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WEIGHTED LONESUM MATRICES AND THEIR GENERATING FUNCTION

Ken Kamano
Department of Mathematics, Osaka Institute of Technology, Osaka, Japan
kamano@ge.oit.ac.jp

Yasuo Ohno[1]
Department of Mathematics, Kinki University, Osaka, Japan
ohno@math.kindai.ac.jp

Shuji Yamamoto[2]
Department of Mathematics, Faculty of Science and Technology, Keio University, Yokohama, Japan
yamashu@math.keio.ac.jp

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Abstract

A lonesum matrix is a $(0,1)$-matrix uniquely determined by its column and row sums, and the sum of its all entries is called the "weight" of it. The generating function of numbers of weighted lonesum matrices of each weight is given. A certain explicit formula for the number of weighted lonesum matrices is also proved.

1. Introduction and Main Theorem

A matrix A is called a $(0,1)$-matrix if each of its entries is either zero or one, and a $(0,1)$-matrix A is called lonesum if it is uniquely determined by its column and row sums. For positive integers m and n, we denote by $L(m,n)$ the number of $m \times n$ lonesum matrices. Further, we define $L(m,0) = L(0,n) = 1$ for all non-negative integers m and n.

Brewbaker [1] proved that

$$L(m,n) = B_n^{(-m)} \tag{1}$$

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for any non-negative integers m and n. Here, $B_n^{(k)}$ is the n-th *poly-Bernoulli number* of index k defined by

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}} = \sum_{n\geq 0} B_n^{(k)} \frac{x^n}{n!}, \tag{2}$$

where $\mathrm{Li}_k(x)$ denotes the k-th polylogarithm (cf. [2]). Shikata [5] gave another proof of equality (1). To obtain the equality, Shikata proved a generating function representation of $L(m,n)$ as follows:

$$\sum_{m\geq 0, n\geq 0} L(m,n) \frac{x^m}{m!} \frac{y^n}{n!} = (e^{-x}+e^{-y}-1)^{-1}. \tag{3}$$

For any lonesum matrix A, Nanbara [3] introduced the *weight* of A as the sum of all entries in A, and did numerical experiments to some extent. For positive integers m, n and a non-negative integer k, we denote by $L(m,n,k)$ the number of $m\times n$ lonesum matrices with weight k. Further, we define $L(m,0,0) = L(0,n,0) = 1$ and $L(m,0,k) = L(0,n,k) = 0$ for any non-negative integers m,n and positive integer k.

By definition, we have $L(m,n,k) = L(n,m,k)$ for $0 \leq k \leq mn$, and

$$L(m,n) = \sum_{k=0}^{mn} L(m,n,k).$$

The following is the main theorem of this paper, which gives a generating function of $L(m,n,k)$.

Theorem 1. *Let x, y and q be indeterminates satisfying*

$$qx = xq, \quad qy = yq \quad \text{and} \quad yx = qxy. \tag{4}$$

Then the following identity holds:

$$\sum_{m,n,k\geq 0} L(m,n,k) \frac{x^m}{m!} \frac{y^n}{n!} q^k = (e^{-x}+e^{-y}-1)^{-1}. \tag{5}$$

It is surprising that the right-hand sides of (3) and (5) coincide with each other. Hence, by setting $q = 1$ in the equality (5), we naturally obtain the original generating function (3) of $L(m,n)$.

2. Proof of Theorem 1

An $m\times n$ $(0,1)$-matrix $A = (a_{i,j})$ is called a *Ferrers matrix* if A satisfies the condition

$$\begin{aligned} a_{i,j} &\geq a_{i+1,j} \quad (1 \leq i < m-1,\ 1 \leq j \leq n), \\ a_{i,j} &\geq a_{i,j+1} \quad (1 \leq i \leq m,\ 1 \leq j < n-1). \end{aligned}$$

This condition means that all 1 entries of A are placed at the upper left of A (see Fig. 1 in the proof of Theorem 1).

The following result is due to Ryser [4] (see also [5, Theorem 3.2]).

Proposition 2. *Let A be an $m \times n$ $(0,1)$-matrix. Then the following conditions are equivalent:*

(i) A is a lonesum matrix.

(ii) A has no minor of the form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *or* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(iii) A is obtained from a Ferrers matrix by permutations of columns and rows.

We can prove our main theorem from this proposition.

Proof of Theorem 1. For a Ferrers matrix $A = (a_{i,j})$, we define m_i and n_j $(1 \leq i, j \leq l)$ by the numbers of columns and rows as indicated in Figure 1. Here we have

$$\begin{gathered} m_1, \ldots, m_{l-1} > 0, \quad m_l \geq 0, \quad m_1 + \cdots + m_l = m, \\ n_1 \geq 0, \quad n_2, \ldots, n_l > 0, \quad n_1 + \cdots + n_l = n. \end{gathered}$$

Then the weight of A is expressed as

$$m_1 n_1 + m_2(n_1 + n_2) + \cdots + m_l(n_1 + \cdots + n_l) = \sum_{l \geq i \geq j \geq 1} m_i n_j.$$

The number of lonesum matrices obtained from the matrix A is equal to

$$\frac{m!}{m_1! \cdots m_l!} \frac{n!}{n_1! \cdots n_l!}.$$

Therefore we have

$$\begin{aligned} &\sum_{m,n,k \geq 0} L(m,n,k) \frac{x^m}{m!} \frac{y^n}{n!} q^k \\ &= \sum_{l=1}^{\infty} \sum_{\substack{m_1, \ldots, m_{l-1} > 0, m_l \geq 0 \\ n_1 \geq 0, n_2, \ldots, n_l > 0}} \frac{x^{m_1 + \cdots + m_l}}{m_1! \cdots m_l!} \frac{y^{n_1 + \cdots + n_l}}{n_1! \cdots n_l!} q^{\sum_{i \geq j} m_i n_j}. \end{aligned} \tag{6}$$

By the commutation relations (4), we have

$$x^{m_1 + \cdots + m_l} y^{n_1 + \cdots + n_l} q^{\sum_{i \geq j} m_i n_j} = y^{n_1} x^{m_1} y^{n_2} x^{m_2} \cdots y^{n_l} x^{m_l}.$$

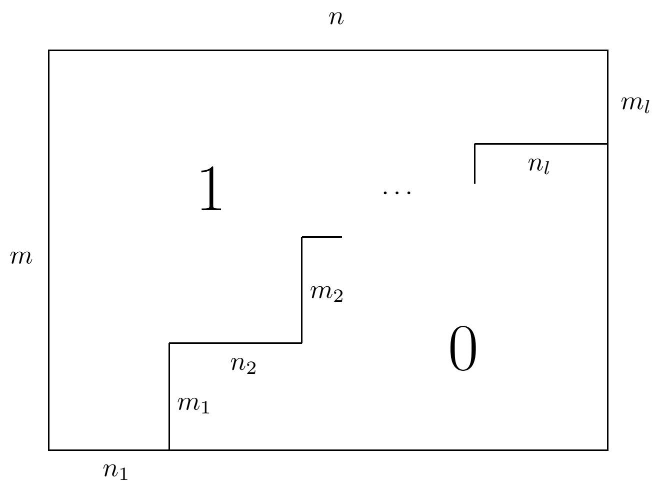

Figure 1: Ferrers matrix A

Hence the right-hand side of (6) equals

$$\begin{aligned}\sum_{l=1}^{\infty}\sum_{\substack{m_1,\ldots,m_{l-1}>0,\\ m_l\geq 0\\ n_1\geq 0,n_2,\ldots,n_l>0}}\frac{y^{n_1}}{n_1!}\frac{x^{m_1}}{m_1!}\frac{y^{n_2}}{n_2!}\frac{x^{m_2}}{m_2!}\cdots\frac{y^{n_l}}{n_l!}\frac{x^{m_l}}{m_l!} &= \sum_{l=1}^{\infty}e^y((e^x-1)(e^y-1))^{l-1}e^x\\ &= e^y\big\{1-(e^x-1)(e^y-1)\big\}^{-1}e^x\\ &= \big\{e^{-x}(1-(e^x-1)(e^y-1))e^{-y}\big\}^{-1}\\ &= (e^{-x}+e^{-y}-1)^{-1}.\end{aligned}$$

Now the proof is complete. □

3. A Recurrence Relation and an Explicit Formula

In this section we investigate the following polynomial in q:

$$L_q(m,n) := \sum_{k=0}^{mn} L(m,n,k)q^k, \quad (m,n\geq 0).$$

When $q = 1$, we have

$$L_1(m,n) = \sum_{k=0}^{mn} L(m,n,k) = L(m,n).$$

This polynomial $L_q(m,n)$ satisfies the following recurrence relation:

Proposition 3. *For $(m,n) \neq (0,0)$, we have*

$$L_q(m,n) = -\sum_{k=0}^{m-1} \binom{m}{k} (-1)^{m-k} L_q(k,n) - \sum_{l=0}^{n-1} \binom{n}{l} (-q^m)^{n-l} L_q(m,l). \tag{7}$$

Proof. By Theorem 1, the identity

$$(e^{-x} + e^{-y} - 1) \sum_{m \geq 0, n \geq 0} L_q(m,n) \frac{x^m}{m!} \frac{y^n}{n!} = 1$$

holds. Since $e^{-y} x^m = x^m e^{-q^m y}$, we have

$$\sum_{m \geq 0, n \geq 0} L_q(m,n) \frac{x^m}{m!} (e^{-x} + e^{-q^m y}) \frac{y^n}{n!} = 1 + \sum_{m \geq 0, n \geq 0} L_q(m,n) \frac{x^m}{m!} \frac{y^n}{n!}.$$

Comparing the coefficients of both sides, we obtain that

$$\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} L_q(k,n) + \sum_{l=0}^{n} \binom{n}{l} (-q^m)^{n-l} L_q(m,l) = L_q(m,n)$$

for $(m,n) \neq (0,0)$. The formula (7) immediately follows from this equation. $\square$

Now we give an explicit formula for the polynomial $L_q(m,n)$. This formula can be proved by using Proposition 3, but we will prove it directly to keep the proof simple.

Theorem 4. *For integers $m \geq 0$ and $n \geq 0$, we have*

$$L_q(m,n) = \sum_{\lambda \vdash m} (-1)^{m+l(\lambda)} \binom{m}{\lambda} \Big(\sum_{i=0}^{l(\lambda)} q^{\lambda_1 + \cdots + \lambda_i} \Big)^n. \tag{8}$$

Here $\lambda \vdash m$ means that λ is a composition of m, i.e., $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a sequence of positive integers such that $\lambda_1 + \cdots + \lambda_l = m$. For such λ, $l = l(\lambda)$ denotes its length and $\binom{m}{\lambda} = \frac{m!}{\lambda_1! \cdots \lambda_l!}$ the associated multinomial coefficient.

Proof. We denote the right-hand side of (8) by $T_q(m,n)$. By Theorem 1, it suffices to show that

$$\sum_{m \geq 0, n \geq 0} T_q(m,n) \frac{x^m}{m!} \frac{y^n}{n!} = (e^{-x} + e^{-y} - 1)^{-1} \tag{9}$$

under the commutation relations (4). We have

$$\begin{aligned}
&\sum_{n\geq 0} T_q(m,n)\frac{x^m}{m!}\frac{y^n}{n!}\\
&= \frac{x^m}{m!}\sum_{\lambda\vdash m}(-1)^{m+l(\lambda)}\binom{m}{\lambda}\sum_{n\geq 0}\left(\sum_{i=0}^{l(\lambda)} q^{\lambda_1+\cdots+\lambda_i}\right)^n\frac{y^n}{n!}\\
&= \frac{x^m}{m!}\sum_{\lambda\vdash m}(-1)^{m+l(\lambda)}\binom{m}{\lambda}e^{\sum_{i=0}^{l(\lambda)} q^{\lambda_1+\cdots+\lambda_i}y}\\
&= \sum_{\lambda\vdash m}(-1)^{l(\lambda)}\frac{(-x)^{\lambda_l}\cdots(-x)^{\lambda_1}}{\lambda_l!\cdots\lambda_1!}e^{\sum_{i=0}^{l(\lambda)} q^{\lambda_1+\cdots+\lambda_i}y}\\
&= \sum_{\lambda\vdash m}(-1)^{l(\lambda)}e^y\frac{(-x)^{\lambda_l}}{\lambda_l!}e^y\cdots e^y\frac{(-x)^{\lambda_1}}{\lambda_1!}e^y
\end{aligned}$$

(here we used the relation $(-x)^i e^{q^i y} = e^y(-x)^i$). Consequently we obtain that

$$\begin{aligned}
\sum_{m\geq 0, n\geq 0} T_q(m,n)\frac{x^m}{m!}\frac{y^n}{n!} &= \sum_{l\geq 0} e^y \overbrace{(1-e^{-x})e^y\cdots(1-e^{-x})e^y}^{l}\\
&= e^y\left\{1-(1-e^{-x})e^y\right\}^{-1}\\
&= \left\{e^{-y}-(1-e^{-x})\right\}^{-1}\\
&= (e^{-x}+e^{-y}-1)^{-1}
\end{aligned}$$

and this completes the proof. □

Examples. We have:

$L_q(1,n) = (1+q)^n$,
$L_q(2,n) = -(1+q^2)^n + 2(1+q+q^2)^n$,
$L_q(3,n) = (1+q^3)^n - 3(1+q+q^3)^n - 3(1+q^2+q^3)^n + 6(1+q+q^2+q^3)^n$.

By putting $q=-1$ in the cases $m=2$ and 3, we obtain the following alternating sum formulas:

Corollary 5. *For any integer $n \geq 1$, we have*

$$\sum_{k=0}^{2n} L(2,n,k)(-1)^k = 2-2^n, \tag{10}$$

$$\sum_{k=0}^{3n} L(3,n,k)(-1)^k = -3((-1)^n+1) = \begin{cases} 0 & \text{if } n \text{ is odd,}\\ -6 & \text{if } n \text{ is even.}\end{cases} \tag{11}$$

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