(1-x^3)\cdots(1-x^{2\alpha-1})} \ .$$

After taking logarithm, we observe that

$$\begin{aligned}\log(Q_k(x)) < -\sum_{j=1}^{\alpha}\log(1-x^{2j-1}) &= \sum_{j=1}^{\alpha}\sum_{m=1}^{\infty}\frac{x^{(2j-1)m}}{m} \\ &= \sum_{m=1}^{\infty}\frac{1}{m}\sum_{j=1}^{\alpha}x^{(2j-1)m} \\ &= \sum_{m=1}^{\infty}\frac{1}{m}\frac{x^m}{1-x^{2m}}(1-x^{2\alpha m}) \ .\end{aligned}$$

Now we let $x = e^{-u}, u > 0$, to find that

$$\begin{aligned}\log(Q_k(e^{-u})) < \sum_{m=1}^{\infty}\frac{e^{-mu}(1-e^{-2\alpha mu})}{m(1-e^{-2mu})} &= \sum_{m=1}^{\infty}\frac{1-e^{-2\alpha mu}}{m(e^{mu}-e^{-mu})} \\ &< \frac{1}{2u}\sum_{m=1}^{\infty}\frac{1-e^{-2\alpha mu}}{m^2} \\ &= \frac{1}{2u}\left(\frac{\pi^2}{6} - Li_2(e^{-2\alpha u})\right) \ .\end{aligned}$$

We exploit the unimodal and symmetry properties of $Q_k(x)$ to obtain that for all $0 < x < 1$, and $n \leq \frac{k(k+1)}{4}$,

$$Q_k(x) \geq Q_k(n)(x^n + x^{n+1} + \cdots + x^{\frac{k(k+1)}{2}-n}) = Q_k(n)x^n\frac{1-x^{\frac{k(k+1)}{2}-2n+1}}{1-x} \ .$$

Therefore, we realize that

$$\begin{aligned}\log(Q_k(n)) &< nu + \log(1-e^{-u}) - \log(1-e^{-(\frac{k(k+1)}{2}-2n+1)u}) \\ &\quad + \frac{1}{2u}\left(\frac{\pi^2}{6} - Li_2(e^{-2\alpha u})\right) \\ &< nu + \log(u) - \log(1-e^{-(\frac{k(k+1)}{2}-2n+1)u}) \\ &\quad + \frac{1}{2u}\left(\frac{\pi^2}{6} - Li_2(e^{-2\alpha u})\right) \ .\end{aligned}$$

Here we have applied the simple estimation $1 - e^{-x} < x$, that is valid for $x > 0$. Now we let $u = \frac{1}{\lambda\sqrt{n}}$, $\lambda > 0$, and estimate the best λ. Substitute u by $u = \frac{1}{\lambda\sqrt{n}}$ in the right hand side of the inequality to find that

$$Q_k(n) < \frac{e^{(\frac{1}{\lambda}+\frac{\pi^2\lambda}{12})\sqrt{n}}}{\lambda\sqrt{n}} \frac{e^{-\frac{1}{2}\lambda\sqrt{n}Li_2(e^{-2\alpha u})}}{(1-e^{-(\frac{k(k+1)}{2}-2n+1)\frac{1}{\lambda\sqrt{n}}})} \ .$$

Calculate the best possible λ to minimize the multiple of $\sqrt{n}$ at the first exponential term; it turns out that $\lambda = \frac{2\sqrt{3}}{\pi}$ and $u = \frac{\pi}{2\sqrt{3n}}$. Hence we conclude that

$$Q_k(n) < \frac{\pi e^{\pi\sqrt{n/3}}}{2\sqrt{3n}} \frac{e^{-\frac{1}{\pi}\sqrt{3n}Li_2(e^{-\pi\frac{\alpha}{\sqrt{3n}}})}}{(1-e^{-(\frac{k(k+1)}{2}-2n+1)\frac{\pi}{2\sqrt{3n}}})} \ . \tag{7}$$

Note that for $x \geq 0$, $1 + x \leq e^x$, or $e^{-x} \leq 1/(1+x)$. Subtracting both sides from 1, gives us the estimate

$$\frac{x}{1+x} \leq 1 - e^{-x};$$

the proof is now complete when we apply this inequality to the right hand side of (7) for $x = (\frac{k(k+1)}{2} - 2n + 1)\frac{\pi}{2\sqrt{3n}}$. □

Fix n and let $k \to \infty$; since $A(k,n)$ tends to $\frac{\pi}{2\sqrt{3}}$, and $e^{-\frac{1}{\pi}\sqrt{3n}Li_2(e^{-\pi\alpha/\sqrt{3n}})}$ tends to 1, we are able to determine a very nice upper bound for $Q(n)$.

Corollary 2. *Let $Q(n)$ denote the number of unrestricted partitions into distinct parts. Then, we have*

$$Q(n) < \frac{\pi e^{\pi\sqrt{n/3}}}{2\sqrt{3n}} \ .$$

Remark. It is clear that for all feasible amounts of k, n, the value of $e^{-\pi\frac{\alpha}{\sqrt{3n}}}$ is small enough to make $Li_2(x) > x$ a good estimation; hence we conclude that

$$Q_k(n) < \frac{A(k,n)}{\sqrt{n}} e^{(\pi/\sqrt{3}-e^{-\pi\frac{\alpha}{\sqrt{3n}}}\sqrt{3}/\pi)\sqrt{n}} \ .$$

3. Simple Lower Bounds for $Q(n)$

Analytic methods, like the saddle point method (see [4], pp. 541-608) are excellent for asymptotic estimations or finding upper bounds, but they seem poor to derive lower bounds. Likewise, the dilogarithm scheme is not applicable to find a lower bound for $Q_k(n)$. However, we are able to find a lower bound for $Q(n)$ by applying other methods. First, we take a detour and prove the convexity of $Q(n)$.

Lemma 3. *If $n > 3$, then $Q(n) \le \frac{1}{2}\{Q(n+1) + Q(n-1)\}$.*

Assuming $n > 3$, we need to show that $Q(n+1) - Q(n) \ge Q(n) - Q(n-1)$. Consider a partition of n into distinct parts and increase the greatest summand by 1; we obtain a partition of $n+1$ into distinct parts in which the two greatest summands differ by at least 2. Conversely, we can delete a 1 from the greatest summand of such partition and obtain a partition of n with distinct parts (for single part partitions of $n, n+1$ there is a similar correspondence).

Therefore, there is a bijection between the entire set of partitions of n into distinct summands, and set of distinct part partitions of $n+1$ for which the greatest summand is at least 2 more than the previous one (this set includes the single part partition of $n+1$). Hence, we find that $Q(n+1) - Q(n)$ is the cardinality of the set of all partitions of $n+1$ into (more than 1) distinct summands with the greatest summand exactly 1 more than the previous summand. Denote this set by Y, and the analogous set pertaining to n by X.

Decompose X into two disjoint sets, one consisting of those partitions that contain 1, say X_1, and the other one including all partitions without 1 in their summands, say X_2. Partition Y in a similar way, and assume that $(1, \lambda_1, \cdots, k_1 - 1, k_1) \in X_1$, $(\lambda'_1, \cdots, k_2 - 1, k_2) \in X_2$ (note that since $n > 3$, $k_1, k_2 > 2$). Define the two mappings σ_1, σ_2 in the following way:

$$\sigma_1 : X_1 \to Y_2, \quad \sigma_1[(1, \lambda_1, \cdots, k_1 - 1, k_1)] = (\lambda_1, \cdots, k_1, k_1 + 1) ,$$
$$\sigma_2 : \ X_2 \to Y_1, \quad \sigma_2[(\lambda'_1, \cdots, k_2 - 1, k_2)] = (1, \lambda'_1, \cdots, k_2 - 1, k_2) .$$

It is quite straightforward to see that σ_1 is an injection from X_1 into Y_2, and σ_2 is a bijection between X_2, Y_1. Therefore, $|X_1| \le |Y_2|$, $|X_2| = |Y_1|$, and we conclude that $|X| \le |Y|$.

The recurrence equation (3) together with the convexity of $Q(n)$ leads us to the following lower bound.

Theorem 4. *If $n > 0$, then $Q(n)$ satisfies the following inequality:*

$$Q(n) > e^{-\frac{7}{12}} \sum_{k=1}^{\infty} \frac{(7/12)^k}{k!} \binom{n+k-1}{n} .$$

Proof. Starting with the equation (3), we divide the right hand sum into parts, each consisting of four consecutive terms with the first one index in the form $4t+1$. If $k = 4t + 1 > 1$, then we have

$$\begin{aligned}\sum_{j=0}^{3} \sigma^o(k+j)Q(n-k-j) &\ge (k+1)Q(n-k) + \frac{k+2}{3}Q(n-k-1) \\ &\quad + (k+3)Q(n-k-2) + Q(n-k-3) \\ &\ge \frac{7}{12}\sum_{j=0}^{3}(k+j)Q(n-k-j) .\end{aligned}$$

To acquire the last inequality, we have applied the monotonicity of $Q(n), n \geq 0$ (the last inequality also holds for the last part which may have less than 4 terms).

For $k = 1$, we could write that

$$\sum_{j=1}^{4} \sigma^o(j)Q(n-j) = Q(n-1) + Q(n-2) + 4Q(n-3) + Q(n-4)$$
$$> \frac{7}{12}\sum_{j=1}^{4} jQ(n-j) + \frac{1}{3}Q(n-1) \ .$$

Here, we have exploited the monotonicity of $Q(n)$ and the fact $Q(n-1)+Q(n-3) > 2Q(n-2)$, valid for $n > 5$. Thus, we conclude that

$$Q(n) \geq \frac{1}{3n}Q(n-1) + \frac{7}{12n}\sum_{k=1}^{n} kQ(n-k), \ n > 5 \ .$$

Now, we define the function $t(n)$ by the recurrence equation

$$t(n) = \frac{7}{12n}\sum_{k=1}^{n} kt(n-k), \ t(0) = 1 \ .$$

A direct computation shows that $Q(i) \geq t(i), \ 1 \leq i \leq 5$. Hence, $Q(i) \geq t(i), \ i \geq 0$. Let $T(x)$ be the generating function of $t(n)$. It is easily seen that $T(x)$ satisfies the equation

$$T(x)\sum_{i=0}^{\infty}(i+1)x^i = \frac{12}{7}T'(x) \ .$$

After solving this differential equation, it turns out that

$$T(x) = T_0 e^{\frac{7}{12-12x}} = T_0 \sum_{k=0}^{\infty} \frac{(7/12)^k}{k!}(1-x)^{-k} \ .$$

Since $t(0) = 1$, the constant T_0 is equal to $e^{-\frac{7}{12}}$. Thus, we have the following formula for $t(n), \ n > 0$:

$$t(n) = e^{-\frac{7}{12}} \sum_{k=1}^{\infty} \frac{(7/12)^k}{k!}\binom{n+k-1}{n} \ ,$$

now the proof is complete. □

Let $q_k(n)$ denote the number of partitions of an integer n into exactly k distinct parts (note that $q_k(n)$ is quite different from $Q_{k-1}(n-k)$). Clearly, $Q(n) = \sum_{k=1}^{a} q_k(n), \ a = \left\lfloor \frac{1}{2}(-1+\sqrt{8n+1}) \right\rfloor$. It is easily verified that $q_k(n) = p_k\left(n - \binom{k}{2}\right)$. Since

$$p_k(n) \geq \frac{1}{k!}\binom{n-1}{k-1}$$

(see [2], pp. 56-57), we obtain a finite sum lower bound for $Q(n)$:

$$Q(n) = \sum_{k=1}^{a} p_k\left(n - \tbinom{k}{2}\right) \geq \sum_{k=1}^{a} \frac{1}{k!}\binom{n-\binom{k}{2}-1}{k-1} . \tag{8}$$

Remark. To improve the lower bound series in Section 3, one may sort terms of the recurrence identity concerning $Q(n)$, modulo 8 or even 16; also a similar argument could be done to derive a lower bound for the partition function $P(n)$. The first lower bound series is a quickly convergent satisfying lower bound. The second lower bound sum, although not as straightforward as the first one, is sharp. In fact, empirical evidence shows that if n is greater than 350000, then the amount of the lower bound series is greater than $e^{0.84\pi\sqrt{n/3}}/n^{3/4}$, and for $n > 12500$, the amount of the second lower bound is greater than $e^{0.93\pi\sqrt{n/3}}/n^{3/4}$.

4. Identities Involving Prime Factors of the Bound Integer

In this section, we find a group of interesting identities which arise from a combinatorial problem. The key idea here is the uniqueness of a basis representation for the cyclotomic field $Q(\zeta_p)$, when you look at it as a $\mathbb{Q}$-vector space. We consider

$$(-x;x)_k = \prod_{j=1}^{k}(1+x^j) = \sum_{n=0}^{\frac{k(k+1)}{2}} Q_k(n)x^n ,$$

and consider p as an odd prime factor of k. Let ζ be the primitive p-th root of unity, i.e., $\zeta = e^{2\pi i/p}$. Let $Q(\zeta)$ be the field extension of ζ over $\mathbb{Q}$. First, we calculate the amount of $(-\zeta;\zeta)_k$ in $Q(\zeta)$. We have

$$(-\zeta;\zeta)_k = \prod_{j=1}^{k}(1+\zeta^j) = \left(\prod_{j=0}^{p-1}(1+\zeta^j)\right)^{k/p} .$$

Since ζ is a primitive root of unity we have

$$P(z) = z^p - 1 = \prod_{j=0}^{p-1}(z-\zeta^j) .$$

Let $z = -1$ in this equation to find that

$$\prod_{j=0}^{p-1}(1+\zeta^j) = 2 .$$

Therefore, we have $(-\zeta;\zeta)_k = 2^{k/p}$.

The polynomial
$$\frac{P(z)}{z-1} = 1 + z + z^2 + \cdots + z^{p-1}$$
is a minimal polynomial for ζ over $\mathbb{Q}$. So we conclude that the set
$$A = \left\{1, \zeta, \zeta^2, \zeta^3, \cdots, \zeta^{p-2}\right\}$$
constitutes a $\mathbb{Q}$-basis for $Q(\zeta)$ as a $\mathbb{Q}$-vector space. Thus,
$$(-\zeta;\zeta)_k = 2^{k/p} = 2^{k/p} \cdot 1 + 0 \cdot \zeta + 0 \cdot \zeta^2 + \cdots + 0 \cdot \zeta^{p-2}$$
is the basis representation of $(-\zeta;\zeta)_k$ in $Q(\zeta)$. On the other hand, let us assume that
$$(-\zeta;\zeta)_k = \prod_{j=1}^{k}(1+\zeta^j) = a_0 + a_1\zeta + a_2\zeta^2 + \cdots + a_{p-1}\zeta^{p-1}\ .$$
In fact, we have expanded the product and reduced the powers modulo p. It is clear that
$$a_i = \sum_{j=0}^{\alpha_i} Q_k(jp+i), \text{ where } \alpha_i = \lfloor k(k+1)/2p - i/p \rfloor\ . \tag{9}$$
The expansion above could be written in the form
$$(a_0 - a_{p-1}) \cdot 1 + (a_1 - a_{p-1}) \cdot \zeta + (a_2 - a_{p-1}) \cdot \zeta^2 + \cdots + (a_{p-2} - a_{p-1}) \cdot \zeta^{p-2},$$
as a representation over the basis A. Since the representation over a basis is unique, we have the following system of equations:
$$a_0 - a_{p-1} = 2^{k/p}, \quad a_i - a_{p-1} = 0,\ 1 \le i \le p-2 \quad \text{and} \quad \sum_{i=0}^{p-1} a_i = 2^k\ ,$$
which leads to the solution
$$a_0 = 2^{k/p} + \frac{2^k - 2^{k/p}}{p}, \quad a_i = \frac{2^k - 2^{k/p}}{p} \text{ for } 1 \le i \le p-1\ .$$
So we have the following result:

Theorem 5. *Let k be a positive integer and p an odd prime factor of it. Then, we have the following identities:*
$$\sum_{j=0}^{\alpha_0} Q_k(jp) = 2^{k/p} + \frac{2^k - 2^{k/p}}{p}, \quad \sum_{j=0}^{\alpha_i} Q_k(jp+i) = \frac{2^k - 2^{k/p}}{p} \textit{ for } 1 \le i < p,$$
where $\alpha_i = \lfloor k(k+1)/2p - i/p \rfloor$.

4.1. Application

Case 1. Let $X = \{1, 2, 3, \cdots, k\}$, and consider p as an odd prime factor of k. We are interested in the number of subsets of X for which the sum of the members is congruent to i modulo p. In fact, $Q_k(i), Q_k(p+i), Q_k(p+2i), \cdots$ are equal to the numbers of subsets of X for which the sum of the members are $i, p+i, p+2i, \cdots$, respectively. Looking at equation (9) makes it clear that each a_i in the expansion of

$$(-\zeta;\zeta)_k = \prod_{j=1}^{k}(1+\zeta^j) = a_0 + a_1\zeta + a_2\zeta^2 + \cdots + a_{p-1}\zeta^{p-1}$$

describes the number of subsets of X for which the sum of the members is congruent to i modulo p. So if we denote the sum of the members of $S \subseteq X$ by $\sigma(S)$, then we have

$$\#\{S \subseteq X : \sigma(S) \equiv i \pmod{p}\} = \begin{cases} 2^{k/p} + \frac{2^k - 2^{k/p}}{p}, & i = 0 \\ \frac{2^k - 2^{k/p}}{p}, & i \neq 0 \end{cases}.$$

Case 2. In the case $X = \{1, 2, 3, \cdots, k\}$, $k = pt + 1$, there would be a similar argument for the coefficients a_i of $(-\zeta;\zeta)_k$. But in this case we have

$$(-\zeta;\zeta)_k = \left(\prod_{j=0}^{p-1}(1+\zeta^j)\right)^t (1+\zeta) = 2^{(k-1)/p} + 2^{(k-1)/p}\zeta\ .$$

So we have the following system of equations:

$$a_0 - a_{p-1} = 2^{(k-1)/p}, \qquad a_1 - a_{p-1} = 2^{(k-1)/p},$$

$$a_i - a_{p-1} = 0,\ 2 \leq i \leq p-2, \qquad \text{and} \quad \sum_{i=0}^{p-1} a_i = 2^k,$$

with the solutions

$$a_0 = a_1 = \frac{2^k + (p-2)2^{(k-1)/p}}{p}, \quad a_i = \frac{2^k + 2^{(k+p-1)/p}}{p} \text{ for } 2 \leq i \leq p-1\ .$$

Hence, we conclude that

$$\#\{S \subseteq X : \sigma(S) \equiv i \pmod{p}\} = \begin{cases} \frac{2^k + (p-2)2^{(k-1)/p}}{p}, & i = 0, 1 \\ \\ \frac{2^k + 2^{(k+p-1)/p}}{p}, & i \neq 0, 1 \end{cases}.$$

Case 3. In the case $X = \{1, 2, 3, \cdots, k\}$, $k = pt - 1$, we have

$$(-\zeta;\zeta)_k = \prod_{j=1}^{k}(1+\zeta^j) = \left(\prod_{j=0}^{p-1}(1+\zeta^j)\right)^{t-1} \prod_{j=1}^{p-1}(1+\zeta^j) = 2^{(k-p+1)/p}$$

which by a similar argument finally leads us to the following answer:

$$\#\{S \subseteq X : \sigma(S) \equiv i \pmod p\} = \begin{cases} 2^{(k-p+1)/p} + \frac{2^k - 2^{(k-p+1)/p}}{p}, & i = 0 \\ \frac{2^k - 2^{(k-p+1)/p}}{p}, & i \neq 0 \end{cases} .$$

Remark. The problem can be solved for the cases $k = pt \pm (p-1)/2$ in a similar way.

5. Concluding Remarks

Real Integral form of $Q_k(n)$. Consider $Q_k(z)$ az a complex variable generating function of $Q_k(n)$. The fact that it has no singularities at $z = 1$ makes it possible to find a real integral form of $Q_k(n)$. Since $Q_k(z)$ is analytic over $\mathbb{C}$, we find that

$$Q_k(n) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{Q_k(z)}{z^{n+1}} dz \ . \tag{10}$$

Substituting z by $e^{i\theta}$, it follows that

$$Q_k(n) = \frac{2^{k-1}}{\pi} \int_0^{2\pi} \cos\left[\frac{1}{4}(k^2 + k - 4n)\theta\right] \left(\prod_{j=1}^{k} \cos\frac{j\theta}{2}\right) d\theta \ . \tag{11}$$

This formula enables us to study the behavior of $Q_k(n)$ for various amounts of k, n, on the background of basic calculus.

Let $q_k(n, r)$ denote the number of partitions of n with exactly r distinct parts, each $\leq k$. If $p_k(n, r)$ denotes the number of partitions of n with exactly r parts, each $\leq k$, it is easily seen that $q_k(n, r) = p_{k-r+1}\left(n - \binom{r}{2}, r\right)$, and $p_k(n, r) = p(k - 1, r, n - r)$, where $p(k, r, n)$ is the coefficient of q^n in the Gaussian polynomial

$$\begin{bmatrix} k + r \\ r \end{bmatrix}_q$$

(see [2], pp. 33-36). Since $Q_k(n)$ is the sum of $q_k(n, r)$'s, having a lower bound on Gaussian polynomials coefficients leads to a finite sum lower bound for $Q_k(n)$.

In Section 4, if we had considered a factor m of k that was not necessarily prime, then we would have had to deal with the cyclotomic polynomial

$$\Phi_m(X) = \prod_{\zeta \in U'_m} (X - \zeta) \ , \tag{12}$$

where U'_m is the subset of primitive m-th roots of unity in the set of complex numbers (to learn more about cyclotomic fields see [3, pp. 140-148]. In this case, there are difficulties with a basis representation; also we have more variables than equations. However, it still is possible to obtain some new identities.

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