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A GENERALIZED DIAGONAL WYTHOFF NIM

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Abstract

The P-positions of the 2-pile take-away game of Wythoff Nim lie on two *beams* of slope $\frac{\sqrt{5}+1}{2}$ and $\frac{\sqrt{5}-1}{2}$ respectively. We study extensions to this game where a player may also remove simultaneously pt tokens from either of the piles and qt from the other, where $p < q$ are given positive integers and where t ranges over the positive integers. We prove that for certain pairs (p, q) the P-positions are identical to those of Wythoff Nim, but for $(p, q) = (1, 2)$ they do not even lie on two beams. By several experimental results, we conjecture a classification of all pairs (p, q) for which Wythoff Nim's beams of P-positions transform via a certain *splitting* behavior, similar to that of going from 2-pile Nim to Wythoff Nim.

1. Introduction

We study generalizations of the 2-player impartial take-away games of 2-pile Nim [2] and Wythoff Nim [7, 8, 9, 10, 11, 12, 15]. A background on impartial (take-away) games can be found in for example [1, 4]. We use some standard terminology for such games without draws. A position is a previous-player win, a *P-position*, if none of its options are P-positions; otherwise it is a next-player win, an *N-position*. We follow the conventions of *normal play*, that is, a player who is not able to move loses and the other player wins. Thus, given an impartial game, we get a recursive characterization of the set of all P-positions.

Let $\mathbb{N}$ denote the positive integers and $\mathbb{N}_0$ the nonnegative integers. A legal move in 2-pile Nim is to remove an arbitrary number of tokens from precisely one of the piles, at least one token and at most the whole pile. These type of moves can be coded in the form $(0, t)$, $(t, 0)$, $t \in \mathbb{N}$. It is easy to see that the P-positions of this game are those where the pile heights are equal, that is (x, x), for $x \in \mathbb{N}_0$, [2]. We regard these positions as an infinite P-*beam* of slope 1, with its source at the origin. See Figures 1 and 2. In the game of Wythoff Nim a player may move as in Nim and also *diagonally*, that is a player may make the move (t, t), $t \in \mathbb{N}$, that is remove the same number of tokens from each pile but at most a whole pile. Let

$$\phi = \frac{\sqrt{5}+1}{2}$$

denote the *Golden ratio*. It is well-known [15] that a position of this game is P if and only if it belongs to the set

$$\mathcal{P}(\mathrm{WN}) = \{(\lfloor \phi x \rfloor, \lfloor \phi^2 x \rfloor), (\lfloor \phi^2 x \rfloor, \lfloor \phi x \rfloor) \mid x \in \mathbb{N}_0\}.$$

Thus, in the transformation from 2-pile Nim to Wythoff Nim, the Nim-beam of P-positions has *split* into two distinct beams, with sources at the origin, of slopes ϕ and $1/\phi$ respectively. The intuitive meaning of the term *split*, defined formally in Section 4, is that there is an infinite sector, in-between two infinite regions of P-positions, which is void of P-positions.

This geometrical observation of the splitting of P-beams, going from Nim to Wythoff Nim, has motivated us to ask the following intuitive question. Does this splitting behavior continue in some meaningful way if we adjoin, to the game of Wythoff Nim, some *generalized diagonal* moves of the form

$$(pt, qt) \text{ and } (qt, pt), \tag{1}$$

where $p, q \in \mathbb{N}$ are fixed game parameters and where t ranges over $\mathbb{N}$, and then play the new game with both the old and the new moves? That is, in addition to the rules of Wythoff Nim, a legal move is to remove simultaneously pt tokens from either of the piles and qt from the other of course restricted by the number of tokens in the respective pile. Does the answer depend on the specific values of p and q? We let (p, q)GDWN, $0 < p < q$, denote this *Generalized Diagonal Wythoff Nim* extension and $\mathcal{P}(p, q)$ its set of P-positions. See Figure 1 for the rules of $(1, 2)$GDWN and its first few P-positions. In Figure 2 we give a macroscopic view of the corresponding P-beams. Given such a game, we define the sequences $a = a(p, q) = (a_n)_{n\in\mathbb{N}}$ and $b = b(p, q) = (b_n)_{n\in\mathbb{N}}$ via

$$\mathcal{P}(p, q) = \{(a_n, b_n), (b_n, a_n) \mid n \in \mathbb{N}\} \cup \{(0, 0)\} \tag{2}$$

where the ordered pairs of the form (a_n, b_n), the *upper* P-positions, are distinct, where $a_n \le b_n$, for all $n \ge 1$ and the sequence a is non-decreasing. For a technical reason we omit the terminal P-position $(a_0, b_0) = (0, 0)$ in the definition of a and b. Here we have used that, since the moves of (p, q)GDWN are *symmetric*, that is (m_1, m_2) is a move if and only if (m_2, m_1) is, the P-positions are also symmetric. The purpose of this paper is to investigate some properties of the sequences a and b from (2).

Our main theorem considers a necessary condition for a splitting of Wythoff Nim's upper P-beam for the case $(1, 2)$GDWN. (See also Conjecture 11 on page 15.)

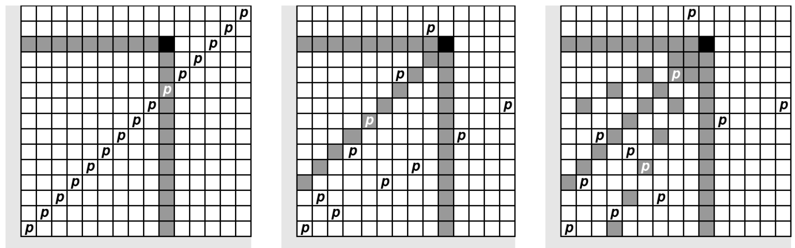

Figure 1: The figures illustrate typical moves (in dark gray) and initial P-positions of Nim, Wythoff Nim, and $(1, 2)$GDWN respectively. The black square is a given game position. The white P's represent the winning options from this position.

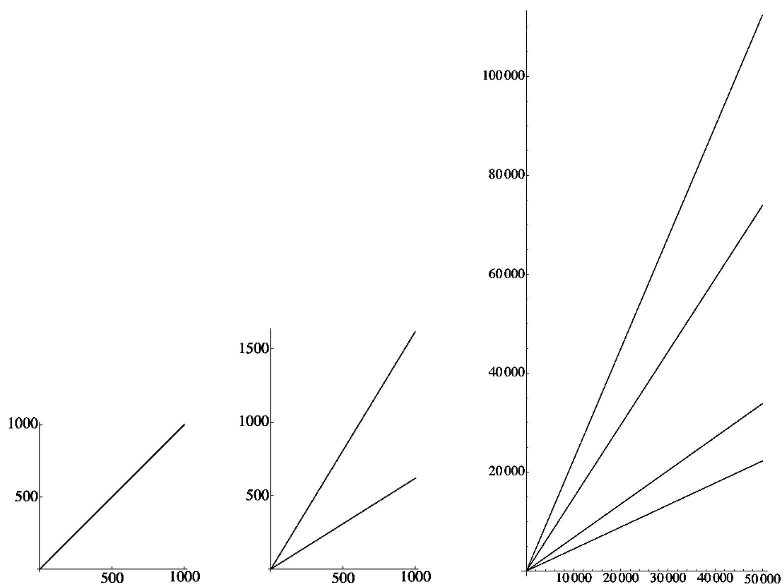

Figure 2: These figures give the initial P-positions of the games Nim, Wythoff Nim and $(1, 2)$GDWN. The left-most figure illustrates 2-pile Nim's single P-beam of slope 1. Then, in the middle we illustrate Wythoff Nim's pair of P-beams with slopes ϕ and $1/\phi$ respectively and, at last, we present the initial P-positions of $(1, 2)$GDWN, where our computations, for all x-coordinates ≤ 50000, seem to suggest that each one of Wythoff Nim's P-beams has split into two new distinct P-beams. See also Figures 3, 4 and 9.

Theorem 1. *Let $\mathcal{P}(1,2)$ define the sequences $a = a(1,2)$ and $b = b(1,2)$. Then the limit*

$$\lim_{n\in\mathbb{N}} \frac{b_n}{a_n}$$

does not exist.

In Section 2 we give the proof of Theorem 1. In Section 3, we prove that a certain subfamily of (p,q)GDWN games have identical P-positions as Wythoff Nim. In Section 4 we state two conjectures, supported by various experimental data and figures. Finally, in Section 5 we discuss some further directions for future research.

2. A Resolution of Theorem 1

Two sequences of positive integers are said to be *complementary* if each positive integer occurs precisely once in precisely one of these sequences (see, e.g., [7]). The following is a basic result concerning the sequences defined in (2).

Proposition 2. *Let $\mathcal{P}(p,q) = \{(a_i,b_i),(b_i,a_i)\} \cup \{(0,0\}$ define the sequences a and b, of which a is non-decreasing and, for all $n \in \mathbb{N}$, $a_n \le b_n$. Then*

(i) $a_1 = 1$,

(ii) $a_i < b_i$, *for all* $i > 0$,

(iii) $a_i \ne a_{i+1}$, *for all i, that is a is strictly increasing,*

(iv) $a_i \ne b_j$ *for all* $i > 0$ *and* $j > 0$,

(v) *for each $n \in \mathbb{N}$ there is an i such that either $a_i = n$ or $b_i = n$,*

that is, a and b are complementary.

Proof. By $(0,0) \in \mathcal{P}(p,q)$ and by the Nim-type moves, there can be no P-position of the form $(0,x), x \in \mathbb{N}$. For (i), notice that there is a least x such that $(1,x)$ is P. Namely, if $(p,q) = (1,2)$ then $x = 3$, otherwise $x = 2$. Clearly (ii) follows from (iv), but it makes sense to start with (ii). Thus, suppose $a_i = b_i$ for some $i > 0$. Then $(a_i,b_i) \to (0,0)$ is a legal diagonal-type move, which contradicts the definition of P. For (iii), suppose that $a_i = a_{i+1}$ for some i. Then, by (i), $i > 0$ and so either $(a_{i+1},b_{i+1}) \to (a_i,b_i)$ or $(a_i,b_i) \to (a_{i+1},b_{i+1})$ is a legal Nim-type move, but either case is ridiculous by definition of P. For (iv), suppose that there were integers $i > j > 0$ such that $a_i = b_j$. Then $(a_i,b_i) \to (b_j,a_j)$ is a Nim-type move since, by (ii), $b_i > a_i = b_j > a_j$, a contradiction. For (v), let $n \in \mathbb{N}$. Then, each position of the form

$$(a_m,b_m), a_m < n \text{ or } (b_m,a_m), b_m < n \tag{3}$$

can, by the rules of (p,q)GDWN, be reached by at most four positions of the form $(n,\cdot)$. By (iii) and (iv), there are at most n positions of the form in (3). We conclude that there exists a least $x \le 4n$ such that the position (n,x) does not have a P-position as an option. Then (n,x) is P and hence of the desired form. □

Let $\mathbb{R}$ denote the real numbers and let $f, g : \mathbb{N}_0 \to \mathbb{R}$. We use the notation $f(N) \ll g(N)$ if $f(N) < g(N)$ for all sufficiently large N. (And analogously for $\gg$), where the term sufficiently large is explained by each surrounding context.

We have use for a well-known and general result (often phrased in terms of bipartite graphs). Simple as it is, it turns out to be a very useful tool.

Lemma 3. *Let $A = \{A_i\}$ and $B = \{B_i\}$ denote sets of positive integers satisfying $A_i < A_{i+1}$, $A_i \le B_i$, for all $i \in \mathbb{N}$ and, for each $N \in \mathbb{N}$, there is precisely one i such that*

$$A_i = N \text{ or } B_i = N, \tag{4}$$

(possibly both). Then, for all $N \in \mathbb{N}$,

$$\#(A \cap \{1,\dots,N\}) \ge \frac{N}{2}.$$

Proof. Put $n = n(N) := \#(A \cap \{1,\dots,N\})$ and note that $A_n \le N < A_{n+1}$. Suppose on the contrary that

$$2n < N \tag{5}$$

for some $N \in \mathbb{N}$. By definition, $B_{n+1} \ge A_{n+1} > N$, so that there are at most n numbers from the B-sequence less than or equal to N. This gives

$$2n \ge \#(A \cap \{1,\dots,N\}) + \#(B \cap \{1,\dots,N\}) \ge N > 2n,$$

where the last inequality is by (5). □

Note that our sequences a and b satisfy the conditions of (A_i) and (B_i) in Lemma 3. The next lemma deepens this result to another powerful tool for our purposes. Roughly spoken, it concerns the structure of P-positions for extensions of Wythoff Nim.

Lemma 4. *Let $A = (A_i)$ and $B = (B_i)$ denote complementary sequences of positive integers satisfying, for all $i, j \in \mathbb{N}$, $A_i < A_{i+1}$, $A_i \le B_i$ and*

$$B_i - A_i = B_j - A_j \text{ if and only if } i = j. \tag{6}$$

Then the set $\{i \mid B_i > \phi A_i\}$ is infinite.

Proof. Let $C \in \mathbb{R}$ with $C > 1$ and define the set

$$S = S(C) := \{i \mid B_i < CA_i\}.$$

For all i, define

$$\delta_i := B_i - A_i \geq 0.$$

Suppose that S contains all but finitely many numbers and let

$$n > \max\{\delta_i \mid i \in (\mathbb{N}_0 \setminus S)\}. \tag{7}$$

By definition of A and B, there is an $x \leq n$ such that $\delta_x \geq n$. Together with the definition of S this gives,

$$1 + \frac{n}{A_x} = \frac{A_x + n}{A_x} \leq \frac{B_x}{A_x} < C,$$

so that

$$n \ll (C-1)A_x \leq (C-1)A_n, \tag{8}$$

since (A_i) is increasing. On the other hand, by Lemma 3, we have that $n \geq \frac{A_n}{2}$, so that we may conclude that $C > \frac{3}{2}$. Denote with $c = C - 1 > \frac{1}{2}$.

For sufficiently large n, by (8) and complementarity, the number of i's such that $B_i < A_n$ is

$$A_n - n \gg (2 - C)A_n = (1 - c)A_n, \tag{9}$$

which, by (6) gives that, there is a least $j \leq n$, such that

$$\delta_j \geq (1 - c)A_n. \tag{10}$$

We may ask, where is this least j? Define the set

$$\rho = \rho(c, n) := \{i \mid A_i \leq cA_n\}.$$

`Case 1` If $j \in \rho$, then

$$C \gg \frac{B_j}{A_j} = 1 + \frac{\delta_j}{A_j} \geq 1 + \frac{1-c}{c}$$

which is equivalent to

$$C^2 > C + 1,$$

which holds if and only if $C > \phi$.

`Case 2:` If $j \notin \rho$, then since B_j is the least number in the B-sequence such that (10) holds, for $i < j$, we get $B_i < (1-c)A_n + A_i \leq A_n$. But then, by (9), $(1-c)A_n \leq \max \rho$. By applying the same argument as in (8), we also have that

$$\max \rho \ll c^2 A_n$$

and so, again, $C > \phi$. □

For the rest of this section, we let $(a_i) = a(1,2)$ and $(b_i) = b(1,2)$.

Proposition 5. *Let $R := \{b_i/a_i \mid i \in \mathbb{N}\}$. Then the following three items hold.*

(i) *The set $(\phi, \infty) \cap R$ is infinite.*

(ii) *Let $C < 2 \leq D$ denote two real constants with $\beta := D - C < 1/2$. Then $([1, C) \cup (D, \infty)) \cap R$ is infinite.*

(iii) *The set $[1, 2] \cap R$ is infinite.*

Proof. Item (i) follows since, by Proposition 2, the sequences a and b satisfy the conditions of A and B in Lemma 4, respectively. (The condition (6) is satisfied, since otherwise a removal of the same number of tokens from each heap would connect two P-positions, which is impossible.)

For item (ii), suppose, for a contradiction, that all but finitely many ratios from R lie in $[C, D]$ and define

$$r := \#\{i \mid b_i/a_i \notin [C, D]\}.$$

Suppose now that $b_i/a_i \in [C, D]$ with $a_i \leq N$, $N \in \mathbb{N}$. Then we get

$$2(N - a_i) + b_i \in I(N) := [CN, DN]. \tag{11}$$

The upper bound follows from $b_i - 2a_i \leq Da_i - 2a_i \leq (D - 2)N$ and the lower is similar.

Denote by $J(N)$ the number of pairs (a_i, b_i) with $a_i \leq N$ such that $b_i/a_i \in [C, D]$. Then, by Lemma 3 and definition of r, for all $N > r$,

$$J(N) \geq \frac{N}{2} - r.$$

This gives that, for all $\epsilon > 0$, for all sufficiently large $N = N_\epsilon$, we have that

$$\frac{J(N) - 1}{N} \geq \frac{1}{2} - \frac{r + 1}{N} > \frac{1}{2} - \epsilon. \tag{12}$$

In particular we may take $\epsilon := 1/2 - D + C > 0$ and define N as N', a fixed integer strictly greater than $\frac{2(r+1)}{1-2(D-C)}$. The number of integer points in $I(N')$ is

$$\lceil DN' \rceil - \lceil CN' \rceil.$$

If we divide this expression by N' and compare with our definition of ϵ, we get

$$\frac{\lceil DN' \rceil - \lceil CN' \rceil}{N'} \leq \frac{1}{2} - \epsilon + \frac{1}{N'}.$$

Hence, by (12), we get that the number of integer points in the interval $I(N')$ is

$$\lceil DN' \rceil - \lceil CN' \rceil < J(N'). \tag{13}$$

Observe now that each pair (a_i, b_i), which is counted by $J(N')$, defines a line of slope 2 which intersects the line $x = N'$ at an integer point, which resides inside the interval $I(N')$. Thus, by (13), the Pigeonhole principle gives that, for some integer $t \in I(N')$, there exists a pair $i < j < N'$ such that

$$\begin{aligned} t &= 2(N' - a_i) + b_i \\ &= 2(N' - a_j) + b_j. \end{aligned}$$

But then $2(a_j - a_i) = b_j - b_i$ so that, by the definition of $(1, 2)$GDWN, there is a move $(a_j, b_j) \to (a_i, b_i)$, which contradicts definition of P. This proves item (ii).

Let us proceed with item (iii). We begin by proving two claims.

Claim 1. Let $N \in \mathbb{N}_0$ be such that $b_N \geq 2a_N$. Then, if there exists a least $k \in \mathbb{N}$ such that $b_{N+k} > 2a_{N+k}$, it follows that $b_{N+k} - 2a_{N+k} = b_N - 2a_N + 1$. (By Table 1 the first such case is $N = 0, k = 1$ and the "γ-row" gives an initial sequence of pairs "(N_i, k_i)" as follows: $(0, 1), (1, 1), (2, 5), (7, 1), (8, 2), (10, 4), (14, k_8)$.)

b_n	0	3	6	5	10	14	17	25	28	18	35	23	31	29	48	32
a_n	0	1	2	4	7	8	9	11	12	13	15	16	19	20	21	22
δ_n	0	2	4	1	3	6	8	14	16	5	20	7	12	9	27	10
γ_n	0	1	2	-3	-4	-2	-1	3	4	-8	5	-9	-7	-11	6	-12
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Table 1: Here (a_n, b_n) represents a P-position of $(1, 2)$GDWN for $0 \leq n \leq 18$. Further, $\delta_n = b_n - a_n$ and $\gamma_n = b_n - 2a_n$.

Proof of Claim 1. Suppose that $N > 1$ is chosen smallest possible such that, unlike the assumption, there is a least $k > 0$ such that

$$\frac{b_{N+k}}{a_{N+k}} > 2,$$

and $b_{N+k} - b_N \neq 2(a_{N+k} - a_N) + 1$. Then, by the minimality of N, we must have

$$b_{N+k} - 1 - 2a_{N+k} > b_N - 2a_N, \tag{14}$$

which, by Proposition 2, implies that $(a_{N+k}, b_{N+k} - 1)$ is N. That is, there exists a $j < N$ such that

$$b_{N+k} - 1 - b_j = \gamma(a_{N+k} - a_j),$$

where $\gamma = 0, \frac{1}{2}, 1$ or 2. Put

$$y := b_N - 2(a_N - a_j).$$

Altogether, by (14), we get

$$b_j - y = (2-\gamma)(a_{N+k} - a_j) + 1 > 0.$$

But, by minimality of N, b_j must be strictly less than y, a contradiction. ⋄

From this point and onwards we assume that $[1,2] \cap R$ is finite (and hence we are going to find a contradiction). The next claim concerns a consequence of this assumption combined with the result in Claim 1.

Claim 2. Suppose that $[1,2] \cap R$ is finite. Then there is an $r \in \mathbb{N}$ such that, for all $N \geq r$, we have

$$b_{N+1} - b_N = 3 \text{ and } a_{N+1} - a_N = 1 \tag{15}$$

or

$$b_{N+1} - b_N = 5 \text{ and } a_{N+1} - a_N = 2. \tag{16}$$

Proof. Since $[1,2] \cap R$ is finite, we get that, for some $s \in \mathbb{N}$, for all $j \geq s$, $b_j/a_j > 2$. By Claim 1, since (a_i) is increasing, this implies that $b_{j+1} \geq b_j + 3$. Further, by definition of a_{N+1}, if N is such that $a_N \geq b_s$, this gives

$$a_{N+1} - a_N \leq 2. \tag{17}$$

Plugging this into the result of Claim 1 we get either (15) or (16). This first part of the proof of Claim 2 implies that both a and $(b_i)_{r\leq i}$ are strictly increasing. Then, by complementarity of a and b it follows that ($\star$) there are infinitely many N's such that (16) holds. ⋄

The remainder of the proof consists of a geometric argument contradicting the 'greedy' definition of the b-sequence. We show (implicitly) that there would be an N-position too much if (iii) fails to hold.

By Claim 2, we can find $r < u < v$ such that (16) holds for both $b_u < b_v$. Define four lines accordingly:

$$\begin{aligned} l_u(x) &= x + b_u, \\ l_{u+1}(x) &= x + b_u + 3, \\ l_v(x) &= b_v, \\ l_{v+1}(x) &= b_v + 5, \end{aligned}$$

These fours lines intersect at the positions (lattice points) $((\alpha_i, \beta_i))_{i\in\{1,2,3,4\}} = ((b_v - b_u - 3, b_v), (b_v - b_u, b_v), (b_v - b_u + 2, b_v + 5), (b_v - b_u + 5, b_v + 5))$, defining the corners of a parallelogram. Denote the set of positions *strictly* inside this parallelogram by $\mathcal{K}$. Then, by inspection

$$\#\mathcal{K} = 8$$

and, by $(\star)$, we may assume that we have chosen v sufficiently large so that, for all $(x, y) \in \mathcal{K}$,

$$1 < \frac{y}{x} < 2. \tag{18}$$

Denote by $\mathcal{L}$ another set of lines satisfying the following conditions. A line l belongs to $\mathcal{L}$ if and only if:

(a) Its slope is either $1/2, 2$ or ∞.

(b) It intersect a point of the form (a_s, b_s) or (b_s, a_s) with $s \geq r$.

(c) It intersects $\mathcal{K}$.

By the definition of $\mathcal{K}$ it follows from (16) that i and j may be defined such that each line of form (b) and (c) is also of the form (a). Again, by $(\star)$ we may assume that we have chosen i and j sufficiently large so that the first part of (b) together with (c) implies $s \geq r$.

Claim 3 There is a game position (lattice point) in the set $\mathcal{K} \setminus \mathcal{L}$.

Clearly, by the definition of (b_i) and by (18), this claim contradicts the assumption that $[1, 2] \cap R$ is finite. (In fact it would imply the existence of an N-position in $\mathcal{K}$ without a P-position as a follower.)

Proof of Claim 3. Let $\mathcal{K}' := \{(0, 0), (1, 0), (1, 1), (2, 1), (2, 2), (3, 2), (3, 3), (4, 3)\}$. Then $\mathcal{K}'$ is simply a linear translation of $\mathcal{K}$. (Namely, given $(x, y) \in \mathcal{K}$, $T(x, y) = x - (b_j - b_i - 1), y - (b_j + 1) \in \mathcal{K}'$.)

Let $\alpha \in \mathbb{R}$. Clearly, the two lines $x + \alpha$ and $x + 3 + \alpha$ can together cover at most three points in $\mathcal{K}'$, namely choose $\alpha = 0$ or 1. The two lines $2x - \alpha$ and $2x - 5 - \alpha$ can cover at most two points in $\mathcal{K}'$, namely we may choose $\alpha = 0, 2$ or 3. (In fact, for the two latter cases it is only the former line that contributes.) On the other hand, the two lines $x/2 + \alpha$ and $x/2 + 5/2 + \alpha$ can cover at most two points in $\mathcal{K}'$, namely, if we choose $\alpha = 0, 1/2$ or 1. (In fact, for these α, it is only the former line that contributes.)

Fix any set of the above six lines, depending only on the choices of α for the respective cases, and denote this set by $\mathcal{L}'$. Then, as we have seen, $\#(\mathcal{L}' \cap \mathcal{K}') \leq 7$.

But, by Claim 2, an instance of $\mathcal{L} \cap \mathcal{K}$ is simply a linear translation of some set $\mathcal{L}' \cap \mathcal{K}'$. $\diamond$

The proposition now follows. $\square$

Let us conclude what has been accomplished so far.

Proof of Theorem 1 Suppose on the contrary that $\alpha := \lim_{n\in\mathbb{N}} \frac{b_n}{a_n}$ exists. Then either

(a) $\alpha \in [1, \phi)$,

(b) $\alpha \in [\phi, 2]$, or

(c) $\alpha \in (2, \infty]$.

By Proposition 5 (i), (a) is impossible. On the other hand (b) is contradicted by Proposition 5 (ii) with, say, $C = \phi$ and $D = 2$. For the last case, Proposition 5 (iii) gives a contradiction. $\square$

3. When GDWN and Wythoff Nim Have the Same P-positions

Let us introduce some more notation. An ordered pair of positive integers belongs to the set $\mathcal{W}$ if it is of the form $(\lfloor \phi t \rfloor, \lfloor \phi^2 t \rfloor)$, a *Wythoff pair*, or of the form $(\lceil \phi t \rceil, \lceil \phi^2 t \rceil)$, a *dual Wythoff pair*, $t \in \mathbb{N}$. Hence

$$\mathcal{W} = \{(1,2), (2,3), (3,5), (4,6), (4,7), (5,8), (6,10), (7,11), \ldots\}.$$

Notice that, for all $t \in \mathbb{N}$, $\lceil \phi^2 t \rceil / \lceil \phi t \rceil < \phi$ and $\lfloor \phi^2 t \rfloor / \lfloor \phi t \rfloor > \phi$.

The main result of this section is the following.

Theorem 6. *Suppose that $(p, q) \notin \mathcal{W}$. Then $\mathcal{P}(p, q) = \mathcal{P}(WN)$ if $1 < \frac{q}{p} < \phi$. That is, with $a = a(p, q)$ and $b = b(p, q)$, for all $n \in \mathbb{N}$, $a_n = \lfloor \phi n \rfloor$ and $b_n = \lfloor \phi^2 n \rfloor$.*

Before proving this theorem we need to develop some facts from combinatorics on Sturmian words. For some background and terminology on this subject we refer to [14, Sections 1 and 2].

We are interested in the (infinite) Sturmian words s and s' on the alphabet $\{0, 1\}$ and the corresponding (non-Sturmian) 'translates' t and t' on $\{1, 2\}$. For all $n \in \mathbb{N}_0$, the n^{th} letter is

$$\begin{aligned} s(n) &:= \lfloor \phi(n+1) \rfloor - \lfloor \phi n \rfloor - \lfloor \phi \rfloor, \\ s'(n) &:= \lceil \phi(n+1) \rceil - \lceil \phi n \rceil - \lfloor \phi \rfloor, \\ t(n) &:= \lfloor \phi(n+1) \rfloor - \lfloor \phi n \rfloor = s(n) + 1 \end{aligned}$$

and

$$t'(n) := \lceil \phi(n+1) \rceil - \lceil \phi n \rceil = s'(n) + 1$$

respectively.

Then s and t (s' and t') are the lower (upper) mechanical words with *slopes* $1/\phi$ and ϕ, respectively, and intercept 0. An *irrational mechanical* word has irrational slope. The *characteristic* word belonging to s and s' is $c = s(1)s(2)s(3)\ldots$. Namely, we have $s(0) = 0$, $s'(0) = 1$ and otherwise, for all $n > 0$, $s(n) = s'(n) = 0$ or $s(n) = s'(n) = 1$. In fact, we have

$$s = 01011010110\ldots$$

and

$$s' = 11011010110\ldots.$$

Let x denote a finite word on $\{0, 1\}$. Then $l(x)$ and $h(x)$ denote the number of letters and 1's in x, respectively. Let α and β be two *factors* of a Sturmian word w. Then w is *balanced* if $l(\alpha) = l(\beta)$ implies $\mid h(\alpha) - h(\beta) \mid \leq 1$. By [14, Section 2], both s and s' are balanced (aperiodic) words. We will also need the following result from the same source.

Lemma 7. ([14]) *Suppose two irrational mechanical words have the same slope. Then their respective set of factors are identical.*

We also use the following notation. Let $x = x_1x_2\ldots x_n$ be a factor of a mechanical word on n letters. Then we define the *sum* of x as $\sum x := x_1 + x_2 + \ldots + x_n$. For example the sum of 2121 equals 6. We let $\xi_n(r)$ denote the unique n-letter *prefix* of an infinite word r. Note that $\sum \xi_n(t) = \sum \xi_n(s) + n$ and $\sum \xi_n(t') = \sum \xi_n(s') + n$, for all n.

Lemma 8. *Let x be any factor of s (or s'). Then*

$$\sum x = \sum \xi_{l(x)}(s) \text{ or } \sum x = \sum \xi_{l(x)}(s').$$

Proof. If two factors of s have the same length and the same height, then, since the number of 1's in the respective factors must be the same, their sums are identical. Therefore, if $h(x) = h(\xi_{l(x)}(s))$, this implies $\sum x = \sum \xi_{l(x)}(s)$.

Assume on the contrary that $h(x) \neq h(\xi_{l(x)}(s))$. On the one hand, for all n, $h(\xi_n(s)) = h(\xi_n(s')) - 1$. On the other hand, the balanced condition implies that if x is a factor of s with a given length, then $h(x)$ takes the value of one of two consecutive integers. It follows that $h(x) = h(\xi_{l(x)}(s'))$. But then, by the initial observation, we are done. □

The following proposition assures that for each pair in $\mathcal{W}$, the P-positions of GDWN are distinct from those of Wythoff Nim. An alternative proof appears in [5].

Proposition 9 ([5]). *Let $p, q \in \mathbb{N}$. Then $(p, q) \in \mathcal{W}$ if and only if there exists a pair $m, n \in \mathbb{N}_0$ with $m < n$ such that*

$$(p, q) = (\lfloor \phi n \rfloor - \lfloor \phi m \rfloor, \lfloor \phi^2 n \rfloor - \lfloor \phi^2 m \rfloor).$$

Proof. Let $(p, q) \in \mathcal{W}$. If $(p, q) = (\lfloor r\phi \rfloor, \lfloor r\phi^2 \rfloor)$, for some $r \in \mathbb{N}$, we may take $m = 0$. If $(p, q) = (\lfloor r\phi \rfloor + 1, \lfloor r\phi^2 \rfloor + 1)$, for some $r \in \mathbb{N}$, then, since s and s' are mechanical with the same slope, by Lemma 7, $\xi_r(s')$ is a factor of s. But then, $\lfloor r\phi \rfloor + 1 = \sum \xi_r(t') = \sum \xi_n(t) - \sum \xi_m(t)$ for some $n - m = r$. This gives $(p, q) = (\lfloor r\phi \rfloor + 1, \lfloor r\phi \rfloor + 1 + r) = (\lfloor n\phi \rfloor - \lfloor m\phi \rfloor, \lfloor n\phi \rfloor - \lfloor m \rfloor + n - m) = (\lfloor n\phi \rfloor - \lfloor m\phi \rfloor, \lfloor \phi^2 n \rfloor - \lfloor \phi^2 m \rfloor)$. For the other direction, let (p, q) and $m < n$ be as in the proposition. Let x denote the factor of s which consists of the $n - m$ last letters in the prefix $\xi_n(s)$. Then, by $l(x) = n - m$ and Lemma 8, we may take $p = \sum \xi_{l(x)}(t) = \lfloor l(x)\phi \rfloor$ or $p = \sum \xi_{l(x)}(t') = \lfloor l(x)\phi \rfloor + 1$. In either case, the assumption gives $q = p + l(x)$, and so $(p, q) \in \mathcal{W}$. □

Proof of Theorem 6. We need to show that, for all n, $(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor) = (a_n, b_n)$, where $a = a(p, q)$ and $b = b(p, q)$ and where $(p, q) \notin \mathcal{W}$ with $1 < p/q < \phi$. The problem of, for each candidate N-position, finding a move of (p, q)GDWN to a candidate P-position was resolved already in [15], namely it suffices to use the Wythoff Nim type moves. In order to assure that no two P-positions can be connected by a move it suffices to use Proposition 9. Namely, we proved, in particular, that there exist integers $0 \le m < n$ such that $(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor) \to (\lfloor \phi m \rfloor, \lfloor \phi^2 m \rfloor)$ is a legal move of (p, q)GDWN only if $(p, q) \in \mathcal{W}$.

Suppose there are integers $0 \le m < n$ such that $(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor) \to (\lfloor \phi^2 m \rfloor, \lfloor \phi m \rfloor)$ is a legal move of (p, q)GDWN. Then $(\lfloor \phi n \rfloor - \lfloor \phi^2 m \rfloor, \lfloor \phi^2 n \rfloor - \lfloor \phi m \rfloor) = (tp, tq)$ for some $t \in \mathbb{N}$. Hence

$$\lfloor \phi n \rfloor = tp + \lfloor \phi^2 m \rfloor$$

and

$$\lfloor \phi^2 n \rfloor = tq + \lfloor \phi m \rfloor.$$

Then $\phi < \lfloor \phi^2 n \rfloor / \lfloor \phi n \rfloor = (tq + \lfloor \phi m \rfloor)/(tp + \lfloor \phi^2 m \rfloor) < q/p$ since $0 \le \lfloor \phi m \rfloor \le \lfloor \phi^2 m \rfloor$ for all m. □

4. Experiments, Conjectures, and Figures

We conjecture a continuation of Theorem 1 if and only if (p, q) is a Wythoff pair or a dual Wythoff pair, as defined in Section 3.

Conjecture 10. Consider the sequences a and b as defined by (p, q)GDWN. If $(p, q) \notin \mathcal{W}$, then the limit

$$\lim_{n \in \mathbb{N}} \frac{b_n}{a_n} \tag{19}$$

exists and equals $\phi = \frac{1+\sqrt{5}}{2}$. Otherwise, if $(p, q) \in \mathcal{W}$, the limit (19) does not exist.

In view of various experimental results, displayed in several figures later in this section, we will next strengthen this conjecture. To this purpose we give a rigorous definition of splitting sequences, introduced in Section 1. Let $\mu \in \mathbb{R}$ with $\mu > 0$. A sequence of ordered pairs of positive integers $((x_i, y_i))_{i \in \mathbb{N}}$ *μ-splits* if there is an $\alpha \in \mathbb{R}$ such that,

- there are at most finitely many i's for which $y_i/x_i \in (\alpha, \alpha + \mu)$,
- there are infinitely many i's for which $y_i/x_i \in [0, \alpha]$
- there are infinitely many i's for which $y_i/x_i \in [\alpha + \mu, \infty)$.

We say that $((x_i, y_i))_{i \in \mathbb{N}}$ *splits* if there is a μ such that $((x_i, y_i))_{i \in \mathbb{N}}$ μ-splits. If $((x_i, y_i))_{i \in \mathbb{N}}$ splits we may define complementary sequences (l_i) and (u_i) such that for all sufficiently large i

$$y_{l_i}/x_{l_i} \in [0, \alpha]$$

and

$$y_{u_i}/x_{u_i} \in [\alpha + \mu, \infty).$$

As we have seen, the P-positions of Nim do not split, but those of Wythoff Nim do. Indeed, the latter 1-splits (with $\alpha = \phi - 1$). See also [13], where we demonstrate a splitting of Wythoff Nim's P-beams in a slightly different context. In that paper the split is produced by a certain *blocking maneuver* (the previous player may block off at most one Wythoff Nim type option at each stage of the game).

The experimental result shown in Figure 2 suggests that the upper P-beam of $(1, 2)$GDWN splits precisely once in the following sense. We say that a sequence of ordered pairs of positive integers $((x_i, y_i))_{i \in \mathbb{N}}$ *splits twice* if the following criteria are satisfied: there are $\alpha, \beta, \mu, \nu \in \mathbb{R}$ with $\mu, \nu > 0$ and $\beta > \alpha + \mu$ such that,

- there are at most finitely many i's for which $y_i/x_i \in (\alpha, \alpha + \mu) \cup (\beta, \beta + \nu)$,
- there are infinitely many i's for which $y_i/x_i \in [0, \alpha]$,
- there are infinitely many i's for which $y_i/x_i \in [\alpha + \mu, \beta]$,
- there are infinitely many i's for which $y_i/x_i \in [\beta + \nu, \infty)$.

See also Section 5. The sequence $((x_i, y_i))_{i \in \mathbb{N}}$ splits *precisely once* if it splits once, but not twice.

Conjecture 11. The sequence of upper P-positions of (p,q)GDWN splits precisely once if and only if $(p,q) \in \mathcal{W}$. Furthermore, if $(p,q)=(1,2)$ or $(p,q)=(2,3)$, then there is a pair of increasing complementary sequences (l_i) and (u_i) such that both $\eta = \lim_{i\to\infty} \frac{b_{l_i}}{a_{l_i}}$ and $\gamma = \lim_{i\to\infty} \frac{b_{u_i}}{a_{u_i}}$ exist with real $1 < \eta < \phi < \gamma \le 3$. See also Table 3 for some conjectured μ-splits.

We support our conjectures with data for several games (p,q)GDWN including all games with $p < q \le 9$. By Section 3, it suffices to investigate the cases for which $q/p > \phi$. Thus, we have explored several thousands of the upper P-positions for the 20 games (p,q)GDWN, where

$$(p,q) \in \{(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(1,9),(2,4),(2,5),\\ (2,6),(2,7),(2,8),(2,9),(3,6),(3,7),(3,8),(3,9),(4,8),(4,9),(5,9)\}. \tag{20}$$

This set consists exclusively of non-$\mathcal{W}$ pairs. Hence, to support Conjectures 10 and 11, we need to show experimental results pointing at a convergence of b_i/a_i to ϕ. Our question is: Given (p,q), is there a reasonably "small" $N_0 \in \mathbb{N}$ such that

$$\frac{b_i}{a_i} \in [1.6175, 1.6185] \text{ for all } i \in [N_0, N_0 + 2000]? \tag{21}$$

For each (p,q) if we find such an N_0 we stop the computation. Our simulations provided a positive answer with $N_0 \le 10000$ for all (p,q) as in (20), except $(1,3),(2,4),(3,6),(5,9)$. If we would have asked the same question, but with the upper bound in (21) exchanged for 1.623, then the answer would have been positive, with $N_0 = 15000$, also for these four pairs. In Figure 5 we display the results for the pair with slowest convergence in the test (20), namely $(2,4)$. In this figure we also display the corresponding ratios for the game $(7,12)$GDWN which also appear to converge, but in an even slower rate.

We also support our conjectures via computations of initial ratios b_i/a_i for some $\mathcal{W}$-pairs in Figures 3, 4 and 6, namely

$$(p,q) \in \{(1,2),(2,3),(3,5),(4,6),(4,7),(5,8),(6,10),(7,11)\}.$$

In Table 3 we conjecture μ-splits for the corresponding upper P-positions. See also Figures 3 and 4 for the games $(1,2)$GDWN and $(2,3)$GDWN.

Then, in Figures 7 and 8, we display some more exotic games. The game $(31,52)$GDWN passes the test in (21) with $N_0 = 5000$, but $(31,51)$GDWN seems so to converge much slower. It does satisfy (21) for $\approx 99\%$ of the ratios for $N_0 = 13000$. On the other hand, in support of our conjectures, the games $(31,50)$GDWN, $(32,52)$GDWN and $(731,1183)$GDWN violate the test in (21) a lot, as far as we have been able to compute.

b_n	0	2	5	7	10	17	14	19	18	20	27	33
a_n	0	1	3	4	6	8	9	11	12	13	16	21
δ_n	0	1	2	3	4	9	5	8	6	7	11	12
n	0	1	2	3	4	5	6	7	8	9	10	11

Table 2: Here (a_n, b_n) represent P-positions of $(2,4)$GDWN. Notice that $(8,13) - (2,1) = (3 \times 2, 3 \times 4)$, so that $(8,13)$ is the first Wythoff pair for which there is a move to a P-position of $(2,4)$GDWN. However, this type of connection of a Wythoff pair to a P-position does not seem to enforce a later split. Our computations rather suggest that the quotient b_n/a_n converges to ϕ in accordance with Conjecture 10 and Figure 5.

(p,q)	μ	lower	upper	i'	total
$(1,2)$	0.73	1.49	2.22	844	31523
$(2,3)$	0.29	1.42	1.71	346	21768
$(3,5)$	0.17	1.57	1.74	118	20565
$(4,6)$	0.10	1.49	1.59	504	22807
$(4,7)$	0.12	1.64	1.76	734	22170
$(5,8)$	0.06	1.58	1.64	167	21237
$(6,10)$	0.07	1.62	1.69	910	20962
$(7,11)$	0.06	1.56	1.61	570	21256

Table 3: We conjecture μ-splits for (p,q)GDWN corresponding to column 1. The index i' indicates the greatest known index such that $b_{i'}/a_{i'}$ lies in between the 'lower' and 'upper' bounds for the conjectured μ-split. The ' total' column gives the number of computed upper P-positions.

Remark 12. Returning to Proposition 5 for a moment, it is clear that (i) may be adapted without any changes for general (p,q)'s. As it stands, however, the combined ideas in (i), (ii) and (iii) do not seem to suffice to prove an actual split of the upper P-positions even for the $(1,2)$ case. Therefore we have chosen not to pursue a generalization of (ii) and (iii) at this point. An interesting observation in support of Conjecture 11 is that $(1,2)$GDWN splits if a positive proportion of the ratios b_i/a_i is greater than 2; that is, if the set $\{i \in \mathbb{N} \mid b_i/a_i > 2\}$ has positive density. This follows from an argument similar to that of the proof of Proposition 5 (iii).

Remark 13. In [12] a restriction, called Maharaja Nim, of the game $(1,2)$GDWN is studied, where precisely the "Knight-type" moves $(1,2)$ and $(2,1)$ are adjoined to the game of Wythoff Nim. In contrast to the main result of this paper, for Maharaja Nim we prove that the P-positions lie on the same beams as in Wythoff Nim. In fact, in [12], we prove that the quotients of the coordinates of the P-positions of Maharaja Nim lie within a bounded distance from the lines $\phi^{-1}x$ and ϕx respectively. In this context it is interesting to observe that the only move on the generalized diagonal $(t,2t)$ which belongs to $\mathcal{W}$ is $(1,2)$. Viewed in a slightly different perspective, we have proved that, for $(1,2)$GDWN, the non-$\mathcal{W}$ pairs on the diagonals $(t,2t)$ and $(2t,t)$, $t>1$, contribute significantly in destroying the asymptotes of the P-positions of Maharaja Nim. For yet another perspective of these type of questions, see Remark 14, Table 2 and Figure 5 for the game $(2,4)$GDWN.

Remark 14. For the main result in Section 4, since $q/p<\phi$, we did not need to consider the possibility of a connection between upper and 'lower' P-positions of Wythoff Nim. On the other hand, whenever $q/p>\phi$, by our simulations, it appears that such interferences are common. See Table 2 for the first such connection for the game $(2,4)$GDWN. It would be interesting to know whether infinitely many P-positions of $(2,4)$GDWN are Wythoff pairs. In case the answer is positive, is there an $n_0 \in \mathbb{N}$ such that, for all $y \geq x \geq n_0$, (x,y) is an upper P-position, if and only it is a Wythoff pair? In this context it would also be interesting to know whether each pair of integers (x,y) with $0<x<y$ satisfies the equality $(tx,ty)=(\lfloor \phi m\rfloor - \lfloor \phi^2 n\rfloor, \lfloor \phi^2 m\rfloor - \lfloor \phi n\rfloor)$ infinitely often for some $m,n,t\in\mathbb{N}$ (and $m>n$). These type of questions are of course relevant for any game (p,q)GDWN satisfying $q/p>\phi$.

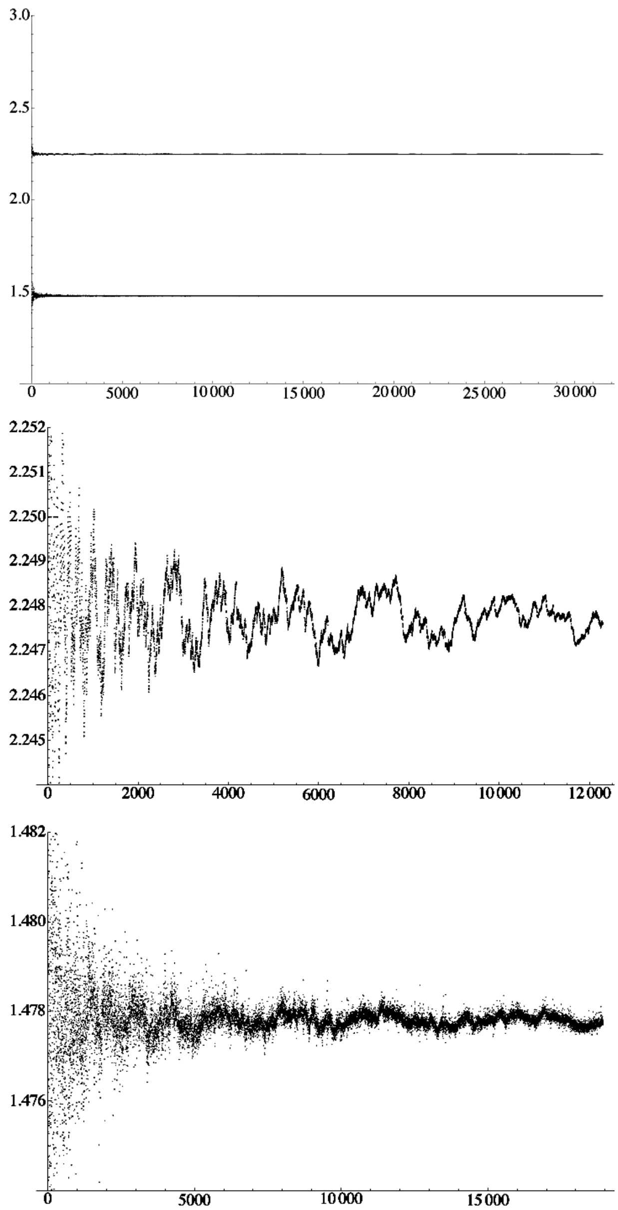

Figure 3: The upper figure illustrates the first ≈ 32000 ratios b_n/a_n for $(1,2)$GDWN. In support of Conjecture 11, there appear to exist two complementary sequences u (lower left) and l (lower right) such that $b_{u_i}/a_{u_i} \to 2.248\ldots$ (roughly 40%) and $b_{l_i}/a_{l_i} \to 1.478\ldots$ (roughly 60%), when $i \to \infty$. (The x-axes in the lower figures are relabeled corresponding to the new sequences.)

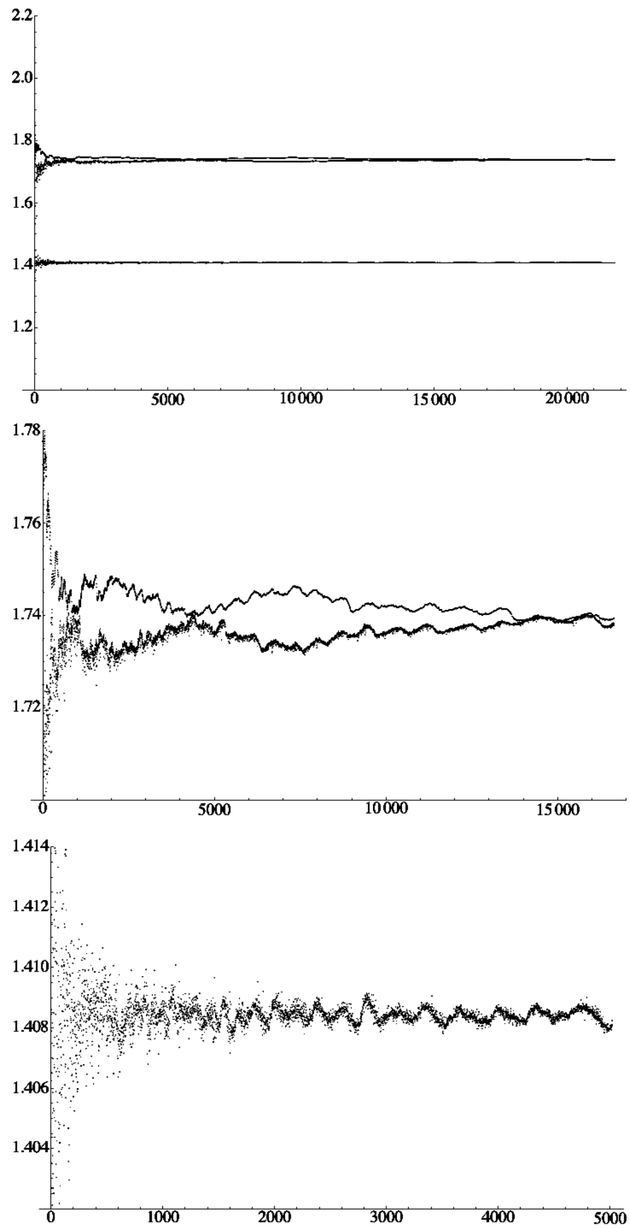

Figure 4: The first ≈ 22000 ratios b_n/a_n for $(2,3)$GDWN. In support of Conjecture 11 there appears to exist a pair of complementary sequences u (lower left figure) and l (lower right figure) such that $b_{u_i}/a_{u_i} \to 1.74\ldots$ (roughly 80%) and $b_{l_i}/a_{l_i} \to 1.408\ldots$ (roughly 20%). (The x-axes in the lower figures are relabeled corresponding to the new sequences.)

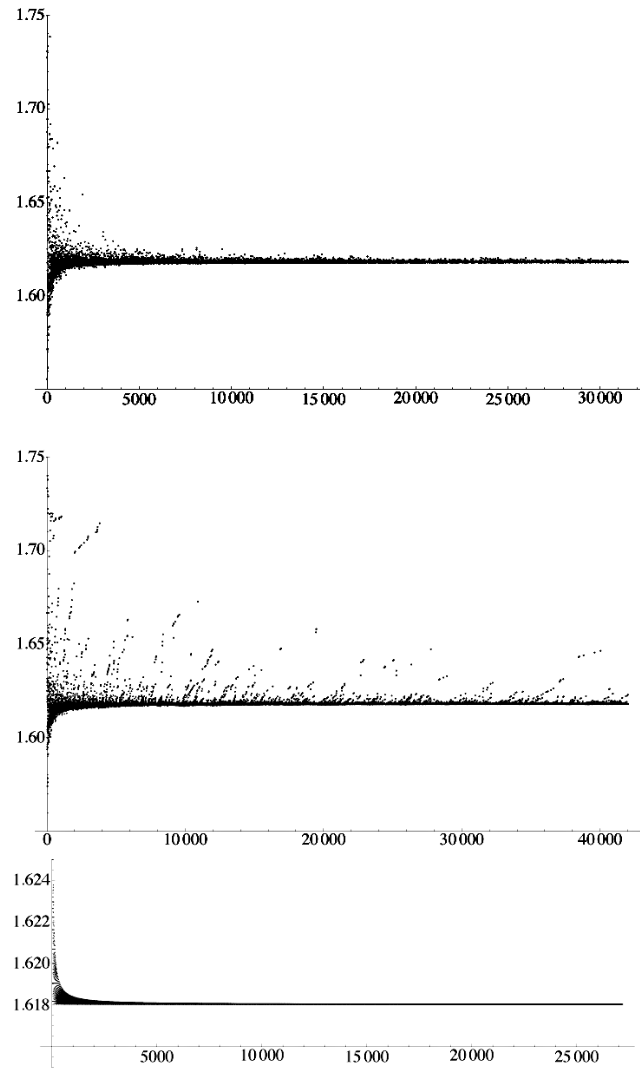

Figure 5: The two upper figures illustrate the conjectured convergence (Conjecture 10) of the ratio b_n/a_n for games (p, q)GDWN, whenever the pair $(p, q) \notin \mathcal{W}$. Here we view $(2, 4)$GDWN and $(7, 12)$GDWN, respectively. The computations include the first ≈ 32000 and ≈ 42000 upper P-positions, respectively. In each case, the upper P-beams do not seem to split, rather $b_n/a_n \to 1.618\ldots$. But, clearly the P-positions are perturbed as compared to the corresponding ratio of the coordinates of the upper P-positions of Wythoff Nim, the lower figure.

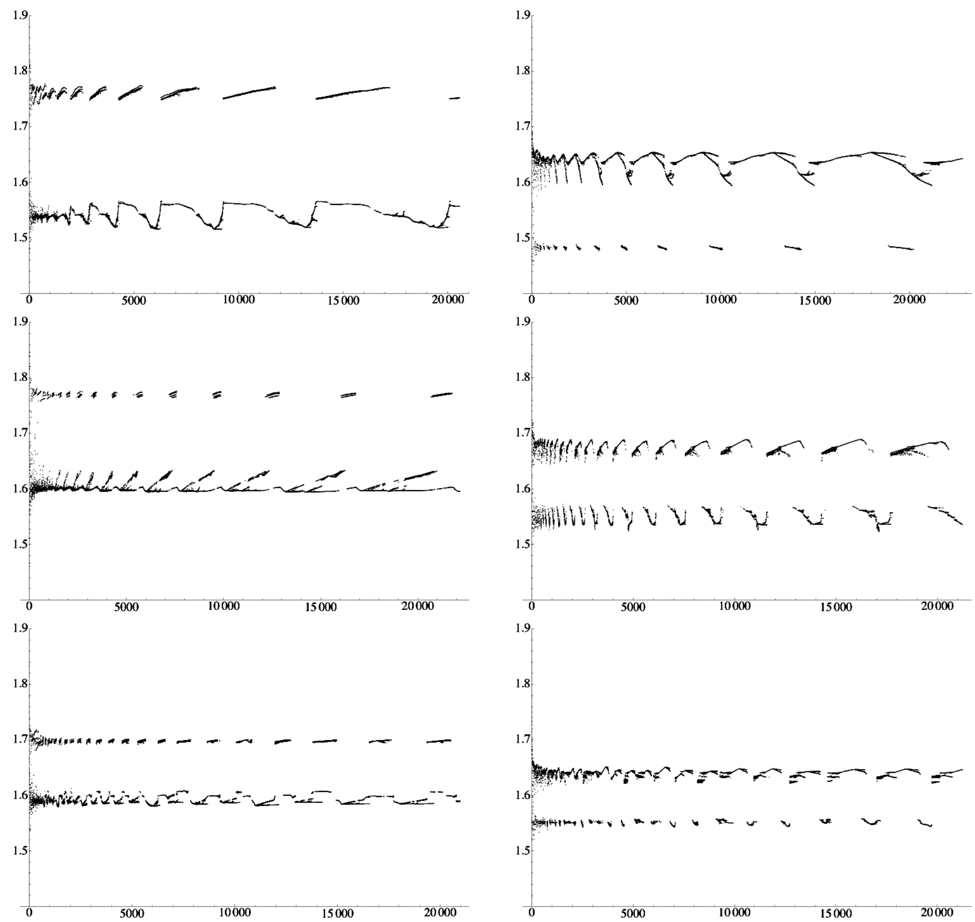

Figure 6: The figures illustrate the first $\approx$ 22000 ratios b_n/a_n for $(3,5)$GDWN, $(4,6)$GDWN, $(4,7)$GDWN, $(5,8)$GDWN, $(6,10)$GDWN and $(7,11)$GDWN, respectively. Our data suggest that the upper P-positions split for all these games, but it appears that the weaker form of our conjecture is more applicable for this case. For example, for the game $(4,6)$GDWN (top right) there seems to be a pair of complementary sequences u and l such that for large i, $1.60\ldots < b_{u_i}/a_{u_i} < 1.66\ldots$ and the quotient is 'drifting back and forth' in this interval, but possibly $b_{l_i}/a_{l_i} \to 1.48\ldots$ as $i \to \infty$. Also $(4,7)$GDWN seems to split asymptotically, namely to a pair of complementary sequences u and l such that b_{l_i}/a_{l_i} is 'drifting' in the interval $[1.59, 1.63]$ for large i, but possibly $b_{u_i}/a_{u_i} \to 1.77\ldots$. Notice that the three games on the left-hand side correspond to Wythoff pairs, whereas those on the right-hand side correspond to dual ditto.

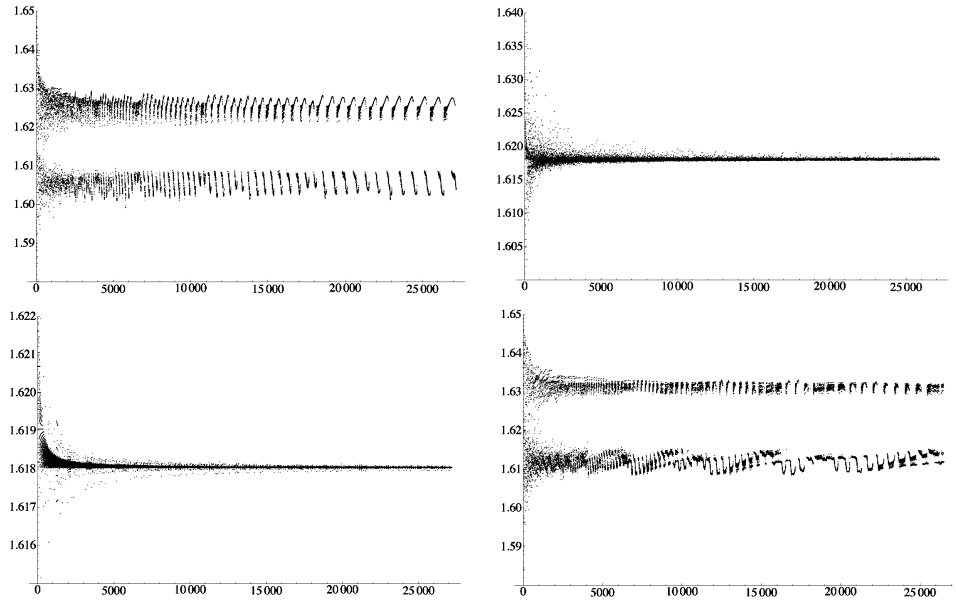

Figure 7: These figures illustrate the first ≈ 27000 ratios b_n/a_n for $(31, 50)$GDWN, $(31, 51)$GDWN, $(31, 52)$GDWN and $(32, 52)$GDWN respectively. Notice that $(31, 51)$ and $(31, 52)$ are non-$\mathcal{W}$ pairs, but $52/31 > 1.677 > 51/31 > 1.645 > \phi$. As in Figure 5, there is perturbation of the P-positions of Wythoff Nim for these two games (more accentuated for $(31, 51)$GDWN).

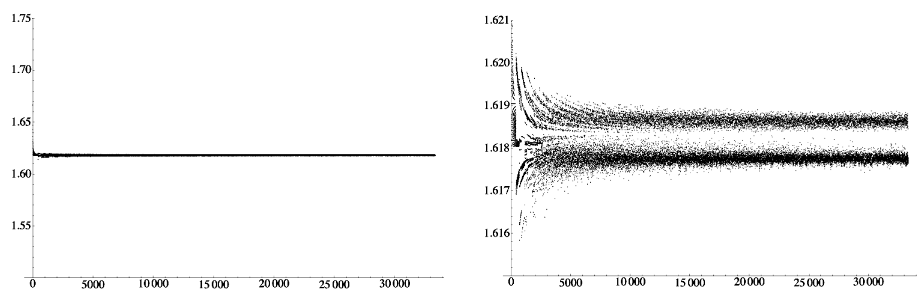

Figure 8: The figure illustrates the first ≈ 33000 ratios b_n/a_n for $(731, 1183)$GDWN. The left-most figure seems to indicate a convergence to ϕ, but the close-up to the right reveals the conjectured split. Namely $(731, 1183) \in \mathcal{W}$ (a Wythoff pair).

5. Some Further Experimental Results

At last, we provide some further motivation for a study of (generalizations of) GDWN-games. It is purely experimental. Let SGDWN denote an extension of GDWN, where S is some finite set of (p, q) pairs defining moves of the forms in (1) and where a move is permitted along any of those diagonals. For example, if $S = \{(1, 2), (2, 3)\}$, then all moves of the forms $(t, 2t), (2t, t), (2t, 3t)$ and $(3t, 2t)$ are permitted in addition to the original Wythoff Nim type moves. The P-beams of three such games are given in Figure 9. As a remark for future investigations, it is easy to verify that an analog of Proposition 2 holds for these games, that is, the 'new' sequences a and b must be complementary.

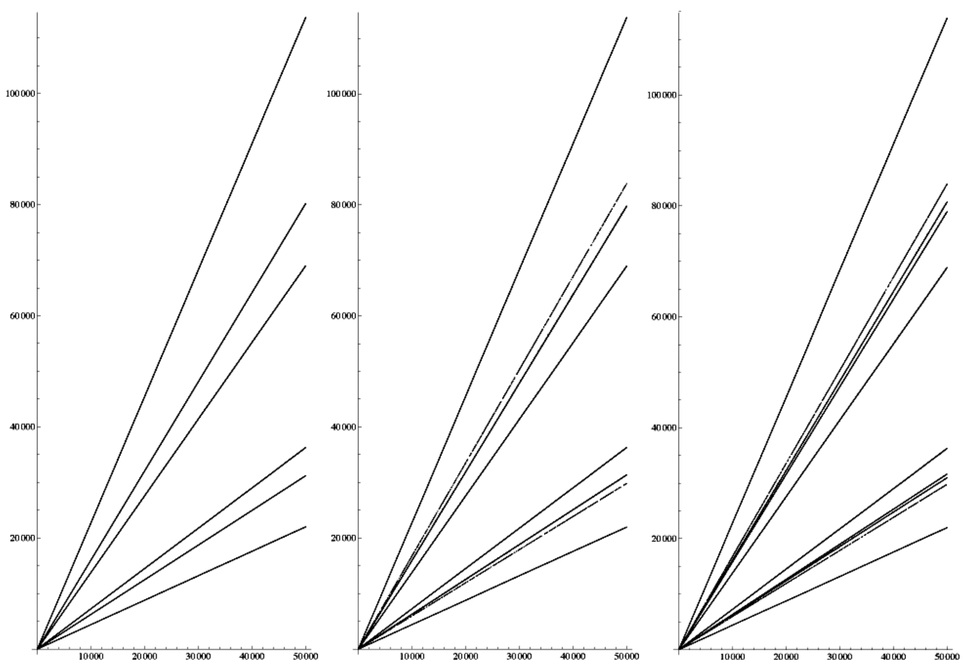

Figure 9: The figures illustrate complete sets of P-positions of $\{(1, 2), (2, 3)\}$GDWN, $\{(1, 2), (2, 3), (3, 5)\}$GDWN and $\{(1, 2), (2, 3), (3, 5), (5, 8)\}$GDWN for all x-coordinates ≤ 50000 (which apparently gives y-coordinates ≤ 120000). Namely, in Section 5 we generalize the GDWN games and permit moves along several generalized diagonals in one and the same game. A mysterious continuation of the splitting of P-beams from Figure 2 seems to have emerged.

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