ON SUPPLEMENTS OF 2×M BOARD IN TOPPLING TOWERS

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Abstract

We investigate an impartial combinatorial game: Toppling Towers. We will give genera of some positions, and provide all tame positions. Furthermore, all positions which are both tame and P-positions for board sizes of $2 \times m$ are shown.

1. Introduction

By a game we mean an impartial combinatorial game, and we restrict our attention to classical impartial games. The theory of such games can be found in [4] and [5].

Given any game G, we say informally that a *P-position* is any position u of G from which the previous player can force a win, that is, the opponent of the player moving from u can force a win. An *N-position* is any position v of G from which the next player can force a win, that is, the player who moves from v can force a win. The set of all P-positions of G is denoted by $\mathcal{P}$, and the set of all N-positions by $\mathcal{N}$. By $\mathrm{Op}(u)$ we denote all the options of u, i.e., the set of all positions that can be reached in one move from the position u. It is clear to see that for each position u of G,

$$u \in \mathcal{P} \text{ if and only if } \mathrm{Op}(u) \subseteq \mathcal{N}, \tag{1}$$

and

$$u \in \mathcal{N} \text{ if and only if } \mathrm{Op}(u) \cap \mathcal{P} \neq \emptyset. \tag{2}$$

Indeed, the player I, beginning from an N-position, will move to a P-position which exists by Eq. (2), and the player II has no choice but to go to an N-position, by Eq. (1). In other words, the set of all positions of every game can be partitioned uniquely into its subsets $\mathcal{P}$ and $\mathcal{N}$.

Recall that the P-positions and N-positions of a game are not "symmetrical". A position in $\mathcal{P}$ requires that all its options are in $\mathcal{N}$, which is a relatively rare event. Singmaster [6,7] proved that "almost all" positions are N-positions. This fact may partially explain why a winning strategy of a game is normally given by characterizing its P-positions rather than its N-positions.

There are two rules in an impartial combinatorial game: *normal play convention* and *misère play convention*. Under the normal play convention, a player *loses* if she has no options. Under the *misère play convention*, a player *wins* if she has no options. The Sprague-Grundy theorem [2] stated that an impartial game under normal play convention is equivalent to a Nim heap of a certain size. However, there is not a similar result under misère play convention. The traditional tool for the analysis of an impartial game under misère play convention is the genus symbol or the misère Grundy value, which was developed by Conway ([4, Chapter 13] and [5, Chapter 12]).

The *Toppling Towers game* is played as follows: Given an $n \times m$ board, along with the placement of $k \leq n \times m$ towers on the board. On her turn, a player can "topple" a tower in *one* of the four cardinal directions. Upon falling, the tower then also topples all contiguous towers in the direction in which it was toppled. Towers that have been toppled are then removed from the board. Under the normal play convention, a player loses if she has no towers to topple. Under the misère play convention, a player wins if she has no towers to topple.

Allen [1] investigated $1 \times m$ board in Toppling Towers game, and proved that

$$\Gamma((\boxtimes)^n) = \begin{cases} 1^{031}, & \text{if } n = 1, \\ n^{n(n\oplus 2)}, & \text{if } n > 1. \end{cases} \tag{3}$$

Moreover, Allen [1] obtained the following results: Let

$$G = (\boxtimes)^{i_1} + (\boxtimes)^{i_2} + \cdots + (\boxtimes)^{i_n} + \sum_{k=1}^{m} \boxtimes \,,$$

where $i_j \geq 2$ for $j \in \{1, 2, \cdots, n\}$, and let $v = 0$ if $m \equiv 0 \bmod (2)$, $v = 1$ if $m \equiv 1 \bmod (2)$. Then

$$G \in \mathcal{P} \text{ under the misère play convention } \Leftrightarrow i_1 \oplus i_2 \oplus \cdots \oplus i_n \oplus v = 0. \tag{4}$$

Thus all P-positions of the $1 \times m$ board in Toppling Towers game under the misère play convention have been determined completely.

Allen [1] also investigated $2 \times m$ board in Toppling Towers game. However, we find that she gave some inaccurate conclusions on tame position for this board size. By means of the theory of genus, we revise some conclusions of [1] in section 3. To do this, section 2 will present more genera of some positions, and give all positions which are both tame and P-positions for board sizes of $2 \times m$.

2. The Genera of Some Positions

Definition 1. Let S be any finite subset of $\mathbb{Z}^{\geq 0} = \{n \in \mathbb{Z} \mid n \geq 0\}$. We define the *minimum excluded* value of S by the smallest nonnegative integer not in S. It will be denoted by $\mathrm{mex}(S)$. In particular, $\mathrm{mex}(\emptyset) = 0$.

Definition 2 ([1]). Given nonnegative integers n and m, their *Nim sum*, denoted by $n \oplus m$, is defined by the *exclusive or* of their binary representation. Equivalently, the Nim sum of n and m can be determined by writing each of them as a sum of distinct powers of two, and then canceling any power of two which occurs an even number of times.

Definition 3 ([1]). Given a game G. We define

$$\mathcal{G}^{+}(G) = \begin{cases} 0, & \text{if } G \text{ has no options,} \\ \mathrm{mex}\{\mathcal{G}^{+}(G') | G' \text{ is an option of } G\}, & \text{else,} \end{cases}$$

and

$$\mathcal{G}^{-}(G) = \begin{cases} 1, & \text{if } G \text{ has no options,} \\ \mathrm{mex}\{\mathcal{G}^{-}(G') | G' \text{ is an option of } G\}, & \text{else.} \end{cases}$$

Definition 4 ([1]). The *genus* of a game G, denoted by $\Gamma(G)$, is defined by the list of the form $g^{g_0 g_1 g_2 g_3 \cdots}$, where

$$\begin{cases} g = \mathcal{G}^{+}(G), \\ g_0 = \mathcal{G}^{-}(G), \\ g_n = \mathcal{G}^{-}(G + \sum\limits_{i=1}^{n} \mathbb{T}) \quad (n \in \mathbb{Z}^{\geq 1}), \end{cases}$$

and $\mathbb{T}$ denotes a Nim heap with 2 tokens.

Definition 5([1]). For a game G, we say that the genus of G, $g^{g_0 g_1 g_2 g_3 \cdots}$, *stabilizes* if there exists an integer $N \in \mathbb{Z}^{\geq 0}$ such that for any integer $n \geq N$, $g_{n+1} = g_n \oplus 2$.

Let G be an impartial game. Then the genus of G stabilizes ([1, Theorem 2.1.5]). If $\Gamma(G) = g^{g_0 g_1 g_2 g_3 \cdots}$ stabilizes at g_N, i.e., $g_{N+1} = g_N \oplus 2$, then the digits in the superscript of $\Gamma(G)$ alternate between g_N and g_{N+1} ([1, Lemma 2.1.4]). Thus we abbreviate the genus symbol $\Gamma(G) = g^{g_0 g_1 g_2 g_3 \cdots}$ to $\Gamma(G) = g^{g_0 g_1 g_2 g_3 \cdots g_N (g_N \oplus 2)}$. Recently, Allen [8] also examined the periodicity of the genus sequences of the heaps of finite quaternary games.

How is the genus $\Gamma(G)$ of G calculated? Lemma 7 gives a kind of method. Lemma 6 presents some properties of operation $\oplus$, which are used in the proofs of the main results. Lemma 6 can be obtained by the properties of Nim sum, while Lemma 7 is a result which was proved in [4].

Lemma 6. *Given $a, b \in \mathbb{Z}^{\geq 0}$ and $a \geq b$. We have*

(1) $a = b \Leftrightarrow a \oplus b = 0$.

(2) $a - b \leqslant a \oplus b \leqslant a + b$.

(3) $a \oplus b = 1 \Leftrightarrow a - b = 1$ and $a \equiv 1$ mod (2).

(4) (i) If $a \equiv 1$ mod (2) and $b \equiv 0$ mod (2), then $a \oplus b \equiv 1$ mod (2).

(ii) If $a \equiv 1$ mod (2) and $b \equiv 1$ mod (2), then $a \oplus b \equiv 0$ mod (2).

(iii) If $a \equiv 0$ mod (2) and $b \equiv 0$ mod (2), then $a \oplus b \equiv 0$ mod (2).

(5) $a \oplus 1 = a + 1$ if a is even, $a \oplus 1 = a - 1$ if a is odd.

Lemma 7 ([4]). *Suppose that G is a game with options G_a, G_b, G_c, G_d, $\cdots$ such that*

$$\begin{array}{rcl} \Gamma(G_a) & = & a^{a_0 a_1 a_2 a_3 \cdots}, \\ \Gamma(G_b) & = & b^{b_0 b_1 b_2 b_3 \cdots}, \\ \Gamma(G_c) & = & c^{c_0 c_1 c_2 c_3 \cdots}, \\ \Gamma(G_d) & = & d^{d_0 d_1 d_2 d_3 \cdots}, \\ & \vdots & . \end{array}$$

Then $\Gamma(G) = g^{g_0 g_1 g_2 g_3 \cdots}$, where

$$\begin{array}{rcl} g & = & mex\{a, b, c, d, \cdots\}, \\ g_0 & = & mex\{a_0, b_0, c_0, d_0, \cdots\}, \\ g_1 & = & mex\{g_0, g_0 \oplus 1, a_1, b_1, c_1, d_1, \cdots\}, \\ g_2 & = & mex\{g_1, g_1 \oplus 1, a_2, b_2, c_2, d_2, \cdots\}, \\ & \vdots & \\ g_n & = & mex\{g_{n-1}, g_{n-1} \oplus 1, a_n, b_n, c_n, d_n, \cdots\}. \end{array}$$

Berlekamp [4] proved the following results: Given an impartial game G with genus $\Gamma(G) = g^{g_0 g_1 g_2 g_3 \cdots}$. Then

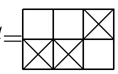
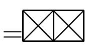
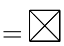
$$\text{under the normal play convention, } G \in \mathcal{P} \Leftrightarrow g = 0, \tag{5}$$

and

$$\text{under the misère play convention, } G \in \mathcal{P} \Leftrightarrow g_0 = 0. \tag{6}$$

Thus, the genus of a game G is the perfect tool to determine the outcome class of G under the normal play convention, or under the misère play convention. In the literature, when this is the only information which is sought, the genus of a game is abbreviated to g^{g_0}. However, we will not use this convention as we are interested in whether a game is *tame* or not which cannot be determined by the base g and the first superscript g_0 of the genus.

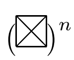
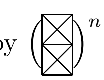
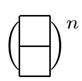
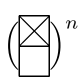
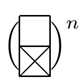
Definition 8 ([1]). Given an impartial game G under the misère play convention. G is *tame* if the following conditions hold: (1) $\Gamma(G) \in \{0^{120}, 1^{031}, k^{k(k\oplus 2)} | k = 0, 1, 2, \cdots\}$; (2) For every option G' of G, G' is tame. An impartial game G is *wild* if it is not tame.

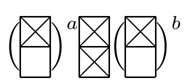
Since a tower can only effect contiguous towers, we see that a Toppling Towers game can be thought of as the *disjunctive sum* of each of the contiguous components, ignoring any empty squares. For example, if G= , then we think of this game as the disjunctive sum of the games G_1=  and G_2 = .

How is the genus of the disjunctive sum of two games G_1 and G_2 calculated? Lemma 9 gives partial result.

Lemma 9 ([5]). *Let G and H be two tame games. Then $G + H$ is also a tame game and*

$$\Gamma(G+H) = \begin{cases} \Gamma(H), & \text{if } \Gamma(G) = 0^{120}, \\ 0^{120}, & \text{if } \Gamma(G) = \Gamma(H) = 1^{031}, \\ (n\oplus 1)^{(n\oplus 1)(n\oplus 3)}, & \text{if } \Gamma(G) = 1^{031}, \quad \Gamma(H) = n^{n(n\oplus 2)}, \\ (n\oplus m)^{(n\oplus m)(n\oplus m\oplus 2)}, & \text{if } \Gamma(G) = n^{n(n\oplus 2)}, \quad \Gamma(H) = m^{m(m\oplus 2)}. \end{cases}$$

For $n \in \mathbb{Z}^{\geq 0}$, by $G =$  we represent a row of n towers, and by  we represent two rows of n towers stacked on top of each other. Similarly, we also have the meanings of , , and .

Theorem 10. *Let G=*  *with $a,\ b \in \mathbb{Z}^{\geq 0}$. Then*

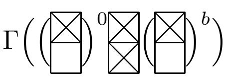
$$\Gamma(G) = \begin{cases} m_0^{m_0(m_0\oplus 2)}, & \text{if } a \equiv 0 \text{ mod } 2 \text{ and } b \equiv 0 \text{ mod } 2, \\ ((a+b+1)\oplus 1)^{((a+b+1)\oplus 1)((a+b+1)\oplus 3)}, & \text{else,} \end{cases}$$

where $m_0 = f(a, b)$ is a unique nonnegative integer depending on both a and b. Moreover, $m_0 \in \{2, \cdots, a+b+2\}$.

Proof. It follows from [1, Theorem 4.3.2] that G is tame. Thus

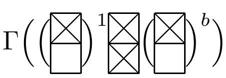
$$\Gamma(G) \in \{0^{120}, 1^{031}, k^{k(k\oplus 2)} | k = 0, 1, 2, \cdots\}.$$

It follows from [1, Proposition 4.3.3] that for $a = 0$ and $b \geq 0$,



$$\Gamma(G) = \Gamma(\cdots) = (b+2)^{(b+2)((b+2)\oplus 2)}. \tag{7}$$

By [1, Proposition 4.3.4], for $a = 1$ and $b \geq 0$, we have



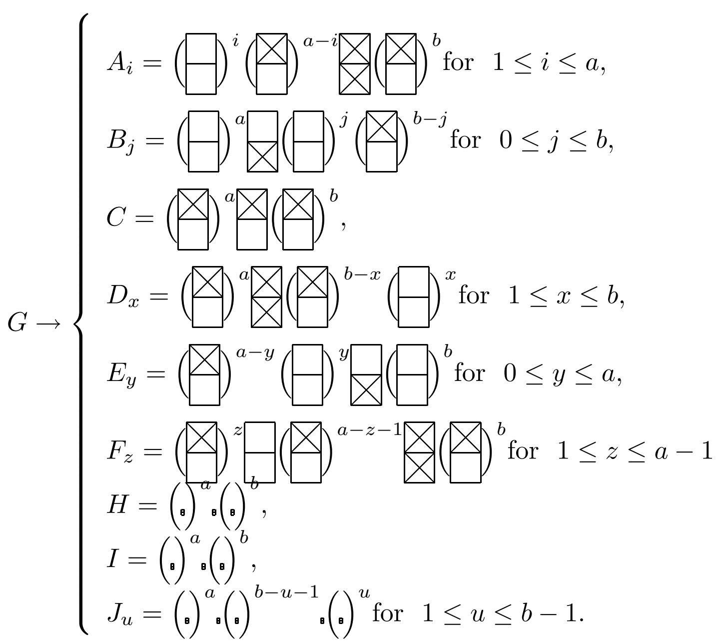
$$\Gamma(G) = \Gamma(\cdots) = ((b+2)\oplus 1)^{((b+2)\oplus 1)((b+2)\oplus 3)}. \tag{8}$$

It is sufficient to consider $a \geq 2$ and $b \geq 2$. We proceed by the induction on $k = a + b$.

For $k = 4$, we have $a = b = 2$. It follows from [1, Appendix B] that $\Gamma(G) = 2^{20}$, which shows the base case.

Suppose that the conclusions of Theorem 10 are true for all $k < n$. We now prove that the conclusions are also true for $k = n$. We distinguish four cases: (i) $a \equiv 1 \bmod (2)$ and $b \equiv 1 \bmod (2)$, (ii) $a \equiv 1 \bmod (2)$ and $b \equiv 0 \bmod (2)$, (iii) $a \equiv 0 \bmod (2)$ and $b \equiv 1 \bmod (2)$ and (iv) $a \equiv 0 \bmod (2)$ and $b \equiv 0 \bmod (2)$.

All options of G are listed as follows:



(i) $a \equiv 1 \bmod (2)$ and $b \equiv 1 \bmod (2)$.

In this case, $(a+b+1) \oplus 1 = a+b$. In order to prove

$$\Gamma(G) = ((a+b+1) \oplus 1)^{((a+b+1)\oplus 1)((a+b+1)\oplus 3)} = (a+b)^{(a+b)((a+b)\oplus 2)},$$

by Lemma 7, we need only to prove the following two facts:

`Fact I` For any $\eta \in \{0^{120}, 1^{031}, m^{m(m\oplus 2)} | m = 2, 3, \cdots, a+b-1\}$, there exists an option G' of G such that $\Gamma(G') = \eta$;

`Fact II` There is no option G' of G such that $\Gamma(G') = (a+b)^{(a+b)((a+b)\oplus 2)}$.

Proof of Fact I.

• Consider E_y. If $y = a$, then $\Gamma(E_a) = \Gamma(\boxtimes) = 1^{031}$ by Eq. (3). If $y = a-1$, then $\Gamma(E_{a-1}) = \Gamma(\boxtimes + \boxtimes) = 0^{120}$ by Lemma 9 and Eq. (3). If $0 \le y \le a-2$, then

$$\Gamma(E_y) = \Gamma((\boxtimes)^{a-y} + \boxtimes) = ((a-y) \oplus 1)^{((a-y)\oplus 1)((a-y)\oplus 3)}.$$

Note that $0 \le y \le a-2$, thus $(a-y) \oplus 1$ ranges from 2 to a by Lemma 6(5).

• Consider D_x. By the induction hypothesis, we have

$$\Gamma(D_x) = ((a+b-x+1) \oplus 1)^{((a+b-x+1)\oplus 1)((a+b-x+1)\oplus 3)}.$$

Note that if $x = 1$, then $(a+b-x+1) \oplus 1 = a+b+1$. The fact $2 \le x \le b$ means $a+1 \le (a+b-x+1) \oplus 1 \le a+b-1$, thus $(a+b-x+1) \oplus 1$ ranges from $a+1$ to $a+b-1$ by Lemma 6(5). $\diamond$

Proof of Fact II.

• Consider A_i. By the induction hypothesis, we have

$$\Gamma(A_i) = ((a+b-i+1) \oplus 1)^{((a+b-i+1)\oplus 1)((a+b-i+1)\oplus 3)}.$$

If $i = 1$, then $(a+b-i+1) \oplus 1 = a+b+1$. If $i = 2$, then $(a+b-i+1) \oplus 1 = a+b-2$. If $3 \le i \le a$, by Lemma 6(2), then

$$(a+b-i+1) \oplus 1 \le (a+b-i+1) + 1 \le a+b-1 < a+b.$$

• Consider B_j. If $j = b$, then $\Gamma(B_b) = 1^{031}$. If $j = b-1$, then $\Gamma(B_{b-1}) = 0^{120}$. If $j = 0$, then

$$\Gamma(B_0) = (b-1)^{(b-1)((b-1)\oplus 2)}.$$

If $1 \le j \le b-2$, then

$$\Gamma(B_j) = ((b-j) \oplus 1)^{((b-j)\oplus 1)((b-j)\oplus 3)}$$

with $(b-j) \oplus 1 \le b-j+1 \le b < a+b$.

- Consider C. It follows from Eq. (3) that $\Gamma(C) = (a+b+1)^{(a+b+1)((a+b+1)\oplus 2)}$.
- Consider F_z. Let $F_{z1} = \left(\,\cdot\,\right)^{z}$ and $F_{z2} = \left(\,\cdot\,\right)^{a-z-1} \cdot \left(\,\cdot\,\right)^{b}$. Then

$\Gamma(F_z) = \Gamma(F_{z1}) + \Gamma(F_{z2})$. Note that $\Gamma(F_{z1}) = z^{z(z\oplus 2)}$. By the induction hypothesis, the fact $b \equiv 1 \bmod (2)$ means

$$\Gamma(F_{z2}) = ((a-z+b)\oplus 1)^{((a-z+b)\oplus 1)((a-z+b)\oplus 3)}.$$

It follows from Lemma 7 that

$$\Gamma(F_z) = (z \oplus (a-z+b) \oplus 1)^{(z\oplus(a-z+b)\oplus 1)(z\oplus(a-z+b)\oplus 3)}.$$

Recall that $a+b \equiv 0 \bmod (2)$, and for any $z \in \{1, 2, \cdots, a-1\}$, $z \oplus (a-z+b) \equiv 0 \bmod (2)$. Thus the fact $z \oplus (a-z+b) \oplus 1 \equiv 1 \bmod (2)$ implies $z \oplus (a-z+b) \oplus 1 \neq a+b$, i.e., $\Gamma(F_z) \neq (a+b)^{(a+b)((a+b)\oplus 2)}$.

- Consider H. $\Gamma(H) = \Gamma((\boxtimes)^a + \boxtimes + (\boxtimes)^b) = (a \oplus b \oplus 1)^{(a\oplus b\oplus 1)(a\oplus b\oplus 3)}$. The given conditions $a \equiv 1 \bmod (2)$ and $b \equiv 1 \bmod (2)$ imply that $a \oplus b \leq a+b-2 < a+b$. Thus $a \oplus b \oplus 1 \leq a+b-2+1 = a+b-1 < a+b$.
- Consider I. $\Gamma(I) = \Gamma((\boxtimes)^a + (\boxtimes)^b) = (a \oplus b)^{(a\oplus b)(a\oplus b\oplus 2)}$. It is easy to see that $a \oplus b \leq a+b-2 < a+b$.
- Consider J_u. Let $J_{u1} = \left(\,\cdot\,\right)^{a} \cdot \left(\,\cdot\,\right)^{b-u-1}$ and $J_{u2} = \left(\,\cdot\,\right)^{u}$. Then $\Gamma(J_u) = \Gamma(J_{u1}) + \Gamma(J_{u2})$. It is easy to see that $\Gamma(J_{u2}) = u^{u(u\oplus 2)}$. By the induction hypothesis, the fact $a \equiv 1 \bmod (2)$ means

$$\Gamma(J_{u1}) = ((a+b-u)\oplus 1)^{((a+b-u)\oplus 1)((a+b-u)\oplus 3)}.$$

It follows from Lemma 7 that

$$\Gamma(J_u) = ((a+b-u)\oplus 1 \oplus u)^{((a+b-u)\oplus 1\oplus u)((a+b-u)\oplus 3\oplus u)}.$$

Note that $a+b \equiv 0 \bmod (2)$, and $(a+b-u)\oplus u \equiv 0 \bmod (2)$ for any $u \in \{1, 2, \cdots, b-1\}$. Thus the fact $(a+b-u)\oplus 1 \oplus u \equiv 1 \bmod (2)$ implies $(a+b-u)\oplus 1\oplus u \neq a+b$, i.e., $\Gamma(J_u) \neq (a+b)^{(a+b)((a+b)\oplus 2)}$.

The above analysis shows that for any option G' of G, $\Gamma(G') \neq (a+b)^{(a+b)((a+b)\oplus 2)}$. The proof of Fact II is completed. $\diamond$

Cases (ii) and (iii) can be proved similarly.

For Case (iv), we have $a \equiv 0 \bmod (2)$ and $b \equiv 0 \bmod (2)$.

In order to prove $\Gamma(G) = m_0^{m_0(m_0\oplus 2)}$ for $m_0 \leq a+b+2$, by Lemma 7, we need only to prove the following two facts:

`Fact I` There exist two options G_1 and G_2 of G such that $\Gamma(G_1) = 0^{120}$ and $\Gamma(G_2) = 1^{031}$.

`Fact II` For any option G' of G, if $\Gamma(G') = n_0^{n_0(n_0 \oplus 2)}$, then $n_0 < a+b+2$.

Proof of Fact I: Consider B_j. If $j = b$, then $\Gamma(B_b) = 1^{031}$. If $j = b-1$, then $\Gamma(B_{b-1}) = 0^{120}$. $\diamond$

Proof of Fact II:

• Consider A_i. If $i \equiv 0 \bmod (2)$, then $a - i \equiv 0 \bmod (2)$. By the induction hypothesis, there exists an integer $m_0 \in \{2, 3, \cdots, a-i+b+2\}$ such that $\Gamma(A_i) = m_0^{m_0(m_0 \oplus 2)}$. Note that $m_0 \le a-i+b+2 \le a+b+1 < a+b+2$.

If $i \equiv 1 \bmod (2)$, then $a - i \equiv 1 \bmod (2)$. By the induction hypothesis,

$$\Gamma(A_i) = ((a-i+b+1) \oplus 1)^{((a-i+b+1)\oplus 1)((a-i+b+1)\oplus 3)}.$$

It follows Lemma 6(2) that $(a-i+b+1) \oplus 1 \le a-i+b+1+1 < a+b+2$.

• Consider B_j. If $j = b$, then $\Gamma(B_b) = 1^{031}$. If $j = b-1$, then $\Gamma(B_{b-1}) = 0^{120}$. If $0 \le j \le b-2$, then

$$\Gamma(B_j) = ((b-j) \oplus 1)^{((b-j)\oplus 1)((b-j)\oplus 3)}.$$

It follows Lemma 6(2) that $(b-j) \oplus 1 \le b-j+1 \le b+1 < a+b+2$.

• $\Gamma(C) = (a+b+1)^{(a+b+1)((a+b+1)\oplus 2)}$.

• D_x, $1 \le x \le b$, is similar to A_i, $1 \le i \le a$.

• Consider E_y. If $y = a$, then $\Gamma(E_a) = 1^{031}$. If $y = a-1$, then $\Gamma(E_{a-1}) = 0^{120}$. If $0 \le y \le a-2$, then

$$\Gamma(E_y) = ((a-y) \oplus 1)^{((a-y)\oplus 1)((a-y)\oplus 3)}.$$

It follows Lemma 6(2) that $(a-y) \oplus 1 \le a-y+1 \le a+1 < a+b+2$.

• Consider F_z. Let $F_{z1} = \left(\cdot\right)^{z}$ and $F_{z2} = \left(\cdot\right)^{a-z-1} \cdot \left(\cdot\right)^{b}$. Then $\Gamma(F_z) = \Gamma(F_{z1}) + \Gamma(F_{z2})$. Note that $\Gamma(F_{z1}) = \begin{cases} 1^{031}, & \text{if } z = 1, \\ z^{z(z\oplus 2)}, & \text{if } z > 1. \end{cases}$

If $z \equiv 0 \bmod (2)$, then $a - z - 1 \equiv 1 \bmod (2)$. By the induction hypothesis,

$$\Gamma(F_{z2}) = ((a-z+b) \oplus 1)^{((a-z+b)\oplus 1)((a-z+b)\oplus 3)}.$$

It follows from Lemma 9 that

$$\Gamma(F_z) = (z \oplus (a-z+b) \oplus 1)^{(z\oplus(a-z+b)\oplus 1)(z\oplus(a-z+b)\oplus 3)}.$$

Note that $z \oplus (a-z+b) \oplus 1 \le a+b+1 < a+b+2$.

If $z \equiv 1 \bmod (2)$, then $a-z-1 \equiv 0 \bmod (2)$. By the induction hypothesis, there exists an integer $m_0 \in \{2, 3, \cdots, a-z-1+b+2\}$ such that $\Gamma(F_{z2}) = m_0^{m_0(m_0\oplus 2)}$. It follows from Lemma 9 that $\Gamma(F_z) = (m_0 \oplus z)^{(m_0\oplus z)(m_0\oplus z\oplus 2)}$. Note that $m_0 \oplus z \le (a-z-1+b+2) \oplus z \le a+b+1 < a+b+2$.

- $\Gamma(H) = (a \oplus b \oplus 1)^{(a \oplus b \oplus 1)(a \oplus b \oplus 3)}$, and $a \oplus b \oplus 1 \leq a + b + 1 < a + b + 2$.
- $\Gamma(I) = (a \oplus b)^{(a \oplus b)(a \oplus b \oplus 2)}$, and $a \oplus b \leq a + b < a + b + 2$.
- Consider J_u. Let $J_{u1} = \left(\centerdot\right)^{z}$ and $J_{u2} = \left(\centerdot\right)^{a} \centerdot \left(\centerdot\right)^{b-u-1}$. Then

$\Gamma(J_u) = \Gamma(J_{u1}) + \Gamma(J_{u2})$. Note that $\Gamma(J_{u1}) = \begin{cases} 1^{031}, & \text{if } u = 1, \\ u^{u(u \oplus 2)}, & \text{if } u > 1. \end{cases}$

If $u \equiv 0 \bmod (2)$, then $b - u - 1 \equiv 1 \bmod (2)$. By the induction hypothesis,

$$\Gamma(J_{u2}) = ((a - u + b) \oplus 1)^{((a-u+b) \oplus 1)((a-u+b) \oplus 3)}.$$

It follows from Lemma 9 that

$$\Gamma(J_u) = ((a + b - u) \oplus 1 \oplus u)^{((a+b-u) \oplus 1 \oplus u)((a+b-u) \oplus 3 \oplus u)}.$$

Note that $(a + b - u) \oplus 1 \oplus u \leq a + b + 1 < a + b + 2$.

If $u \equiv 1 \bmod (2)$, then $b - u - 1 \equiv 0 \bmod (2)$. By the induction hypothesis, there exists an integer $m_0 \in \{2, 3, \cdots, a - u - 1 + b + 2\}$ such that $\Gamma(J_{u2}) = m_0^{m_0(m_0 \oplus 2)}$. It follows from Lemma 9 that $\Gamma(J_u) = (m_0 \oplus u)^{(m_0 \oplus u)(m_0 \oplus u \oplus 2)}$. Note that $m_0 \oplus u \leq (a + b - u - 1 + 2) \oplus u \leq a + b + 1 < a + b + 2$. $\diamond$

The proofs of Theorem 10 are completed. $\square$

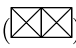
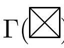
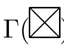
3. Tame Positions in Toppling Towers Game

Theorem 11. *Let* $G_1 = \begin{array}{|c|c|}\hline \times & \times \\ \hline \times & \times \\ \hline \end{array}$ *and* $G_2 = \begin{array}{|c|c|c|}\hline & \times & \times \\ \hline \times & \times & \\ \hline \end{array}$. *Then* G_1 *and* G_2 *are tame and* $\Gamma(G_1) = 0^{02}$, $\Gamma(G_2) = 1^{031}$.

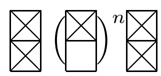
Proof. G_1 has two options up to symmetry: $A = \begin{array}{|c|c|}\hline & \times \\ \hline \times & \times \\ \hline \end{array}$ and $B = \begin{array}{|c|c|}\hline & \\ \hline \times & \times \\ \hline \end{array}$. We now calculate $\Gamma(A)$. A has three options: $A_1 = \begin{array}{|c|c|}\hline & \\ \hline \times & \times \\ \hline \end{array}$, $A_2 = \begin{array}{|c|c|}\hline & \times \\ \hline & \\ \hline \end{array}$ and $A_3 = \begin{array}{|c|c|}\hline & \times \\ \hline \times & \\ \hline \end{array}$.

By Eq. (3), we have $\Gamma(A_1) = \Gamma(B) = 2^{20}$ and $\Gamma(A_2) = 1^{031}$. It follows Lemma 9 that $\Gamma(A_3) = \Gamma(\boxtimes) + \Gamma(\boxtimes) = 0^{120}$. By Lemma 7, we have $\Gamma(A) = 3^{31}$ and $\Gamma(G_1) = 0^{02}$. Note that the options A and B of G_1 are tame. Hence, G_1 is tame.

G_2 has four options: $C = \begin{array}{|c|c|c|}\hline & & \times \\ \hline \times & \times & \\ \hline \end{array}$, $D = \begin{array}{|c|c|c|}\hline & & \\ \hline \times & \times & \\ \hline \end{array}$, $E = \begin{array}{|c|c|c|}\hline & \times & \times \\ \hline & \times & \\ \hline \end{array}$, and $F = \begin{array}{|c|c|c|}\hline & & \times \\ \hline \times & & \\ \hline \end{array}$. It follows from Lemma 9 and Eq. (3) that $\Gamma(D) = 2^{20}$, $\Gamma(E) = \Gamma(A) =$

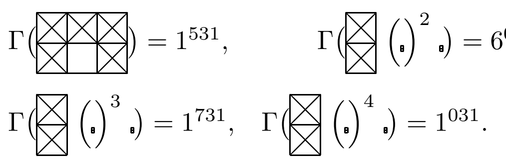
3^{31}, $\Gamma(C) = \Gamma($$) + \Gamma($$) = 3^{31}$, and $\Gamma(F) = \Gamma($$) + \Gamma($$) = 0^{120}$. By Lemma 7, we have $\Gamma(G_2) = 1^{031}$. Note that the options C, D, E and F of G_2 are tame. Hence, G_2 is tame. $\square$

Allen [1] proved that the game G defined by Theorem 12 is tame if and only if $n \equiv 0 \bmod (4)$. Theorem 12 obtains the explicit representation of $\Gamma(G)$ which contains Allen's result.

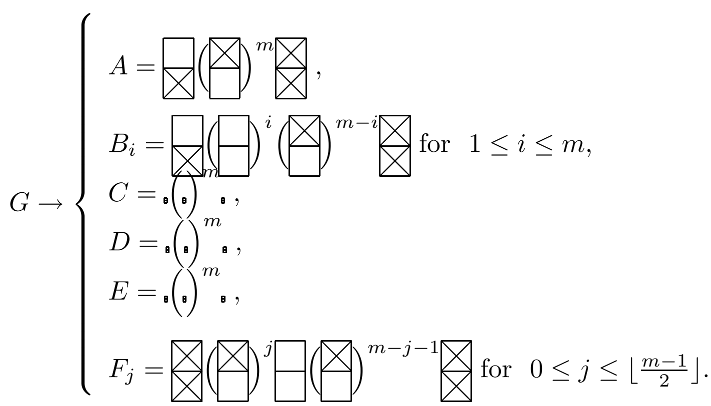
Theorem 12. *Let* $G =$  *for* $n \in \mathbb{Z}^{\geq 1}$. *Then*

$$\Gamma(G) = \begin{cases} 1^{031}, & \text{if } n \equiv 0 \bmod (4), \\ 1^{(n+4)31}, & \text{if } n \equiv 1 \text{ or } 3 \bmod (4), \\ (n+4)^{0(n+2)((n+2)\oplus 2)}, & \text{if } n \equiv 2 \bmod (4). \end{cases}$$

Proof. We proceed by the induction on n. Allen [1] determined the following four genera, which show the base cases:



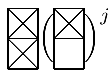
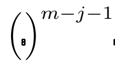
Suppose the conclusions of Theorem 12 are true for $\forall\, n < m$. We now consider $n = m$. All options of G, up to symmetry, can be listed as follows:



The genus of each option of G can be determined:

• $\Gamma(A) = a^{a_0 a_1 a_2 a_3 \cdots} = ((m+2) \oplus 1)^{((m+2)\oplus 1)((m+2)\oplus 3)}$, by Lemma 9 and Eq. (7).

• $\Gamma(B_i) = b_i^{b_{i0} b_{i1} b_{i2} b_{i3} \cdots} = ((m-i+2) \oplus 1)^{((m-i+2)\oplus 1)((m-i+2)\oplus 3)}$ for $1 \leq i \leq m$, by Lemma 9 and Eq. (7).

- $\Gamma(C) = c^{c_0 c_1 c_2 c_3 \cdots} = 0^{120}$, by Lemma 9 and Eq. (3).
- $\Gamma(D) = d^{d_0 d_1 d_2 d_3 \cdots} = (m+3)^{(m+3)((m+3)\oplus 2)}$, by Eq. (7).
- $\Gamma(E) = e^{e_0 e_1 e_2 e_3 \cdots} = (m+2)^{(m+2)((m+2)\oplus 2)}$, by Eq. (7).
- Let $F'_{j1} =$  and $F'_{j2} =$ . It follows from Eq. (7) that

$$\Gamma(F'_{j1}) = (j+2)^{(j+2)((j+2)\oplus 2)},$$

$$\Gamma(F'_{j2}) = (m-j+1)^{(m-j+1)((m-j+1)\oplus 2)}.$$

By Lemma 9, we have

$$\Gamma(F_j) = f_j^{f_{j0} f_{j1} f_{j2} f_{j3} \cdots} = ((m-j+1)\oplus(j+2))^{((m-j+1)\oplus(j+2))((m-j+1)\oplus(j+2)\oplus 2)}$$

for $0 \le j \le \lfloor \frac{m-1}{2} \rfloor$.

It is easy to see that all options A, B, C, D, E and F are tame, and none of the genera for A, B, D, E and F is 1^{031}. We distinguish the following three cases:

`Case 1.` $m \equiv 0 \bmod (4)$.

In this case, G is tame by [1, Theorem 4.3.5]. Thus the genus

$$\Gamma(G) \in \{0^{120}, 1^{031}, n^{n(n\oplus 2)} | n \in \mathbb{Z}^{\ge 0}\}.$$

Let $\Gamma(G) = g^{g_0 g_1 g_2 g_3 \cdots}$. We will prove that $g = 1$ and $g_0 = 0$. Hence we have $\Gamma(G) = 1^{031}$.

In fact, $m \ge 1$, $m+2 \ge 3$ and $m+3 \ge 4$. Thus

$$\begin{aligned} &a = a_0 = (m+2)\oplus 1 = m+3 \ne 0 \text{ or } 1,\\ &d = d_0 = m+3 \ne 0 \text{ or } 1,\\ &e = e_0 = m+3 \ne 0 \text{ or } 1.\end{aligned}$$

Note that for $1 \le i \le m$, we have $m-i+2 \ge 2$, and so

$$b_i = b_{i0} = (m-i+2)\oplus 1 \ge 2.$$

It remains to examine $\Gamma(F_j)$. Recall that $0 \le j \le \lfloor \frac{m-1}{2} \rfloor = \frac{m-2}{2}$. We will prove that for $0 \le j \le \frac{m-2}{2}$,

$$f_j = f_{j0} = (m-j+1)\oplus(j+2) \ne 0 \text{ or } 1.$$

In fact, if $0 \le j \le \frac{m-4}{2}$, by Lemma 6(2), we have

$$f_j = f_{j0} = (j+2)\oplus(m-j+1) \le (m-j+1)+(j+2) = m+3,$$

and

$$f_j = f_{j0} = (j+2)\oplus(m-j+1) \ge (m-j+1)-(j+2) \ge 3.$$

If $j = \frac{m-2}{2}$, then $j+2 = \frac{m+2}{2}$ is odd, and $m-j+1 = \frac{m+4}{2}$ is even, and $\frac{m+4}{2} - \frac{m+2}{2} = 1$. It follows from Lemma 6 that $(\frac{m+2}{2} \oplus \frac{m+4}{2}) \neq 0$ or 1.

Note that $c = 0$ and $c_0 = 1$. By Lemma 7, we have

$$\begin{aligned} g &= \operatorname{mex}\{a, b_i(1 \le i \le m), c, d, e, f_j(0 \le j \le \tfrac{m-2}{2})\} = 1, \\ g_0 &= \operatorname{mex}\{a_0, b_{i0}((1 \le i \le m)), c_0, d_0, e_0, F_{j0}((0 \le j \le \tfrac{m-2}{2}))\} = 0. \end{aligned}$$

`Case 2.` $m \equiv 1$ or 3 mod (4).

In this case, G is wild by [1, Theorem 4.3.5]. The genus of each option of G can be determined:

`Subase 2.1.` $\Gamma(A) = a^{a_0 a_1 a_2 a_3 \cdots} = (m+1)^{(m+1)((m+1)\oplus 2)}$; $\Gamma(C) = c^{c_0 c_1 c_2 c_3 \cdots} = 0^{120}$; $\Gamma(D) = d^{d_0 d_1 d_2 d_3 \cdots} = (m+3)^{(m+3)((m+3)\oplus 2)}$.

`Subase 2.2.` Consider B_i. $\Gamma(B_1) = (m+2)^{(m+2)((m+2)\oplus 2)}$. For $2 \le i \le m$, we have $2 \le (m-i+2) \oplus 1 \le m$. Thus

$$\begin{aligned} &\{\Gamma(B_i) = b_i^{b_{i0} b_{i1} b_{i2} b_{i3} \cdots} \mid i = 1, 2, 3, \cdots, m\} \\ = \; &\{2^{20}, 3^{31}, 4^{46}, \cdots, m^{m(m\oplus 2)}, (m+2)^{(m+2)((m+2)\oplus 2)}\}. \end{aligned}$$

`Subase 2.3.` Consider F_j. Note that $0 \le j \le \lfloor \frac{m-1}{2} \rfloor = \frac{m-1}{2}$. We will prove that for every integer $j \in \{0, 1, \cdots, \frac{m-1}{2}\}$,

$$\Gamma(F_j) \neq 1^{13} \text{ or } (m+4)^{(m+4)((m+4)\oplus 2)}.$$

In fact, if $0 \le j \le \frac{m-5}{2}$, by Lemma 6, we have

$$f_j = f_{j0} = (j+2) \oplus (m-j+1) \le (j+2) + (m-j+1) = m+3,$$

and

$$f_j = f_{j0} = (j+2) \oplus (m-j+1) \ge (m-j+1) - (j+2) \ge 4.$$

If $j = \frac{m-3}{2}$, by Lemma 6, we have

$$2 \le f_j = f_{j0} = (j+2) \oplus (m-j+1) = \frac{m+1}{2} \oplus \frac{m+5}{2} \le m+3.$$

If $j = \frac{m-1}{2}$, then $f_j = f_{j0} = (j+2) \oplus (m-j+1) = \frac{m+3}{2} \oplus \frac{m+3}{2} = 0$.

We now prove that $\Gamma(G) = 1^{(m+4)31}$. Let $\Gamma(G) = g^{g_0 g_1 g_2 g_3 \cdots}$ and

$$\begin{aligned} \mathcal{M} &= \{a, b_i(1 \le i \le m), c, d, e, f_j(0 \le j \le \tfrac{m-1}{2})\}, \\ \mathcal{M}_0 &= \{a_0, b_{i0}(1 \le i \le m), c_0, d_0, e_0, f_{j0}(0 \le j \le \tfrac{m-1}{2})\}, \\ \mathcal{M}_1 &= \{g_0, g_0 \oplus 1, a_1, b_{i1}(1 \le i \le m), c_1, d_1, e_1, f_{j1}(0 \le j \le \tfrac{m-1}{2})\}, \\ \mathcal{M}_2 &= \{g_1, g_1 \oplus 1, a_2, b_{i2}(1 \le i \le m), c_2, d_2, e_2, f_{j2}(0 \le j \le \tfrac{m-1}{2})\}, \\ \mathcal{M}_3 &= \{g_2, g_2 \oplus 1, a_3, b_{i3}(1 \le i \le m), c_3, d_3, e_3, f_{j3}(0 \le j \le \tfrac{m-1}{2})\}, \\ &\vdots \quad . \end{aligned}$$

By Lemma 7, we have

(g) $0 = c \in \mathcal{M}$ and $1 \notin \mathcal{M}$. Thus $g = \mathrm{mex}(\mathcal{M}) = 1$.

(g_0) $0 = f_{\frac{m-1}{2}0} \in \mathcal{M}_0$, $1 = c_0 \in \mathcal{M}_0$, $m+1 = a_0 \in \mathcal{M}_0$, $m+3 = e_0 \in \mathcal{M}_0$, $m+4 \notin \mathcal{M}_0$, and

$$\{b_{i0} | i = 1, 2, 3, \cdots, m\} = \{2, 3, 4, \cdots, m, m+2\} \subset \mathcal{M}_0.$$

Thus $g_0 = \mathrm{mex}(\mathcal{M}_0) = m+4$.

(g_1) Note that $\Gamma(B_i) = b_i^{b_{i0}b_{i1}b_{i2}b_{i3}\cdots} = ((m-i+2) \oplus 1)^{((m-i+2)\oplus 1)((m-i+2)\oplus 3)}$. For $i = m-1$, $\Gamma(B_i) = 2^{20}$, and thus $b_{(m-1)1} = 0 \in \mathcal{M}_1$. For $i = m$, $\Gamma(B_m) = 3^{31}$, and thus $b_{m1} = 1 \in \mathcal{M}_1$. Moreover, $2 = c_1 \in \mathcal{M}_1$ and $3 \notin \mathcal{M}_1$. Thus $g_1 = \mathrm{mex}(\mathcal{M}_1) = 3$.

(g_2) $0 = c \in \mathcal{M}_2$ and $1 \notin \mathcal{M}_2$. Thus $g_2 = \mathrm{mex}(\mathcal{M}_2) = 1$.

(g_3) For $i = m-1$, $\Gamma(B_{m-1}) = 2^{2020\cdots}$, thus $b_{(m-1)3} = 0 \in \mathcal{M}_3$. For $i = m$, $\Gamma(B_m) = 3^{3131\cdots}$, so $b_{m3} = 1 \in \mathcal{M}_3$. $2 = c_3 \in \mathcal{M}_3$ and $3 \notin \mathcal{M}_3$. Thus $g_3 = \mathrm{mex}(\mathcal{M}_3) = 3$.

$\vdots$.

Hence $\Gamma(G) = 1^{(m+4)31}$.

`Case 3.` $m \equiv 2 \bmod (4)$.

In this case, G is wild by [1, Theorem 4.3.5]. The genus of each option of G can be determined:

`Subase 3.1.` $\Gamma(A) = a^{a_0a_1a_2a_3\cdots} = (m+3)^{(m+3)((m+3)\oplus 2)}$; $\Gamma(C) = c^{c_0c_1c_2c_3\cdots} = 0^{120}$; $\Gamma(D) = d^{d_0d_1d_2d_3\cdots} = (m+3)^{(m+3)((m+3)\oplus 2)}$.

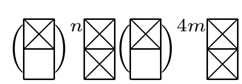
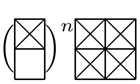
`Subase 3.2.` Consider B_i. For $1 \le i \le m$, we have $2 \le (m-i+2) \oplus 1 \le m+1$. Thus

$$\{\Gamma(B_i) | i = 1, 2, 3, \cdots, m\} = \{2^{20}, 3^{31}, 4^{46}, \cdots, (m+1)^{(m+1)((m+1)\oplus 2)}\}.$$

`Subase 3.3.` Consider F_j. Note that $0 \le j \le \lfloor \frac{m-1}{2} \rfloor = \frac{m-2}{2}$. We will prove that for every integer $j \in \{0, 1, \cdots, \frac{m-2}{2}\}$, $\Gamma(F_j) \notin \{0^{02}, (m+4)^{(m+4)((m+4)\oplus 2)}\}$.

In fact, if $0 \le j \le \frac{m-4}{2}$, by Lemma 6, we have $3 \le (j+2) \oplus (m-j+1) \le m+3$. If $j = \frac{m-2}{2}$, then $j+2 = \frac{m+2}{2} \equiv 0 \bmod (2)$, and $m-j+1 = \frac{m+4}{2} \equiv 1 \bmod (2)$, and $\frac{m+4}{2} - \frac{m+2}{2} = 1$. It follows from Lemma 6 that $\frac{m+2}{2} \oplus \frac{m+4}{2} = 1$.

We can determine $\Gamma(G) = (m+4)^{0(m+2)((m+2)\oplus 4)}$ by the same method as Case 2. $\square$

Allen [1, Corollary 4.3.6] stated that for $G =$  with $n, m \in \mathbb{Z}^{\ge 0}$, G is tame. Let $G_n =$ . Then G_n is obtained by letting $m = 0$ in

G. It follows from [1, Appendix B] that $\Gamma(G_1) = 5^{146}$, so G_1 is wild. For $n \geq 2$, G_n has an option G_1. It follows from the definition of wild position that G_n is wild. Thus Corollary 4.3.7, Corollary 4.3.8 and Theorem 4.3.9 of [1] are inaccurate conclusions. We will revise them, and give our new results.

Denote by $\mathcal{T}$ the set of all tame positions of $2 \times m$ board in the Toppling Towers game. Similarly, denote by $\mathcal{W}$ the set of all wild positions. Obviously, the set of all positions of $2 \times m$ board in the Toppling Towers game can be uniquely partitioned into $\mathcal{T}$ and $\mathcal{W}$. Recall that the set of all positions of a game can be uniquely partitioned into $\mathcal{P}$ and $\mathcal{N}$.

It follows from Eqs. (5) and (6) that the outcome class ($\mathcal{P}$ or $\mathcal{N}$) of a position G can be determined by its genus $\Gamma(G)$ under both the normal and misère play conventions.

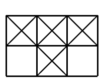
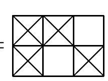
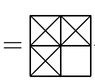
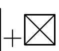
Given a position G, how can we determine $G \in \mathcal{T}$ or $G \in \mathcal{W}$? By Definition 8, if G has an option G' which is wild, then G is wild. Allen [1, Example 2.2.2] proved that all options of G being tame does not imply that G is tame. Lemma 13 gives the answer to the case that G has only tame options.

Lemma 13 ([4]). *Suppose G is a position with only tame options. Then G is wild if and only if among the options of G, the following conditions hold: (1) G has options with genera equal to one, but not both, of 0^{120} or 1^{031}; (2) G has options with genera equal to one, but not both, of 0^{02} or 1^{13}.*

Let

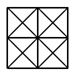
$$\begin{aligned} S &= \{u \mid u \text{ is a position of the } 2 \times m \text{ board in the Toppling Towers game}\}, \\ S^0 &= \{u \in S \mid u \text{ contains no in the up-down direction}\}, \\ S^{>0} &= \{u \in S \mid u \text{ contains at least one in the up-down direction}\}, \end{aligned}$$

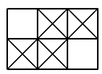
$$\begin{aligned} S^{>0}_{notsum} &= \{u \in S^{>0} \mid u \text{ can not be considered as the disjunctive sum of two or} \\ &\quad \text{more positions}\}, \\ S^{>0}_{sum} &= \{u \in S^{>0} \mid u \text{ can be considered as the disjunctive sum of two or} \\ &\quad \text{more positions}\}. \end{aligned}$$

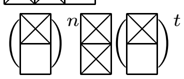
It is easy to see that the set S can be uniquely partitioned into two sets S^0 and $S^{>0}$, by distinguishing that whether there exists a in the up-down direction or not. Similarly, $S^{>0}$ can be uniquely partitioned into two sets $S^{>0}_{notsum}$ and $S^{>0}_{sum}$. For examples, $H_1 =$  $\in S^{>0}_{notsum}$, $H_2 =$  $\in S^{>0}_{sum}$, since $H_2 =$  $+$ .

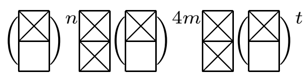
For any position $H \in S^{>0}_{sum}$, H can be thought as the disjunctive sum of $H_1, H_2, \cdots$, H_m, where $H_i \in S^{>0}_{notsum}$. If all of the components H_i are tame, then H is also tame. Thus a key question is that how to determine the outcome class of the position $G \in S^{>0}_{notsum}$. Our Theorem 14 gives the answer to this question.

Theorem 14. *Suppose that $G \in S^{>0}_{notsum}$ is a position. Then $G \in \mathcal{T}$ if and only if $G \in \{G_1, G_2, G_3, G_4\}$, where, up to symmetry,*

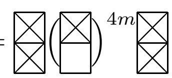
$G_1 =$  ,

$G_2 =$  ,

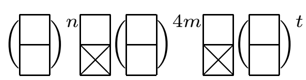
$G_3 =$  *for all* $n, t \in \mathbb{Z}^{\geq 0}$,

$G_4 =$  *for all* $n, t \in \mathbb{Z}^{\geq 0}$ *and* $m \in \mathbb{Z}^{\geq 1}$.

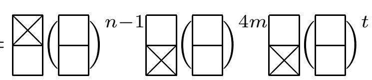
Proof. ($\Leftarrow$) It follows from Theorem 11 that G_1 and G_2 are tame. By Theorem 10 or [1, Theorem 4.3.2], G_3 is tame. We now consider G_4:

If $n = t = 0$, then $G_4 =$  . Thus G_4 is tame by Theorem 12 or by [1, Theorem 4.3.5].

If $n + t > 0$, without loss of generality, let $n \geq 1$. Consider all options of G_4. Note that they are tame and G_4 has options

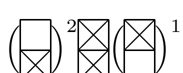
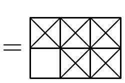
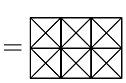
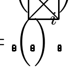
$G_4^{'} =$ 

and

$G_4^{''} =$  .

It follows from Lemma 9 and Eq. (3) that $\Gamma(G_4^{'}) = 0^{120}$ and $\Gamma(G_4^{''}) = 1^{031}$. By Lemma 13, G_4 is tame.

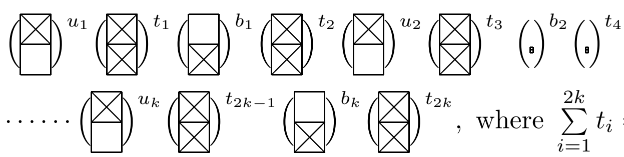
($\Rightarrow$) Firstly, we define the following four structures:

$H_1 =$  , $H_2 =$  , $H_3 =$  ,
$H_4 =$  , where $i \equiv 1$ or 2 or 3 mod (4).

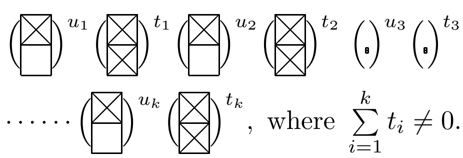
It follows from [1, Appendix B] that $\Gamma(H_1) = 5^{146}$, $\Gamma(H_2) = 5^{146}$ and $\Gamma(H_3) = 2^{1520}$. By the definition of wild, H_1, H_2 and H_3 are wild. By Theorem 12, H_4 is wild.

Secondly, it is enough to show that for any $H \in S^{>0}_{notsum}$ and $H \notin \{G_1, G_2, G_3, G_4\}$, H is wild. We say that a *follower* of position H is a new position which can be reached from H after a finite number of moves. We now prove that one of the structures H_1, H_2, H_3 and H_4 must be a follower of H, and thus H is wild, by Definition 8.

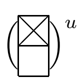
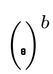
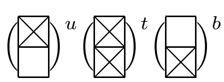
In fact, all of the positions in $S^{>0}_{notsum}$ can be represented in the form K:

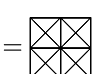
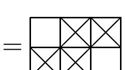
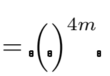
$K =$  , where $\sum_{i=1}^{2k} t_i \neq 0$.

(1) $\sum_{i=1}^{k} u_i = 0$ or $\sum_{i=1}^{k} b_i = 0$. In this case, K has the form K_1:

$K_1 =$  , where $\sum_{i=1}^{k} t_i \neq 0$.

If there exists an integer $j \in \{1, 2, \cdots, k\}$ such that $t_j \geq 3$, then H_3 is a follower of H. If there exists an integer $j \in \{1, 2, \cdots, k\}$ such that $t_j = 2$, then $H \neq G_1$ implies that H_2 is a follower of H. If $t_j \leq 1$ and $j = 1, 2, \cdots, k$, the facts $H \neq G_3$ and $H \neq G_4$ imply that H_4 is a follower of H.

(2) $\sum_{i=1}^{k} u_i \neq 0$ and $\sum_{i=1}^{k} b_i \neq 0$. In this case, H must contain the structures  and . Moreover, $H \in S^{>0}_{notsum}$ implies that H must contain the structure , where $u, b, t \geq 1$. If $t \geq 3$, then H_3 is a follower of H. If $t = 2$, then H_2 is a follower of H. If $t = 1$, then $H \neq G_2$ implies that H_1 is a follower of H. $\square$

Theorem 15. *Let $G^{'} =$* , *$G^{''} =$*  *and $G^{'''} =$* , *where $m \in \mathbb{Z}^{\geq 1}$. Suppose that $G \in S^{>0}_{notsum}$ and G be a tame position. Then*

(1) Under the normal play convention, $G \in \mathcal{P}$ if and only if $G = G^{'}$.

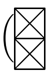
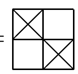
(2) Under the misère play convention, $G \in \mathcal{P}$ if and only if $G = G^{'}, G^{''}$ or $G^{'''}$.

Proof. It follows from Theorem 11 that $\Gamma(G^{'}) = 0^{02}$ and $\Gamma(G^{''}) = 1^{031}$. By Theorem 12, we have $\Gamma(G^{'''}) = 1^{031}$.

(1) ($\Leftarrow$) By Theorem 14, $G^{'} \in \mathcal{T}$. $G^{'}$ is a P-position under the normal play convention by Eq. (5), i.e., $G^{'} \in \mathcal{P}$.

($\Rightarrow$) By Theorem 14, $G \in \mathcal{T}$ implies $G \in \{G_1, G_2, G_3, G_4\}$, where $G_i (i = 1, 2, 3, 4)$ is defined in Theorem 14. We will prove that G_2, G_3, G_4 are N-positions under the normal play convention, thus $G \notin \{G_2, G_3, G_4\}$.

It follows from Eq. (5) that $G_1 = G^{'}$ is a P-position under the normal play convention, and $G_2 = G^{''}$ is an N-position under the normal play convention.

We now consider G_3. Let $\Gamma(G_3) = g^{g_0 g_1 g_2 g_3 \cdots}$. If $n + t = 0$, then $\Gamma(G_3) =$ $\Gamma\Big($  $\Big) = 2^{20}$. Thus G_3 is an N-position under the normal play convention by Eq. (5), as $g \neq 0$. If $n + t > 0$, without loss of generality, let $n \geq 1$. In this case, G_3 has options $G_3^{'} =$  and $G_3^{''} =$. Note that $\Gamma(G_3^{'}) = 0^{120}$ and $\Gamma(G_3^{''}) = 1^{031}$. By Lemma 7, $\mathcal{G}^+(G_3) = g \neq 0$ and $\mathcal{G}^-(G_3) = g_0 \neq 0$. So G_3 is an N-position under the normal play convention.

We now consider G_4. Let $\Gamma(G_4) = g^{g_0 g_1 g_2 g_3 \cdots}$. If $n + t = 0$, then $G_4 = G^{'''}$ is an N-position under the normal play convention by Eq. (5). If $n + t > 0$, without loss of generality, let $n \geq 1$. In this case, G_4 has options $G_4^{'}=$ and $G_4^{''}=$. Note that $\Gamma(G_4^{'})=1^{031}$ and $\Gamma(G_4^{''})=0^{120}$. By Lemma 7, $\mathcal{G}^+(G_4) = g \neq 0$ and $\mathcal{G}^-(G_4) = g_0 \neq 0$. Hence G_4 is an N-position under the normal play convention.

(2) ($\Leftarrow$) $G^{'}$, $G^{''}$ and $G^{'''}$ are P-positions under the misère play convention by Eq. (6). By Theorem 14, $G^{'}$, $G^{''}$ and $G^{''} \in \mathcal{T}$. Thus $G = G^{'}$, $G^{''}$ or $G^{'''} \in \mathcal{T} \cap \mathcal{P}$.

($\Rightarrow$) By Theorem 14, $G \in \mathcal{T}$ implies $G \in \{G_1, G_2, G_3, G_4\}$, where $G_i (i = 1, 2, 3, 4)$ is defined in Theorem 14.

Note that $G_1 = G^{'}$ and $G_2 = G^{''}$ are P-positions under the misère play convention by Eq. (6).

Consider G_3. It follows from case (1) that $\mathcal{G}^-(G_3) \neq 0$. By Eq. (6), G_3 is an N-position under the misère play convention.

Consider G_4. If $n + t > 0$, it follows from case (1) that $\mathcal{G}^-(G_4) \neq 0$, thus G_4 is an N-position under the misère play convention. If $n + t = 0$, then $G_4 = G^{'''}$. Recall that $\mathcal{G}^-(G_4) = \mathcal{G}^-(G^{'''}) = 0$, thus $G^{'''}$ is a P-position under the misère play convention by Eq. (6).

Thus $G = G^{'}$, $G^{''}$ or $G^{'''}$. □

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