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Abstract

Let $S(h,k)$ be the least positive integer such that any 2-coloring of the interval $[1, S(h,k)]$ must admit either

(i) a monochromatic solution to $x_1 + \ldots + x_{h-1} = x_h$ with $x_1 < x_2 < \ldots < x_h$ or

(ii) a monochromatic solution to $x_1 + \ldots + x_{k-1} = x_k$ with $x_1 < x_2 < \ldots < x_k$.

We prove $S(3,3) = 9$, $S(3,4) = 16$, and for all $k \geqslant 5$,

$$S(3,k) = \begin{cases} 3k^2/2 - 7k/2 + 3 & \text{if } k \equiv 0, 1 \pmod 4, \\ 3k^2/2 - 7k/2 + 4 & \text{if } k \equiv 2, 3 \pmod 4. \end{cases}$$

1. Introduction

Let $\mathbb{N}$ denote the set of positive integers and $[a,b] = \{n \in \mathbb{N} : a \leqslant n \leqslant b\}$. A mapping $\chi : [a,b] \to [1,t]$ is called a t-coloring of $[a,b]$. Let L_m denote the system of inequalities given by

$$x_1 + x_2 + \ldots, x_{m-1} = x_m, \qquad x_1 < x_2 < \cdots < x_m.$$

A solution $n_1, n_2, \ldots, n_m$ to L_m is *monochromatic* if $\chi(n_i) = \chi(n_j)$ for $1 \leqslant i, j \leqslant m$. Henceforth, we assume a *two-coloring* ($t = 2$) of the interval and denote each color by *red* and *blue*. Furthermore, a monochromatic solution to L_m such that $\chi(n_1) = \chi(n_2) = \ldots = \chi(n_m) = red$ will be called a "*red* solution," and likewise for a "*blue* solution." Lastly, we define $S(h,k)$ to be the least positive integer such that every coloring of the interval $[1, S(h,k)]$ by the colors *red* and *blue* contains either a *red* solution to L_k or a *blue* solution to L_h.

In the following proofs, we show that $S(3,3) = 9$, $S(3,4) = 16$, and for all $k \geqslant 5$,

$$S(3,k) = \begin{cases} 3k^2/2 - 7k/2 + 3 & \text{if } k \equiv 0,1 \pmod 4, \\ 3k^2/2 - 7k/2 + 4 & \text{if } k \equiv 2,3 \pmod 4. \end{cases}$$

2. The Lower Bound

Lemma 1. (Lower Bound) *For $k \geqslant 3$,*

$$S(3,k) \geqslant N = \begin{cases} 3k^2/2 - 7k/2 + 3 & \textit{if } k \equiv 0,1 \pmod 4, \\ 3k^2/2 - 7k/2 + 4 & \textit{if } k \equiv 2,3 \pmod 4. \end{cases}$$

Proof. Consider a coloring of $\chi : [1, N-1] \to \{blue, red\}$ defined as follows. For $n \in [1, N-1]$, let

$$\chi(n) = \begin{cases} blue & \text{if } n \equiv 1 \pmod 2 \text{ and } n \leqslant k(k-1)/2, \\ blue & \text{if } n \equiv 0 \pmod 2 \text{ and } n \geqslant k(k-1), \\ red & \text{otherwise.} \end{cases}$$

We claim this coloring has no *blue* solution to L_3 and no *red* solution to L_k.

Suppose $n_1 + n_2 = n_3$, where $n_1 < n_2 < n_3$, is a *blue* solution to L_3 on the interval $[1, N-1]$. Suppose $n_2 \equiv 1 \pmod 2$. Then $n_1 < n_2 \leqslant k(k-1)/2$, which implies $n_1 \equiv 1 \pmod 2$ and $n_3 < k(k-1)$. Since $n_3 = n_1 + n_2 \equiv (1+1) \pmod 2 \equiv 0 \pmod 2$, we must have that $n_3 \geqslant k(k-1)$, which is a contradiction. Therefore, $n_2 \equiv 0 \pmod 2$.

Hence, $n_3 > n_2 \geqslant k(k-1)$, which implies $n_3 \equiv 0 \pmod 2$, which then implies $n_1 \equiv 0 \pmod 2$. Therefore, $n_2 > n_1 \geqslant k(k-1)$, which implies $n_3 = n_1 + n_2 \geqslant k(k-1) + (k(k-1)+2) > N-1$, another contradiction implying no such *blue* solution to L_3 exists.

Next, suppose $n_1 + n_2 + \cdots + n_{k-1} = n_k$, where $n_1 < n_2 < \cdots < n_k$, is a *red* solution to L_k on the interval $[1, N-1]$. Let q denote the minimal sum of $k-2$ *red* numbers. Clearly, $q = \sum_{i=1}^{k-2} 2i = k^2 - 3k + 2$.

If $n_{k-1} \leqslant k(k-1)/2$, then $n_i \equiv 0 \pmod 2$ for $i \in [1, k-1]$. This implies $n_k \equiv 0 \pmod 2$, which implies $n_k < k(k-1)$, but this is a contradiction since $k(k-1) > n_k \geqslant q + 2(k-1) = k(k-1)$. Therefore, $n_{k-1} > k(k-1)/2$.

If $k \equiv 0, 1 \pmod 4$, then $n_k > q + k(k-1)/2 = 3k^2/2 - 7k/2 + 2 = N - 1$, a contradiction that implies $k \equiv 2, 3 \pmod 4$.

If $n_{k-1} = k(k-1)/2 + 1$, then $n_i \equiv 0 \pmod 2$ for all $i \in [1, k-1]$, which implies $n_k \equiv 0 \pmod 2$. Since n_k is *red*, $n_k < k(k-1)$, which is a contradiction since $n_k \geqslant q + k(k-1)/2 + 1$. Therefore, $n_{k-1} > k(k-1)/2 + 1$, which implies $n_k \geqslant q + k(k-1)/2 + 2 = 3k^2/2 - 7k/2 + 4 = N$. □

3. The Upper Bound

Throughout this section, let p denote the sum of the first k *red* numbers and let r_i and b_i denote the i^{th} *red* and *blue* numbers, respectively. Then, $r_i < r_j$ and $b_i < b_j$, for all $i < j$.

Lemma 2. *For $n \geqslant 3$, if at least $n+1$ numbers in the interval $[1, 2n]$ are colored blue, then the only coloring that avoids a blue solution to L_3 is given by*

$$\chi(x) = \begin{cases} \textit{red} & \textit{if } x \in [1, n-1], \\ \textit{blue} & \textit{if } x \in [n, 2n]. \end{cases}$$

Proof. Since the case $n = 3$ is trivial, assume $n > 3$.
`Case 1.` $\chi(2n) = \textit{red}$. We proceed via induction on n. For some $n > 3$, assume the claim holds for $n-1$. To avoid a *blue* solution on the interval $[1, 2(n-1)]$, we must have $\chi(x) = \textit{blue}$ for all $x \in [(n-1), 2(n-1)]$ and $\chi(x) = \textit{red}$ for all $x \in [1, n-2]$. Since we need another *blue* in the interval $[1, 2n]$, the number $(2n-1)$ must be *blue*, but then $(n-1) + n = (2n-1)$ is a *blue* solution to L_3.

`Case 2.` $\chi(2n) = \textit{blue}$. By the pigeonhole principle, $\chi(n) = \textit{blue}$, since otherwise one of the pairs $\{x, 2n-x\}$ with $1 \leqslant x < n$ would be all *blue* giving us the *blue* solution $(x) + (2n-x) = 2n$. Now suppose $\chi(1) = \textit{blue}$, which implies the pair $\{n-1, n+1\}$ is all *red*. Hence, some other pair $\{x, 2n-x\}$ with $1 \leqslant x < n-1$, is all *blue* and we get a contradiction. Therefore, $\chi(1) = \textit{red}$; hence $\chi(2n-1) = \textit{blue}$, otherwise some other pair $\{x, 2n-x\}$ with $2 \leqslant x \leqslant n-2$ would be all *blue*, giving us a contradiction as before. But then we must have $\chi(n-1) = \textit{red}$ since $(n-1)+n = (2n-1)$; hence $\chi(n+1) = \textit{blue}$. But then we must have $\chi(n-2) = \textit{red}$ since $(n-2)+(n+1) = (2n-1)$; hence $\chi(n+2) = \textit{blue}$. Continuing in this manner, we get the desired coloring. □

Corollary 1. *To avoid a blue solution to L_3, $r_i \leqslant 2i+1$ for all i.*

Proof. The claim is easily proven for r_1 and r_2. Suppose $r_i > 2i+1$ for some $i \geqslant 3$. This would imply at least $i+1$ numbers are colored *blue* in the interval $[1, 2i]$. Applying Lemma 2 gives us that $\chi(i) = \chi(i+1) = \textit{blue}$. Since there are $2i+1-(i-1) = i+2$ *blue* integers in $[1, 2i+1]$, $\chi(2i+1) = \textit{blue}$ as well, but this yields the *blue* solution $i + (i+1) = 2i+1$. □

Corollary 2. *There exists at least k integers in the interval $[1, N]$ colored red, that is, p exists.*

Proof. If p does not exist, then by Corollary 1, $r_{k-1} \leqslant 2k - 1$, which implies that all numbers in $[2k, N]$ are colored *blue*, while the Pigeonhole Principle ensures that some number in $[1, k]$ is also colored *blue*, say $x \in \{1, 2, \ldots, k\}$. But then $x + 2k = x + 2k \leqslant N$ is a *blue* solution for $k \geqslant 4$, a contradiction. The case $k = 3$ can be done separately by hand varying this argument and handling several cases. □

Corollary 3. *Let $i > b_1$ and $i \geqslant 3$. To avoid a blue solution to L_3, $r_i \leqslant 2i$.*

Proof. Suppose $r_i > 2i$. Then Corollary 1 implies $r_i = 2i + 1$. Therefore, the interval $[1, 2i + 1]$ must contain exactly $i + 1$ *blue* numbers. Since $i \geqslant 3$, Lemma 2 implies that the interval $[1, i - 1]$ is all *red*, but this contradicts the hypothesis $b_1 < i$. □

Lemma 3. (P Lemma) *The following hold:*

(*i*) *If $b_1 = 1$, then $p \leqslant k^2 + k - 12 + r_1 + r_2 + r_3$,*

(*ii*) *If $1 < b_1 \leqslant k$, then $p \leqslant k^2 + k + 1 - b_1(b_1 - 1)/2$,*

(*iii*) *If $b_1 > k$, then $p = k(k + 1)/2$.*

Proof. (*i*) If $b_1 = 1$, then by Corollary 3

$$\begin{aligned} p = r_1 + r_2 + r_3 + \sum_{i=4}^{k} r_i &\leqslant r_1 + r_2 + r_3 + \sum_{i=4}^{k} 2i \\ &= k^2 + k - 12 + r_1 + r_2 + r_3. \end{aligned}$$

(*ii*) If $1 < b_1 \leqslant k$, then by Corollaries 1 and 3

$$\begin{aligned} p = \sum_{i=1}^{b_1-1} r_i + r_{b_1} + \sum_{i=b_1+1}^{k} r_i &\leqslant \sum_{i=1}^{b_1-1} i + (2b_1 + 1) + \sum_{i=b_1+1}^{k} r_i \\ &\leqslant \sum_{i=1}^{b_1-1} i + (2b_1 + 1) + \sum_{i=b_1+1}^{k} 2i \\ &= k^2 + k + 1 - b_1(b_1 - 1)/2. \end{aligned}$$

(*iii*) If $b_1 > k$, then $p = \sum_{i=1}^{k} r_i = \sum_{i=1}^{k} i = k(k + 1)/2$.

□

Given a valid colouring, the upper bound for p can be improved by modifying Lemma 3. For example, for $k \geqslant 6$, if $r_1 = 1$, $b_1 = 2$, $r_2 = 3$, and $r_3 = 4$, then

$$p \leqslant k^2 + k + 1 - b_1(b_1 - 1)/2 - (5 - r_2) - (6 - r_3) = k^2 + k - 4.$$

Fact 1. *If $k \geqslant 6$, then $k^2 + k - 5 \leqslant N$.*

Corollary 4. *If $p - r_j \leqslant N$ for some $j \in [1, k]$, then to avoid a red solution to L_k, $\chi(p - r_i) = blue$ for all $i \in [j, k]$.*

Proof. If $\chi(p - r_i) = red$ for some $i \in [j, k]$, then we get the *red* solution to L_k

$$r_1 + r_2 + \ldots + r_{i-1} + r_{i+1} + r_{i+2} + \ldots + r_k = p - r_i,$$

where $p - r_i \leqslant p - r_j \leqslant N$ (by hypothesis). Hence, $\chi(p - r_i) = blue$ for all $i \in [j, k]$. $\square$

Corollary 5. *If $k \geqslant 6$ and $b_1 > 1$, then $p - r_i \leqslant N$ for all r_i.*

Proof. We have $p - r_i \leqslant p - 1$. In view of Fact 1, if $p \leqslant k^2 + k - 4$, then $p - r_i \leqslant N$ for all r_i. If $b_1 = 2$, then modifying Lemma 3, we get the following cases:

#	1	2	3	4	5	6	7	8	9	10	11	12	n s.t. $p \leqslant n$
1.	r_1	b_1	b_2	b_3	r_2	r_3	r_4	b_4	b_5	r_5	r_6	r_7	k^2+k-4
2.	r_1	b_1	b_2	b_3	r_2	r_3	r_4	b_4	r_5	r_6			k^2+k-4
3.	r_1	b_1	b_2	b_3	r_2	r_3	r_4	r_5	b_4	b_5	r_6		k^2+k-4
4.	r_1	b_1	b_2	b_3	r_2	r_3	r_4	r_5	b_4	r_6			k^2+k-5
5.	r_1	b_1	b_2	b_3	r_2	r_3	r_4	r_5	r_6				k^2+k-6
6.	r_1	b_1	b_2	r_2	r_3	b_3	b_4	r_4	r_5	r_6			k^2+k-5
7.	r_1	b_1	b_2	r_2	r_3	b_3	r_4	r_5					k^2+k-5
8.	r_1	b_1	b_2	r_2	r_3	r_4							k^2+k-4
9.	r_1	b_1	r_2	b_2	b_3	r_3	r_4	b_4	r_5				k^2+k-4
10.	r_1	b_1	r_2	b_2	b_3	r_3	r_4	r_5					k^2+k-5
11.	r_1	b_1	r_2	b_2	r_3	r_4							k^2+k-5
12.	r_1	b_1	r_2	r_3									k^2+k-4

Similarly, if $b_1 = 3$ then modifying Lemma 3, we get the following cases:

#	1	2	3	4	5	6	7	8	9	10	n s.t. $p \leqslant n$
1.	r_1	r_2	b_1	b_2	b_3	b_4	r_3	r_4	r_5	r_6	k^2+k-5
2.	r_1	r_2	b_1	b_2	b_3	r_3	r_4				k^2+k-4
3.	r_1	r_2	b_1	b_2	r_3						k^2+k-4
4.	r_1	r_2	b_1	r_3							k^2+k-5

For $4 \leqslant b_1 \leqslant k$, by Lemma 3 we have $p \leqslant k^2 + k - 5$. For $b_1 > k$, $p = k(k+1)/2$ by part (iii) of Lemma 3. For $k \geqslant 6$ and $b_1 > 1$, we have $p \leqslant k^2 + k - 4$, and hence, using Fact 1

$$p - r_i \leqslant p - 1 \leqslant (k^2 + k - 4) - 1 = k^2 + k - 5 \leqslant N$$

for all r_i with $i \in [1, k]$. □

Remark 1. Combining Corollaries 4 and 5, we see that, for $k \geqslant 6$ and $b_1 > 1$, $\chi(p - r_j) = blue$ for all $j \in [1, k]$.

Lemma 4. (Upper Bound) *For $k \geqslant 6$,*

$$S(3, k) \leqslant N = \begin{cases} 3k^2/2 - 7k/2 + 3 & \textit{if } k \equiv 0, 1 \pmod 4, \\ 3k^2/2 - 7k/2 + 4 & \textit{if } k \equiv 2, 3 \pmod 4. \end{cases}$$

Proof. Suppose to the contrary that N is not an upper bound for $k \geqslant 6$. This occurs if and only if there exists a coloring of $[1, N]$ without a *blue* solution to L_3 and a *red* solution to L_k. Consider the following two cases:

(1) $\chi(1) = blue$ (with $k \geqslant 6$). Suppose $\chi(2) = blue$. Then $r_1 = 3$ and $r_2 \leqslant 5$ (by Corollary 1) to avoid *blue* solutions $1 + 2 = 3$ and $1 + 4 = 5$, respectively. Therefore, by Lemma 3 and Corollary 3, we have

$$p \leqslant k^2 + k - 4 + r_3 \leqslant k^2 + k + 2,$$

which implies (by Fact 1) that $p - r_i \leqslant k^2 + k - 5 \leqslant N$ for $r_i \geqslant 7$.

If $\chi(x) = blue$ for some $x \in [6, 7]$, then this implies $\chi(x+1) = \chi(x+2) = red$ to avoid the *blue* solutions $1+x = x+1$ and $2+x = x+2$. Corollary 4 implies $\chi(p - x - 1) = \chi(p - x - 2) = blue$, and then $1 + (p - x - 2) = p - x - 1$ is a *blue* solution since $p - x - 2 \geqslant k(k + 1)/2 - 9 \geqslant 12 > 1$. Therefore, $\chi(6) = \chi(7) = red$. Considering the present information, it can be shown that $p - r_i \leqslant N$ for $r_i \geqslant 6$. If $r_2 = 4$, then $p \leqslant k^2 + k + 1$, otherwise 7 being *red* means that the estimate for r_4 can be improved by one. Now, Corollary 4 gives $\chi(p - 6) = \chi(p - 7) = blue$. Thus $1 + (p - 7) = p - 6$ is a *blue* solution in view of $p - 8 \geqslant k(k + 1)/2 - 8 \geqslant 13 > 1$. So we conclude that $\chi(2) = red$.

If $b_2 \geqslant 5$, then $r_2 = 3$, $r_3 = 4$, and, by Lemma 3 and Fact 1,

$$p - r_1 \leqslant k^2 + k - 12 + r_2 + r_3 = k^2 + k - 5 \leqslant N,$$

which leads to a contradiction since Corollary 4 gives us the *blue* solution $1 + (p - 4) = p - 3$.

If $b_2 = 4$, then $r_2 = 3$ and $r_3 = 5$, and by Lemma 3 and Fact 1

$$p - r_2 \leqslant k^2 + k - 12 + r_1 + r_3 = k^2 + k - 5 \leqslant N.$$

By Corollary 4, $\chi(p-3) = \chi(p-5) = blue$. To avoid a *blue* solution $1+(p-6) = p - 5$, we need $\chi(p - 6) = red$, but by Corollary 4, this implies $\chi(6) = blue$.

In that case, to avoid the *blue* solution $1 + 6 = 7$, we need $\chi(7) = red$, but that yields the *blue* solution $4 + (p - 7) = p - 3$.

Now, suppose $b_3 > r_3$. If $b_2 = 3$, then $r_2 = 4$, $r_3 = 5$ (since $b_3 > r_3$), and by Lemma 3 and Fact 1, $p - r_2 \leqslant k^2 + k - 5 \leqslant N$, but that yields the *blue* solution $1 + (p - 5) = p - 4$ (by Corollary 4).

Therefore, $b_3 < r_3$, which implies the interval $[3, 5]$ has two *blue* numbers. Since these *blue* numbers cannot be adjacent, the only valid coloring is $\chi(3) = \chi(5) = blue$ and $\chi(4) = \chi(6) = red$. With this coloring, Corollary 1, Lemma 3, Corollary 4, and Fact 1 imply that $\chi(p - r_i) = blue$, for $i \in [3, k]$.

To avoid the *blue* solution $1+(p-r_{i+1}) = p-r_i$, we must have $r_{i+1} > r_i+1$, that is, $\chi(r_i+1) = blue$, for all $i \in [3, k-1]$. Thus $\chi(7) = blue$. Also, to avoid the *blue* solution $1 + b_i = b_{i+1}$, we must have $\chi(b_i + 1) = red$, for all $i > 1$. Thus $\chi(8) = red$, which implies $\chi(9) = blue$, which implies $\chi(10) = red$, and continuing in this manner, we get that, for all $x \in [1, 2k]$,

$$\chi(x) = \begin{cases} red & \text{if } x \equiv 0 \pmod 2, \\ blue & \text{if } x \equiv 1 \pmod 2. \end{cases}$$

Furthermore, for all $x \in [1, 2k - 3]$ with $\chi(x) = blue$, we must have $\chi(x + (2k - 1)) = red$, otherwise we get the *blue* solution $x + (2k - 1) = x + 2k - 1$. This implies $\chi(y) = red$ for all $y \in [2k, 4k - 4]$ with $y \equiv 0 \pmod 2$.

Using the block of even *red* numbers, we can extend the *blue* interval. Clearly, the sum of any $k - 1$ *red* numbers which is less than N must be *blue*. The maximal sum of $k-1$ *red* numbers from the block is $\sum_{i=0}^{k-2}((4k-4)-2i) = 3k^2 - 5k + 2$, which is clearly greater than N. Furthermore, the minimal sum of $k - 1$ *red* numbers from the block is $\sum_{i=1}^{k-1} 2i = k^2 - k$. Since we can always replace a *red* number in the minimal sum by an adjacent even number which is also *red*, and the maximal sum is greater than N, we get that all even numbers greater than or equal to $k^2 - k$ must be *blue*. This yields the extended coloring

$$\chi(x) = \begin{cases} red & \text{if } x \equiv 0 \pmod 2 \text{ and } x \in [2, 4k-4], \\ blue & \text{if } x \equiv 1 \pmod 2 \text{ and } x \in [1, 2k-1], \\ blue & \text{if } x \equiv 0 \pmod 2 \text{ and } x \in [k^2-k, N]. \end{cases}$$

It can easily be shown that $N \equiv 1 \pmod 2$, implying $\chi(N - 1) = blue$. Since $\chi(1) = \chi(3) = blue$, we must have $\chi(N) = \chi(N - 2) = \chi(N - 4) = red$. Let q be the sum of first $k - 2$ *red* numbers. Then $q = k^2 - 3k + 2$. To avoid a *red* solution to L_k, we must have

$$\chi(N - q) = \chi(N - 2 - q) = \chi(N - 4 - q) = blue,$$

since $N-4-q > r_{k-2} = 2(k-2)$.

If $k \equiv 0,1 \pmod 4$, then we get the *blue* solution

$$(N-q)+(N-2-q) = 2(3k^2/2-7k/2+3)-2(k^2-3k+2)-2 = k^2-k.$$

Likewise, if $k \equiv 2,3 \pmod 4$, then we get the *blue* solution

$$(N-q)+(N-4-q) = 2(3k^2/2-7k/2+4)-2(k^2-3k+2)-4 = k^2-k.$$

Therefore, $\chi(1) \neq$ *blue*.

(2) $\chi(1) =$ *red* (with $k \geqslant 6$). Since $k \geqslant 6$ and $b_1 > 1$, by Remark 1, we have $\chi(p-r_i) =$ *blue* for all $i \in [1,k]$. Let a be the minimum *red* number such that $\chi(a-1) =$ *blue*. It can be shown that a exists. Suppose a does not exist, that is, x (say) *red* numbers are followed by $N-x$ *blue* numbers. If $x \geqslant k(k-1)/2$, then we have a *red* solution $1+2+\cdots+(k-1) = k(k-1)/2$, or else we have a *blue* solution $k(k-1)/2+(k(k-1)/2+1) = k^2-k+1 \leqslant N$.

If $a < r_k$, then $\chi(p-a) =$ *blue*, which gives us a potential *blue* solution $(a-1)+(p-a) = p-1$. In order for it to be a valid solution, we must have that $a-1 \neq p-a$. However, since $a = r_i$ for some $i \in [1,k]$, this has already been proven in Corollary 4 ($p-r_i > r_k$ for all $i \in [1,k]$).

The argument of the previous paragraph yields a contradiction unless $p-r_k = p-a = a-1$, in which case $k+(k+1) = 2k+1 \leqslant \sum_{i=1}^{k-1} i = p-r_k = p-a = a-1$, from which it is clear in view of the definition of a that $k+(k+1) = 2k+1$ is a *blue* solution.

Suppose $b_1 \leqslant 3k/2$. To avoid the *blue* solution, $b_1+(b_1+1) = 2b_1+1$, either b_1+1 or $2b_1+1$ must be *red*, which implies $a \leqslant 2b_1+1 \leqslant 3k+1$. Now consider,

$$\begin{aligned}(p-r_k)-1+a &= \sum_{i=1}^{k-1} i+(a-1) = k(k-1)/2+(a-1)\\ &\leqslant k(k-1)/2+(3k+1)-1 = (k^2+5k)/2 \leqslant N.\end{aligned}$$

Since $\chi(a) =$ *red*, to avoid the *red* solution $a+\sum_{i=2}^{k-1} i = (p-r_k)-1+a$, we have $\chi(p-r_k+a-1) =$ *blue*, which yields the potential *blue* solution

$$(p-r_k)+(a-1) = (p-r_k)-1+a.$$

To be a valid solution, we must have $a-1 \neq p-r_k$. If $a-1 = p-r_k$, then $p-r_k+1 = a$, which implies $\chi(p-r_k+1) =$ *red*. However, this is a contradiction since $\chi(p-r_{k-1}) =$ *blue* and $r_{k-1} = r_k-1$.

Therefore, $b_1 > 3k/2$, which implies $\chi(x) =$ *red* for all $x \in [1,3k/2]$. Using this *red* interval, we can create another *blue* interval. The minimum sum of

$k-1$ *red* numbers in this interval is $k(k-1)/2$. If k is even, then the maximal sum is k^2-1, and if k is odd, then since only integers are colored, we only know that the interval $[1,(3k-1)/2]$ is colored all *red*, in which case, the maximal sum of $k-1$ *red* integers is $k^2-k/2-1/2$.

Since every integer in this new interval can be represented by a sum of $k-1$ *red* numbers, the interval $[k(k-1)/2, k^2-k/2-1/2]$ must be colored *blue* to avoid a *red* solution. Since k^2-k+1 is in the *blue* interval, we have the *blue* solution $k(k-1)/2+(k(k-1)/2+1)=k^2-k+1$.

Hence, for $k \geqslant 6$, every coloring of $[1,N]$ has a *blue* solution to L_3 or a *red* solution to L_k. □

4. The Cases $3 \leqslant k \leqslant 5$

In this section, we formally prove the exact values of $S(3,3)$ and $S(3,4)$, and provide the computer proof for the exact value of $S(3,5)$.

Lemma 5. $S(3,3)=9$.

Proof. Let $\chi(1)=\chi(2)=\chi(4)=\chi(8)=red$ and $\chi(3)=\chi(5)=\chi(6)=\chi(7)=blue$. This coloring has no *red* or *blue* solution to L_3. Therefore, $S(3,3)>8$.

Suppose to the contrary that $S(3,3)>9$. Without loss of generality, let *blue* be the color used 5 or more times from 1 to 9. If $\chi(9)=red$, then Lemma 2 gives us the *red* solution $1+2=3$. Therefore, $\chi(9)=blue$. We are left with two cases.

`Case 1.` $\chi(8)=blue$. To avoid the *blue* solution $1+8=9$, we have $\chi(1)=red$. If $\chi(5)=blue$, then we must have $\chi(3)=red$ (to avoid the *blue* solution $3+5=8$) and $\chi(4)=red$ (to avoid the *blue* solution $4+5=9$). But then we have the *red* solution $1+3=4$. Therefore, $\chi(5)=red$. To avoid the *red* solutions $1+4=5$ and $1+5=6$, we must have $\chi(4)=\chi(6)=blue$. Then $\chi(2)=red$ (to avoid the *blue* solution $2+4=6$), which implies $\chi(3)=blue$ (to avoid the *red* solution $1+2=3$), which gives the *blue* solution $3+6=9$.

`Case 2.` $\chi(8)=red$. If $\chi(7)=red$, then Lemma 2 gives us the *red* solution $1+7=8$. Therefore, $\chi(7)=blue$, which leads to a contradiction after a chain of implications:

$\chi(2)=red$ (to avoid the *blue* solution $2+7=9$),
$\chi(6)=blue$ (to avoid the *red* solution $2+6=8$),
$\chi(1)=red$ (to avoid the *blue* solution $1+6=7$),
$\chi(3)=blue$ (to avoid the *red* solution $1+2=3$),
and hence the *blue* solution $3+6=9$. □

Lemma 6. $S(3,4)=16$.

Proof. For all $x \in [1, 15]$, let $x \in [6, 12]$ be *blue* and x be *red* otherwise. This coloring has no *blue* solution to L_3 and no *red* solution to L_4. Therefore, $S(3, 4) > 15$.

Suppose to the contrary that $S(3, 4) > 16$. Then suppose $\chi(1) = blue$. Corollary 3 implies $r_i \leqslant 2i$, for all $i \geqslant 3$.

Since $r_1+r_2+r_4 \leqslant 3+5+8 = 16$, we have $\chi(r_1+r_2+r_3) = \chi(r_1+r_2+r_4) = blue$. If $r_4 > r_3 + 2$, we get the *blue* solution $1 + (r_3 + 1) = r_3 + 2$, and if $r_4 = r_3 + 1$, we get the *blue* solution $1 + (r_1 + r_2 + r_3) = r_1 + r_2 + r_4$. Hence, $r_4 = r_3 + 2$. To avoid the *blue* solution $2 + (r_1 + r_2 + r_3) = r_1 + r_2 + r_4$, we must have $\chi(2) = red$, that is, $r_1 = 2$, which implies $r_1 + r_3 + r_4 \leqslant 2 + 6 + 8 = 16$. Thus $\chi(r_1 + r_3 + r_4) = blue$.

Applying the same reasoning to r_3 as we did to r_4, we get that $r_3 = r_2 + 2$. Then to avoid the *blue* solution $4 + (r_1 + r_2 + r_3) = r_1 + r_3 + r_4$, we must have $\chi(4) = red$. If $\chi(3) = red$, then $r_2 = 3$, and so $r_3 = 4$. But $r_3 \neq r_2 + 1$. Therefore, $\chi(3) = blue$, which implies $r_2 = 4$, and so $r_3 = 6$ and $r_4 = 8$. Then we must have $\chi(5) = \chi(7) = \chi(12) = blue$, but then we get the *blue* solution $5 + 7 = 12$. Therefore, $\chi(1) = red$.

Corollary 1 gives us that $r_2 = 2, 3, 4$, or 5. We handle these four cases separately.

`Case 1.` $r_2 = 5$. This implies $\chi(2) = \chi(3) = \chi(4) = blue$. Therefore, $\chi(6) = red$ and $\chi(7) = red$ to avoid the *blue* solutions $2 + 4 = 6$ and $3 + 4 = 7$, respectively. Hence, $\chi(12) = blue$ and $\chi(14) = blue$ to avoid the *red* solutions $1+5+6 = 12$ and $1 + 6 + 7 = 14$, respectively. But, then we get the *blue* solution $2 + 12 = 14$.

`Case 2.` $r_2 = 4$. This implies $\chi(2) = \chi(3) = blue$. Therefore, $\chi(5) = red$ (to avoid the *blue* solution $2 + 3 = 5$), which implies $\chi(10) = blue$ (to avoid the *red* solution $1 + 4 + 5 = 10$). Therefore, $\chi(7) = red$ and $\chi(12) = red$ to avoid the *blue* solutions $3+7 = 10$ and $2+10 = 12$, respectively. But then we get the *red* solution $1 + 4 + 7 = 12$.

`Case 3.` $r_2 = 3$. This implies $\chi(2) = blue$. If $r_4 = 9$, then Lemma 2 implies $\chi(2) = red$. Thus $r_4 \leqslant 8$.

If $r_3 \geqslant 6$, then $\chi(4) = \chi(5) = blue$, which implies $r_3 = 6$ (to avoid the *blue* solution $2 + 4 = 6$). Therefore, $\chi(7) = red$ (to avoid the *blue* solution $2 + 5 = 7$), which implies $\chi(14) = blue$ (to avoid the *red* solution $1+6+7 = 14$) and $\chi(16) = blue$ (to avoid the *red* solution $3 + 6 + 7 = 16$). But then we get the *blue* solution $2 + 14 = 16$. Therefore, $r_3 \leqslant 5$.

This implies $3+r_3+r_4 \leqslant 16$, which gives us that $\chi(1+r_3+r_4) = \chi(3+r_3+r_4) = blue$, but then we get the *blue* solution $2 + (1 + r_3 + r_4) = 3 + r_3 + r_4$.

`Case 4.` $r_2 = 2$. Suppose $\chi(7) = red$. Then $\chi(4) = blue$ (to avoid the *red* solution $1 + 2 + 4 = 7$) and $\chi(10) = blue$ (to avoid the *red* solution $1 + 2 + 7 = 10$). This implies $\chi(6) = red$ and $\chi(14) = red$ to avoid the *blue* solutions $4 + 6 = 10$ and $4 + 10 = 14$, respectively. But then we get the *red* solution $1 + 6 + 7 = 14$. Therefore, $\chi(7) = blue$.

Suppose $\chi(3) = blue$. Then $\chi(4) = red$ and $\chi(10) = red$ to avoid the *blue* solutions $3 + 4 = 7$ and $3 + 7 = 10$, respectively. This implies $\chi(13) = blue$ and $\chi(16) = blue$ to avoid the *red* solutions $1 + 2 + 10 = 13$ and $2 + 4 + 10 = 16$, respectively. Therefore, $\chi(3) = red$, which leads to a contradiction after a chain of implications:

$\chi(6) = blue$ (to avoid the *red* solution $1 + 2 + 3 = 6$),
$\chi(13) = red$ (to avoid the *blue* solution $6 + 7 = 13$),
$\chi(9) = blue$ (to avoid the *red* solution $1 + 3 + 9 = 13$),
$\chi(16) = red$ (to avoid the *blue* solution $7 + 9 = 16$),
and hence the *red* solution $1 + 2 + 13 = 16$. □

4.1. Computer Assisted Proof for $S(3, 5)$

Let us write a coloring of $[1, n]$ as a bit-string of length n where the i-th bit is zero if $\chi(i) = blue$, and one if $\chi(i) = red$.

4.1.1. S(3,5)=23

By Lemma 1, the lower bound is $S(3, 5) > 22$. We consider all of the ten colorings of $[1, 22]$ (obtained by computer search) without a *blue* solution to L_3 and a *red* solution to L_5.

1. For each of the following four colorings

   0010110111111111111110,

   0010110111111011111110,

   0010110111111011011110, and

   0010110111101111011110,

   if $\chi(23) = blue$, then we have a *blue* solution $1 + 22 = 23$ to L_3; and

   if $\chi(23) = red$, then we have a *red* solution $3 + 5 + 6 + 9 = 23$ to L_5.

2. For each of the following four colorings

   0010111011111111111101,

   0010111011111110111101,

   0010111011111101111101, and

   0010111011111011111101,

   if $\chi(23) = blue$, then we have a *blue* solution $2 + 21 = 23$ to L_3; and

   if $\chi(23) = red$, then we have a *red* solution $3 + 5 + 6 + 9 = 23$ to L_5.

3. For each of the following two colorings

   0101010101111111101010, and

   0101010101111111111010,

   if $\chi(23) = blue$, then we have a *blue* solution $1 + 22 = 23$ to L_3; and

   if $\chi(23) = red$, then we have a *red* solution $2 + 4 + 6 + 11 = 23$ to L_5.

Therefore, $S(3, 5) = 23$.

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